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A theorem of representation for Hilbert algebras

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Abstract. The main scope of this paper is to prove the following theorem of representation for a Hilbert algebra A : There exist a complete residuated lattice $L_r(A)$ which is a $G-$ algebra and an injective morphism of Hilbert algebras $i_A : A \to L_r(A)$. As a consequence, we deduce that the free Hertz algebra H_A over A (see [15]) is isomorphic with a Hertz subalgebra of $L_r(A)$. Also, I give the description of the elements of $L_r(A)$.

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1. Introduction

The concept of Hilbert algebras was introduced in the 50's by Henkin and Skolem for investigations in intuitionistic and other non-classical logics, as an algebraic counterpart of Hilbert's positive implicative propositional calculus ([16]). Hilbert algebras were intensively studied by A. Diego ([5]) and this theory was further developed by Busneag ([3]). BCK algebras were introduced by Iséki in 1966 ([9], [11]) to give an algebraic framework for Meredit's implicational logic BCK. Since Iséki's definition, these algebras have been studied by several authors. For further information see for example [2], [4], [7], [10], [12] and the references given there.

The paper is organized as follows: In Section 2 we recall the basic definitions and some results relative to BCK algebras; also we put in evidence some rules of calculus in Hilbert and BCK algebras (which we need in Section 3). In the final of Section 2 we put in evidence a theorem of embedding for Hilbert algebras into complete integral residuated lattices which is $G-$ algebra (Theorem 2.2).

In Section 3 we give a characterization of the elements of the complete integral residuated lattice $L_r(A)$ from Section 2.

2. Preliminaries

In this paper the symbols $\Rightarrow$ and $\Leftrightarrow$ are used for logical implication and respectively logical equivalence.

Definition 2.1. *(*[4], [10]*) A* BCK algebra *is an algebra* $(A, \to, 1)$ *of type (2,0) such that the following axioms are verified for every* $x, y, z \in A$*:*
(a_1) $x \to x = 1$*;*
(a_2) *If* $x \to y = y \to x = 1$, *then* $x = y$;
$(\mathbf{B})$ $(x \to y) \to ((y \to z) \to (x \to z)) = 1$;
$(\mathbf{C})$ $x \to (y \to z) = y \to (x \to z)$;

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(**K**) $x \to (y \to x) = 1$.

The relation $a \leq b$ iff $a \to b = 1$ is a partial order on A (called the *natural order* on A); with respect to this order 1 is the largest element of A.

For examples of BCK algebras see [4] and [10].

A *Hilbert algebra (*[3], [5], [10]*)* is a BCK algebra $(A, \to, 1)$ which verifies one of the following equivalent conditions for all $x, y \in A$:

(a_3)**:** $x \to (x \to y) = x \to y$;

(a_4)**:** $(x \to y) \to ((y \to x) \to x) = (y \to x) \to ((x \to y) \to y)$.

In a BCK algebra we have ([4], [7], [10], [12]) the following rules of calculus for $x, y, z \in A$:

(c_1) $x \leq y \to x$;

(c_2) $x \leq (x \to y) \to y$;

(c_3) $((x \to y) \to y) \to y = x \to y$;

(c_4) If $x \leq y$, then for every $z \in A$, $z \to x \leq z \to y$ and $y \to z \leq x \to z$;

(c_5) $x \to y \leq (z \to x) \to (z \to y) \leq z \to (x \to y)$;

(c_6) $x \to y \leq (y \to z) \to (x \to z)$.

If A is a Hilbert algebra, then

(c_7) $x \to (y \to z) = (x \to y) \to (x \to z)$.

If A is a BCK algebra and $x_1, ..., x_n, x \in A$ $(n \geq 1)$ we define $(x_1, ..., x_n; x) = x_1 \to (x_2 \to ...(x_n \to x)...)$.

Following (**C**) we deduce that if σ is permutation of $(1, 2, ..., n)$, then for every $x, y, x_1, ..., x_n \in A$:

(c_8) $(x_{\sigma(1)}, ..., x_{\sigma(n)}; x) = (x_1, ..., x_n; x)$;

(c_9) $(x_1, ..., x_n; x \to y) = x \to (x_1, ..., x_n; y)$.

If A is a Hilbert algebra then :

(c_{10}) $(x_1, ..., x_n; x \to y) = (x_1, ..., x_n; x) \to (x_1, ..., x_n; y)$.

For a BCK algebra A, two elements $x, y \in A$ and a natural number $n \geq 1$ we denote $x \to_n y = (x, x, ..., x; y)$, where n indicates the number of occurrences of x. Clearly, if A is a Hilbert algebra, then $x \to_n y = x \to y$, for every $n \geq 1$.

A *deductive system* (or *i-filter*) of a BCK algebra A is a nonempty subset $D \subseteq A$ such that:

(a_5) $1 \in D$;

(a_6) If $x, x \to y \in D$, then $y \in D$.

It is clear that if D is a deductive system, $a \leq b$ and $a \in D$, then $y \in D$ (that is, D is increasing subset of A).

We denote by $Ds(A)$ the set of all deductive systems of A (clearly, $\{1\}, A \in Ds(A)$).

For a nonempty subset $X \subseteq A$, the *deductive system generated by* X will be denoted by $[X)$. It is known ([7], [12]) that $[X) = \{x \in A : (x_1, ..., x_n; x) = 1$, for some $x_1, ..., x_n \in X$ $\}$. In particular for $a \in A, [\{a\}) \overset{not}{=} [a) = \{x \in A : a \to_n x = 1$, for some $n \geq 1\}$.

If $D \in Ds(A)$ and $a \in A \backslash D$, then $[D \cup \{a\}) \overset{not}{=} D(a) = \{x \in A : a \to_n x \in D$, for some $n \geq 1\}$.

In particular, if A is a Hilbert algebra, then for $X = \{x_1, ..., x_n\}, [X) = \{x \in A : (x_1, ..., x_n; x) = 1\}$ and if $D \in Ds(A)$ and $a \in A \backslash D$, then $D(a) = \{x \in A : a \to x \in D\}$.

Remark 2.1. *If A is a Hilbert algebra, then if $X = \{x_1, ..., x_m\}$ and $Y = \{y_1, ..., y_n\}$, $[X \cup Y) = [X)(y_1, ..., y_n) = [Y)(x_1, ..., x_m)$*

(where $[X)(y_1, ..., y_n) = (...([X)(y_1))(y_2)...)(y_n))$.

For a BCK algebra A we let $W(A)$ denote the set of all words $\mathcal{X} = x_1x_2...x_n$ $(n \geq 1)$ over A.

For any word $\mathcal{W} = x_1x_2...x_n \in W(A)$ and an element $a \in A$, we shall write $\mathcal{W} \to a = (x_1, x_2, ..., x_n; a) \in A$.

Remark 2.2. *If* $\mathcal{W} \in W(A)$*, then* $\mathcal{W} \to a = 1 \Rightarrow a \in [\mathcal{W})$*. If* A *is a Hilbert algebra, then* $\mathcal{W} \to a = 1 \Leftrightarrow a \in [\mathcal{W})$.

From (**C**) we deduce that for $\mathcal{X}$,$\mathcal{Y} \in W(A)$ and $a \in A$, then:

(c_{12}) $\mathcal{X} \to (\mathcal{Y} \to a) = \mathcal{Y} \to (\mathcal{X} \to a) = (\mathcal{X}\mathcal{Y}) \to a$, where $\mathcal{X}\mathcal{Y} \in W(A)$ stand for concatenation of $\mathcal{X}$ and $\mathcal{Y}$.

Let $Fin(W(A))$ be the set of all finite non-empty subsets of $W(A)$.

One readily sees ([13]) that the relation ρ_A defined on $Fin(W(A))$ by the stipulation $\{\mathcal{X}_1, ..., \mathcal{X}_n\}\ \rho_A \{\mathcal{Y}_1, ..., \mathcal{Y}_n\}$ iff for all $\mathcal{W} \in W(A)$ and $a \in A$ we have

$\mathcal{W} \to (\mathcal{X}_i \to a) = 1$ for all $i = 1, 2, ..., m$ iff $\mathcal{W} \to (\mathcal{Y}_j \to a) = 1$ for all $j = 1, 2, ..., n$ is an equivalence on $Fin(W(A))$; the ρ_A -class of $\{\mathcal{X}_1, ..., \mathcal{X}_n\}$ will be briefly denoted as $< \mathcal{X}_1, ..., \mathcal{X}_n >$. Further, we equip the quotient set $M_A \overset{not}{=} Fin(W(A))/\rho_A$ with two binary operations $\sqcap$ and $\star$, as follows:

$$< \mathcal{X}_1, ..., \mathcal{X}_m > \sqcap < \mathcal{Y}_1, ..., \mathcal{Y}_n > = < \mathcal{X}_1, ..., \mathcal{X}_m, \mathcal{Y}_1, ..., \mathcal{Y}_n >,$$

$$< \mathcal{X}_1, ..., \mathcal{X}_m > \star < \mathcal{Y}_1, ..., \mathcal{Y}_n > = < \mathcal{X}_i\mathcal{Y}_j : i = 1, 2, ..., m, j = 1, 2, ..., n > .$$

Definition 2.2. *By a* meet-semilattice-orderd monoid *we mean an algebra* $(M, \wedge, \bullet, e)$ *such that :*

(a_7) $(M, \wedge)$ *is a meet-semilattice;*

(a_8) $(M, \bullet, e)$ *is a monoid;*

(a_9) $(x \wedge y) \bullet z = (x \bullet z) \wedge (y \bullet z)$ *and* $z \bullet (x \wedge y) = (z \bullet x) \wedge (z \bullet y)$ *for every* $x, y, z \in A$.

If the identity element e is the least element of M (that is, e play the role of 0), then M is called *dually integral*.

In [13] it is proved the following result:

Proposition 2.1. *For every* BCK *algebra* A*, the structure* $(M_A, \sqcap, \star, < 1 >)$ *is a dually integral meet-semilattice-orderd monoid.*

Remark 2.3. *In* [13]*, the above result is obtained for the case of a pseudo* BCK *algebra* A; *if* A *is a* BCK *algebra, then the operation* $\star$ *is commutative. Indeed, if* $\alpha = < \mathcal{X}_1, ..., \mathcal{X}_m >, \beta = < \mathcal{Y}_1, ..., \mathcal{Y}_n > \in M_A$*, then* $\alpha \star \beta = < \mathcal{X}_i\mathcal{Y}_j : i = 1, 2, ..., m, j = 1, 2, ..., n >$ *. If* $\mathcal{W} \in W(A)$ *and* $a \in A$*, then* $\mathcal{W} \to (\mathcal{X}_i\mathcal{Y}_j \to a) = 1$ *iff* $\mathcal{W} \to (\mathcal{X}_i \to (\mathcal{Y}_j \to a)) = 1$ *iff* $\mathcal{W} \to (\mathcal{Y}_j \to (\mathcal{X}_i \to a)) = 1$ *iff* $\mathcal{W} \to (\mathcal{Y}_j\mathcal{X}_i \to a)) = 1$*,for all* $i = 1, 2, ..., m$ *and* $j = 1, 2, ..., n$*, so* $\alpha \star \beta = \beta \star \alpha$.

Lemma 2.1. *Let* $(M, \wedge, \bullet, e)$ *a dually integral meet-semilattice-orderd (commutative) monoid. Then for every* $x, y \in M$:

(c_{13})**:** $x \leq x \bullet y, y \leq x \bullet y$;

(c_{14})**:** $x \leq x \bullet x$.

Proof. (c_{13}). We have $x \bullet (y \wedge e) = (x \bullet y) \wedge (x \bullet e) \Rightarrow x \bullet e = (x \bullet y) \wedge x \Rightarrow x = (x \bullet y) \wedge x \Rightarrow x \leq x \bullet y$.

(c_{14}). Clearly. ∎

Remark 2.4. *It is worth noticing that the partial order $\sqsubseteq$ associated with the meet operation $\sqcap$ on M_A we have $<\mathcal{X}_1, ..., \mathcal{X}_m> \sqsubseteq <\mathcal{Y}_1, ..., \mathcal{Y}_n>$ iff for all $\mathcal{W} \in W(A)$ and $a \in A$, $\mathcal{W} \to (\mathcal{X}_i \to a) = 1$ for all $i = 1, 2, ..., m$, then $\mathcal{W} \to (\mathcal{Y}_j \to a) = 1$ for all $j = 1, 2, ..., n$.*

Corollary 2.1. *If A is a Hilbert algebra, then $\alpha \star \alpha = \alpha$ for every $\alpha \in M_A$.*

Proof. By (c_{14}) we deduce that $\alpha \sqsubseteq \alpha \star \alpha$. To prove that $\alpha \star \alpha \sqsubseteq \alpha$,
let $\alpha = <\mathcal{X}_1, ..., \mathcal{X}_m> \in M_A$, $\mathcal{W} \in W(A)$ and $a \in A$ such that $\mathcal{W} \to (\alpha \star \alpha \to a) = 1$. Since $\alpha \star \alpha = <\mathcal{X}_1\mathcal{X}_1, \mathcal{X}_1\mathcal{X}_2, ..., \mathcal{X}_2\mathcal{X}_2, ..., \mathcal{X}_{n-1}\mathcal{X}_n, \mathcal{X}_n\mathcal{X}_n>$, then in particular we have $\mathcal{W} \to (\mathcal{X}_i\mathcal{X}_i \to a) = 1$ for all $i = 1, 2, ..., m$.

Since A is a Hilbert algebra, then for all $i = 1, 2, ..., m$ we have $\mathcal{W} \to (\mathcal{X}_i \to (\mathcal{X}_i \to a)) = 1 \Rightarrow \mathcal{W} \to (\mathcal{X}_i \to a) = 1$, hence $\alpha \star \alpha \sqsubseteq \alpha$, so $\alpha \star \alpha = \alpha$. ∎

We recall that if $(M, \wedge)$ is a meet-semilattice, then $F \subseteq M$ is a *filter* ([1]) if $x, y \in F \Rightarrow x \wedge y \in F$ and if $x \leq y$ and $x \in F \Rightarrow y \in F$.

For $(M, \wedge, \bullet, e)$ a dually integral meet-semilattice-orderd monoid, let $\mathcal{F}(M)$ the set of all filters of $(M, \wedge)$ augmented by Ø.

Let us introduce the following notation for $F, G \in \mathcal{F}(M)$:

$$F \vee G = \text{the filter generated by } F \cup G = \{a \in M : x \wedge y \leq a \text{ for some } x, y \in F \cup G\},$$

$$F \odot G = \{a \in M : x \bullet y \leq a \text{ for some } x \in F \text{ and y} \in G\},$$

$$\begin{aligned} F \quad \to \quad G &= \{a \in M : \{a\} \bullet F \subseteq G\} = \\ &= \quad \{a \in M : \text{ if } x \in M \text{ and } x \geq a \bullet f \text{ with } f \in F, \text{ then } x \in G\}. \end{aligned}$$

We recall ([6], [14]) that an *integral residuated lattice* is an algebra $(L, \vee, \wedge, \odot, \to, 0, 1)$ such that $(L, \vee, \wedge, 0, 1)$ is a bounded lattice, $(L, \odot, 1)$ is a (commutative) monoid whose identity 1 is the greatest element of the lattice and $x \odot a \leq y$ iff $a \leq x \to y$ for all $a, x, y \in L$.

Remark 2.5. *([6], [14]) If $(L, \vee, \wedge, \odot, \to, 0, 1)$ is an integral residuated lattice then $(L, \to, 1)$ is a BCK algebra.*

In [13] it is proved the following result:

Lemma 2.2. *If A is a BCK algebra, then $(\mathcal{F}(M_A), \vee, \cap, \odot, \to, O, M_A)$ is a complete integral residuated lattice.*

For $a \in A$, we put $i_A(a) = \{<\mathcal{X}_1, ..., \mathcal{X}_m> \in M_A : \mathcal{X}_i \to a = 1$, for all $i = 1, 2, ..., m\}$.

In [13] it is proved the following result:

Theorem 2.1. *If A is a BCK algebra, then the map $i_A : A \to L_r(A) = \mathcal{F}(M_A)$ is an injective morphism of BCK algebras. Moreover, if for $a, b \in A$ there exists $a \vee b$ in A, then $i_A(a \vee b) = i_A(a) \vee i_A(b)$.*

Taking as guide-line the case of BL algebras (see [8], Definition 4.2.12), an integral residuated lattice L is a G *-algebra* if $x \odot x = x$,for every $x \in L$.

We have the following results:

Proposition 2.2. *([14]) Let $(L, \vee, \wedge, \odot, \to, 0, 1)$ is an integral residuated lattice. Then the following are equivalent:*
(i)**:** *L is a G -algebra;*
(ii)**:** *$x \odot y = x \wedge y$, for every $x, y \in L$;*
(iii)**:** *$x \odot (x \to y) = x \wedge y$, for every $x, y \in L$.*

Proposition 2.3. *([14])For an integral residuated lattice* $(L, \vee, \wedge, \odot, \rightarrow, 0, 1)$ *the following are equivalent:*
(i)**:** $(L, \rightarrow, 1)$ *is a Hilbert algebra;*
(ii)**:** $(L, \vee, \wedge, \odot, \rightarrow, 0, 1)$ *is a* G *-algebra.*

Lemma 2.3. *If* A *is a Hilbert algebra, then the integral residuated lattice* $L_r(A)$ *is a* G*-algebra.*

Proof. We must prove that for $F \in L_r(A), F \odot F = F$. Since $L_r(A)$ is an integral residuated lattice, then $F \odot F \subseteq F$ ([6], [14]). If $\alpha \in F$, by Corollary 2.1, $\alpha = \alpha \star \alpha$, hence $\alpha \in F \odot F \Rightarrow F \subseteq F \odot F$, so $F = F \odot F$. ∎

From Lemma 2.3 and Theorem 2.1 we obtain the following theorem of representation for Hilbert algebras:

Theorem 2.2. *If* A *is a Hilbert algebra, then there exist a complete integral residuated lattice* $L_r(A)$ *which is a* $G-$ *algebra and an injective morphism of Hilbert algebras* $i_A : A \rightarrow L_r(A)$*. Moreover, if for* $a, b \in A$ *there exists* $a \vee b$ *in* A*, then* $i_A(a \vee b) = i_A(a) \vee i_A(b)$*.*

Remark 2.6. *For others theorems of representation for Hilbert algebras, see* [3], [5].

3. A characterization of the elements of $L_r(A)$

If $(S, \wedge)$ is a meet-semilattice, for a nonempty subset $M \subseteq S$, by $[M)$ we denote the filter of S generated by M.

We have ([1]): $[M) = \{x \in S : x_1 \wedge ... \wedge x_n \leq x$ for some $x_1, ..., x_n \in M\}$. In particular, if $M = \{a\}, [\{a\}) \overset{not}{=} [a) = \{x \in S : a \leq x\}$.

Remark 3.1. *We recall ([1]) that if* $(S, \wedge)$ *is a meet-semilattice then:*
(i)**:** *If* $a, b \in S$ *and* $a \leq b \Rightarrow [b) \subseteq [a)$*;*
(ii)**:** *If* $a_1, a_2, ..., a_n \in S$ *then* $[a_1 \wedge a_2 \wedge ... \wedge a_n) = [a_1) \vee [a_2) \vee ... \vee [a_n)$*.*

Lemma 3.1. *If* A *is a BCK algebra, then for every* $a \in A, i_A(a) = [< a >)$*.*

Proof. If $< \mathcal{X}_1, ..., \mathcal{X}_n > \in [< a >) \Rightarrow < a > \sqsubseteq < \mathcal{X}_1, ..., \mathcal{X}_n >$. Since $1 \rightarrow (a \rightarrow a) = 1 \Rightarrow 1 \rightarrow (\mathcal{X}_i \rightarrow a) = 1$, for $i = 1, 2, ..., n \Rightarrow < \mathcal{X}_1, ..., \mathcal{X}_n > \in i_A(a) \Rightarrow [< a >) \subseteq i_A(a)$.

Conversely, let $< \mathcal{X}_1, ..., \mathcal{X}_n > \in i_A(a)$, that is, $\mathcal{X}_i \rightarrow a = 1$, for $i = 1, 2, ..., n$. To prove $< a > \sqsubseteq < \mathcal{X}_1, ..., \mathcal{X}_n >$, let $\mathcal{W} = a_1 a_2 ... a_m \in W(A)$ and $x \in A$ such that $\mathcal{W} \rightarrow (a \rightarrow x) = 1$.

For $i \in \{1, 2, ..., n\}$ consider $\mathcal{X}_i = x_1 ... x_t \in W(A)$. ¿From $\mathcal{W} \rightarrow (a \rightarrow x) = 1 \Rightarrow a \rightarrow (\mathcal{W} \rightarrow x) = 1 \Rightarrow a \leq (a_1, ..., a_m; x) \overset{(c_3)}{\Rightarrow} (x_1, ..., x_t; a) \leq (x_1, ..., x_t, a_1, ..., a_m; x) \Rightarrow$
$(x_1, ..., x_t, a_1, ..., a_m; x) = 1 \Rightarrow \mathcal{X}_i \rightarrow (\mathcal{W} \rightarrow a) = 1 \overset{(C)}{\Rightarrow} \mathcal{W} \rightarrow (\mathcal{X}_i \rightarrow a) = 1$, for $i = 1, 2, ..., n \Rightarrow < \mathcal{X}_1, ..., \mathcal{X}_n > \in [< a >) \Rightarrow i_A(a) = [< a >)$. ∎

Lemma 3.2. *Let* A *be a BCK algebra and* $< \mathcal{X}_1, ..., \mathcal{X}_n > \in M_A$*. Then*

$$[< \mathcal{X}_1, ..., \mathcal{X}_n >) = [< \mathcal{X}_1 >) \vee ... \vee [< \mathcal{X}_n >).$$

Proof. We have $[< \mathcal{X}_1, ..., \mathcal{X}_n >) = [< \mathcal{X}_1 > \sqcap ... \sqcap < \mathcal{X}_n >) \overset{\text{Remark3.1},(ii)}{=}$
$[< \mathcal{X}_1 >) \vee ... \vee [< \mathcal{X}_n >)$. ∎

Lemma 3.3. *If* A *is a Hilbert algebra and* $a_1, a_2, ..., a_n \in A$*, then* $[< a_1 a_2 ... a_n >) = [< a_1 >) \cap ... \cap [< a_n >)$*.*

Proof. It is suffice to prove that for two elements $a, b \in A$, we have the equality $[<ab>) = [<a>) \cap [<b>)$.

Indeed, $<a> \star <b> = <ab>$ and since $<a>, <b> \sqsubseteq <a> \star <b> = <ab>$ we deduce that $[<ab>) \subseteq [<a>), [<b>) \Rightarrow [<ab>) \subseteq [<a>) \cap [<b>)$.

To prove the converse inclusion, let $<\mathcal{X}_1, ..., \mathcal{X}_n> \in [<a>) \cap [<b>)$. Then $<a>, <b> \subseteq <\mathcal{X}_1, ..., \mathcal{X}_n>$.

Consider $\mathcal{W} \in W(A)$ and $a \in A$ such that $\mathcal{W} \to (ab \to x) = 1$. Then $\mathcal{W} \to (a \to (b \to x)) = 1$. Since $<a> \subseteq <\mathcal{X}_1, ..., \mathcal{X}_n>$, then $\mathcal{W} \to (\mathcal{X}_i \to (b \to x)) = 1$, for all $i = 1, 2, ..., n \Rightarrow \mathcal{W} \to (b \to (\mathcal{X}_i \to x)) = 1$, for all $i = 1, 2, ..., n \Rightarrow$

$\mathcal{W} \to (\mathcal{X}_i \to (\mathcal{X}_i \to x)) = 1$, for all $i = 1, 2, ..., n \Rightarrow \mathcal{W} \to (\mathcal{X}_i \to x) = 1$, for all $i = 1, 2, ..., n \Rightarrow <ab> \sqsubseteq <\mathcal{X}_1, ..., \mathcal{X}_n> \Rightarrow <\mathcal{X}_1, ..., \mathcal{X}_n> \in [<ab>) \Rightarrow [<a>) \cap [<b>) \subseteq [<ab>) \Rightarrow [<a>) \cap [<b>) = [<ab>)$. ∎

Corollary 3.1. *If A is a Hilbert algebra and $\mathcal{W} = a_1 a_2 ... a_n \in W(A)$, then*

$$[<\mathcal{W}>) = i_A(a_1) \cap ... \cap i_A(a_n).$$

Proof. By Lemma 3.3 we deduce that $[<\mathcal{W}>) = [<a_1 a_2 ... a_n>) =$
$[<a_1>) \cap ... \cap [<a_n>) = i_A(a_1) \cap ... \cap i_A(a_n)$. ∎

From the above results we obtain the following theorem of characterization for the elements of $L_r(A)$ when A is a Hilbert algebra:

Theorem 3.1. *Let A be a Hilbert algebra. Then for $F \in L_r(A) = \mathcal{F}(M_A)$ we have* $F = \bigvee_{<\mathcal{X}_1, ..., \mathcal{X}_n> \in F} [(\bigcap_{x \in \mathcal{X}_1} i_A(x)) \vee ... \vee (\bigcap_{x \in \mathcal{X}_n} i_A(x))]$.

Proof. For $F \in L_r(A) = \mathcal{F}(M_A)$ we have $F = \bigvee_{<\mathcal{X}_1, ..., \mathcal{X}_n> \in F} [<\mathcal{X}_1, ..., \mathcal{X}_n>) \overset{Lemma 3.2}{=}$

$= \bigvee_{<\mathcal{X}_1, ..., \mathcal{X}_n> \in F} [[<\mathcal{X}_1>) \vee ... \vee [<\mathcal{X}_n>)] \overset{Lemma 3.3}{=}$

$= \bigvee_{<\mathcal{X}_1, ..., \mathcal{X}_n> \in F} [(\bigcap_{x \in \mathcal{X}_1} i_A(x)) \vee ... \vee (\bigcap_{x \in \mathcal{X}_n} i_A(x))]$. ∎

Definition 3.1. *A* Hertz algebra *is a Hilbert algebra A with the property that for every $x, y \in A$, the infimum $x \wedge y$ (relative to the natural ordering) exists in A (that is, A is meet-semilattice relative to the natural order) and for every $x, y \in A$ we have the relation:*

(P): $x \to (y \to (x \wedge y)) = 1$.

In [15] it is proved the equivalence of above definition with:

Definition 3.2. *A* Hertz algebra *is an algebra $(A, \to, \wedge)$ of type $(2, 2)$ satisfying the following axioms:*

(a_{10})**:** $x \to x = y \to y$;
(a_{11})**:** $(x \to y) \wedge y = y$;
(a_{12})**:** $x \wedge (x \to y) = x \wedge y$;
(a_{13})**:** $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$.

Definition 3.3. *If A is a Hilbert algebra, a Hertz algebra H_A (together with an injective morphism of Hilbert algebras $\varphi_A : A \to H_A$) is said to be free over A if:*

(a_{14})**:** *H_A is generated (as a Hertz algebra) by $\varphi_A(A)$;*
(a_{15})**:** *For every Hertz algebra H and every morphism of Hilbert algebras $f : A \to H$, there exists a unique morphism of Hertz algebras $f' : H_A \to H$ such that $f' \circ \varphi_A = f$.*

Theorem 3.2. *([15]) For every Hilbert algebra A, there exists the free Hertz algebra H_A over A, unique up to an isomorphism of Hertz algebras.*

In what follow we only recall the construction of the Hertz algebra H_A (using the model and notations from [15]).

Let $\mathcal{F}(A)$ the set of all finite and nonempty subsets of A and $I = \{1\}$.

For $\mathcal{X} = \{x_1, x_2, ..., x_m\}$ and $\mathcal{Y} = \{y_1, ..., y_n\} \in \mathcal{F}(A)$ we define

$$\mathcal{X} \to \mathcal{Y} = \bigcup_{1 \leq j \leq n} \{(x_1, x_2, ..., x_m; y_j)\} \text{ and } \mathcal{X} \wedge \mathcal{Y} = \mathcal{X} \cup \mathcal{Y}.$$

Consider the relation θ_A on $\mathcal{F}(A)$ defined for $\mathcal{X}, \mathcal{Y} \in \mathcal{F}(A)$ by

$$\mathcal{X}\theta_A\mathcal{Y} \text{ iff } \mathcal{X} \to \mathcal{Y} = \mathcal{Y} \to \mathcal{X} = \mathcal{I}.$$

Then θ_A is an equivalence relation on $\mathcal{F}(A)$ compatible with the operations $\to$ and $\wedge$.

For $\mathcal{X} \in \mathcal{F}(A)$ we denote by $[\mathcal{X}]_{\theta_A}$ the equivalence class of $\mathcal{X}$ modulo θ_A and by $H_A = \mathcal{F}(A)/\theta_A$.

For $a \in A$ we define $\varphi_A : A \to H_A, \varphi_A(a) = [\{a\}]_{\theta_A}$. Then $(H_A, \to, \mathbf{1})$ is the free Hertz algebra over A (where for $\mathcal{X}, \mathcal{Y} \in \mathcal{F}(A), [\mathcal{X}]_{\theta_A} \to [\mathcal{Y}]_{\theta_A} = [\mathcal{X} \to \mathcal{Y}]_{\theta_A}, [\mathcal{X}]_{\theta_A} \wedge [\mathcal{Y}]_{\theta_A} = [\mathcal{X} \wedge \mathcal{Y}]_{\theta_A}$ and $\mathbf{1} = [\{1\}]_{\theta_A}$).

If H is a Hertz algebra and $f : A \to H$ is a morphism of Hilbert algebras, then $f' : H_A \to H$, $f'([\mathcal{X}]_{\theta_A}) = \bigwedge_{i=1}^{m} f(x_i)$ $(\mathcal{X} = \{x_1, x_2, ..., x_m\})$ is the unique morphism of Hertz algebras such that $f' \circ \varphi_A = f$.

For a Hilbert algebra A I want to re-write the relation θ_A using the notation from Section 2.

So, we can consider an element $\mathcal{X} = \{x_1, x_2, ..., x_m\} \in \mathcal{F}(A)$ as the word $\mathcal{X} = x_1 x_2 ... x_n \in W(A)$ and for $a \in A$, $\mathcal{X} \to a = (x_1, x_2, ..., x_n; a) \in A$.

Lemma 3.4. *If A is a Hilbert algebra, then $\rho_A = \theta_A$.*

Proof. Clearly, for $\mathcal{X} = \{x_1, x_2, ..., x_m\}$, $\mathcal{Y} = \{y_1, ..., y_n\} \in \mathcal{F}(A)$, $\mathcal{X}\theta_A\mathcal{Y}$ iff $\mathcal{X} \to y_j = \mathcal{Y} \to x_i = 1$ for every $i = 1, 2, ..., m, j = 1, 2, ..., n \Leftrightarrow x_i \in [\mathcal{Y})$ and $y_j \in [\mathcal{X})$ for every $i = 1, 2, ..., m, j = 1, 2, ..., n \Leftrightarrow [\mathcal{Y}) = [\mathcal{X})$.

Suppose $\mathcal{X}\rho_A\mathcal{Y}$ (that is, if $\mathcal{W} \in W(A), a \in A$, then $\mathcal{W} \to (\mathcal{X} \to a) = 1$ iff $\mathcal{W} \to (\mathcal{Y} \to a) = 1$). Since $1 \to (\mathcal{X} \to x_i) = 1$ for every $i = 1, 2, ..., m$, then $1 \to (\mathcal{Y} \to x_i) = 1$ for every $i = 1, 2, ..., m \Rightarrow [\mathcal{X}) \subseteq [\mathcal{Y})$. Analogously we deduce $[\mathcal{Y}) \subseteq [\mathcal{X})$, so $[\mathcal{X}) = [\mathcal{Y})$, hence $\mathcal{X}\theta_A\mathcal{Y}$.

Suppose that $\mathcal{X}\theta_A\mathcal{Y}$ (hence $[\mathcal{X}) = [\mathcal{Y})$) and consider $\mathcal{W} \in W(A)$ and $a \in A$ such that $\mathcal{W} \to (\mathcal{X} \to a) = 1$. Then $a \in [\mathcal{W} \cup \mathcal{X}) = [\mathcal{X})(\mathcal{W})$. Since $[\mathcal{X}) = [\mathcal{Y}) \Rightarrow a \in [\mathcal{Y})(\mathcal{W}) \Rightarrow a \in [\mathcal{Y} \cup \mathcal{W}) \Rightarrow \mathcal{W} \to (\mathcal{Y} \to a) = 1 \Rightarrow \mathcal{X}\rho_A\mathcal{Y}$. ∎

Corollary 3.2. *If A is a Hilbert algebra, then $H_A = \mathcal{F}(A)/\theta_A = W(A)/\rho_A$.*

Theorem 3.3. *If A is a Hilbert algebra, then there exist an injective morphism of Hertz algebras $\Psi_A : H_A \to L_r(A)$ such that $\Psi_A \circ \varphi_A = i_A$.*

Proof. The existence of $\Psi_A : H_A \to L_r(A)$ is assured by Theorem 3.2 and for $\mathcal{X} = \{x_1, x_2, ..., x_m\} \in \mathcal{F}(A)$, $\Psi_A([\mathcal{X}]_{\theta_A}) = \bigwedge_{i=1}^{m} i_A(x_i)$.

To prove the injectivity of Ψ_A, consider $\mathcal{Y} = \{y_1, ..., y_n\} \in \mathcal{F}(A)$ such that $\Psi_A([\mathcal{X}]_{\theta_A}) = \Psi_A([\mathcal{Y}]_{\theta_A}) \Leftrightarrow \bigwedge_{i=1}^{m} i_A(x_i) = \bigwedge_{j=1}^{n} i_A(y_j)$. Then for every $j = 1, 2, ..., n$:

$\overset{m}{\underset{i=1}{\wedge}} i_A(x_i) \leq i_A(y_j) \Rightarrow (\overset{m}{\underset{i=1}{\wedge}} i_A(x_i)) \to i_A(y_j) = 1 \Rightarrow (i_A(x_1), ..., i_A(x_m); i_A(y_j)) = 1 \Rightarrow i_A((x_1, ..., x_m; y_j)) = 1 \Rightarrow (x_1, ..., x_m; y_j) = 1 \Rightarrow [\mathcal{Y}) \subseteq [\mathcal{X})$ and analogously $[\mathcal{X}) \subseteq [\mathcal{Y})$, hence $[\mathcal{X}) = [\mathcal{Y})$, that is, Ψ_A is injective. ∎

Corollary 3.3. *If A is a Hilbert algebra, then the free Hertz algebra H_A over A is isomorphic with a Hertz subalgebra of $L_r(A)$.*

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