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Note on tense SHn–algebras

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Abstract. In this article, we continue the study of tense SHn–algebras [5]. These algebras constitute a generalization of tense Łukasiewicz–Moisil algebras [4]. In particular, we describe a discrete duality for tense SHn–algebras bearing in mind the results indicated by Orłowska and Rewitzky in [13], for SHn–algebras. In addition, we introduce a propositional calculus and prove this calculus has tense SHn–algebras as algebraic counterpart. Finally, the duality mentioned above allowed us to show the completeness theorem for this calculus.

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1. Introduction

Classical tense logic is a logical system obtained from bivalent logic by adding the tense operators G (*it is always going to be the case that*) and H (*it has always been the case that*)(see [1, 10]). Starting with other logical systems and adding appropiate tense operators, we produce new tense logics (see [2, 3, 4]).

On the other hand, the propositional SHn–logics were introduced by Iturrioz in [7]. In [7], she gave a lattice–based semantics for these logics, by means of symmetrical Heyting algebras of order n, or for short SHn–algebras. There are two motivations for the study of SHn–logics and SHn–algebras. On the one hand, SHn–algebras are intended to provide an exact algebraic approach to the n–valued counterpart of the symmetrical modal propositional calculus introduced by Moisil in 1942. The symmetrical modal propositional calculus is obtained by the addition of one unary connective (a negation, characterized by the double negation law and the contraposition rule) to the alphabet of the intuitionistic propositional calculus. On the other hand, it has been shown that Post and Łukasiewicz–Moisil algebras are both Heyting algebras with operators. In both Łukasiewicz–Moisil and Post algebras a symmetry can be expressed in terms of the primitive operations [11, 12], cf. also [8]. This led to the study of more general algebras, called SHn–algebras [7, 8], which are Heyting algebras with a symmetry and which, in addition, have $n-1$ unary operations that satisfy certain properties, or, alternatively, can be seen as Łukasiewicz–Moisil algebras with a generalized negation [8]. In [9], Iturrioz and Orłowska give a completeness theorem for SHn–logics with respect to a Kripke–style semantics. Also a discrete duality for SHn–algebras is given by Orłowska and Rewitzky in [13].

This paper is devoted to the tense propositional SHn–calculus, a logical system obtained from the propositional SHn–logic by adding the tense operators G and H. The algebraic basis of this logic consists of tense SHn–algebras, algebraic structures studied in our paper [5].

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2. Preliminaries

SHn–logics were introduced by Iturrioz [7] and further studied by other authors [9, 16]. The language of SHn–logics is a propositional language, whose formulae are built from propositional variables taken from a set V, with operations $\vee$ (disjunction), $\wedge$ (conjunction), $\to$ (intuitionistic implication), $\sim, \neg$ (a De Morgan, resp. an intuitionistic negation), and a family $\{S_i : i = 1, \ldots, n-1\}$ of unary operations (which, intuitively, represent degrees of truth). The following Hilbert style axiomatization of SHn–logics is given in [7].

Axioms:

(A1) $\alpha \to (\beta \to \alpha)$
(A2) $(\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$
(A3) $(\alpha \wedge \beta) \to \alpha$
(A4) $(\alpha \wedge \beta) \to \beta$
(A5) $(\alpha \to \beta) \to ((\alpha \to \gamma) \to (\alpha \to (\beta \wedge \gamma)))$
(A6) $\alpha \to (\alpha \vee \beta)$
(A7) $\beta \to (\alpha \vee \beta)$
(A8) $(\alpha \to \gamma) \to ((\beta \to \gamma) \to ((\alpha \vee \beta) \to \gamma))$
(A9) $\sim\sim \alpha \leftrightarrow \alpha$
(A10) $S_i(\alpha \wedge \beta) \leftrightarrow S_i(\alpha) \wedge S_i(\beta)$
(A11) $S_i(\alpha \to \beta) \leftrightarrow (\bigwedge_{k=i}^{n} S_k(\alpha) \to S_k(\beta))$
(A12) $S_i(S_j(\alpha)) \leftrightarrow S_j(\alpha)$, for every $i, j = 1, \ldots, n-1$
(A13) $S_1(\alpha) \to \alpha$
(A14) $S_i(\sim \alpha) \leftrightarrow \sim S_{n-i}(\alpha)$, for $i = 1, \ldots, n-1$
(A15) $S_1(\alpha) \vee \neg S_1(\alpha)$,

where $\neg\alpha = (\alpha \to\sim (\alpha \to \alpha)$ and $\alpha \leftrightarrow \beta$ is an abbreviation for $(\alpha \to \beta) \wedge (\beta \to \alpha)$.

Inference rules:

(R1) $\dfrac{\alpha, \ \ \alpha \to \beta}{\beta}$ (R2) $\dfrac{\alpha \to \beta}{\sim \beta \to \sim \alpha}$ (R3) $\dfrac{\alpha \to \beta}{S_1\alpha \to S_1\beta}$.

The SHn–algebras constitute the algebraic counterpart of the SHn–logics. The Lindembaum–Tarski algebra of the SHn–logics is an SHn–algebra (see [7], p. 300).

We shall recall the definition of SHn–algebras.

Definition 2.1. *A symmetrical Heyting algebra of order n (SHn–algebra for short) is an algebra $(W, \vee, \wedge, \to, \sim, \{S_i\}_{i=1}^{n-1}, 0, 1)$ where $(W, \vee, \wedge, \to, 0, 1)$ is a Heyting algebra and the unary operations $\sim$ and S_i (for $i = 1, \cdots, n-1$) satisfy the following axioms:*

(n1) $\sim\sim x = x$,
(n2) $\sim (x \vee y) =\sim x\wedge \sim y$,
(S1) $S_i(x \wedge y) = S_i(x) \wedge S_i(y)$,
(S2) $S_i(x \to y) = \bigwedge_{k=i}^{n}(S_k(x) \to S_k(y))$,
(S3) $S_i(S_j(x)) = S_j(x)$, *for every* $i, j = 1, \ldots, n-1$,
(S4) $S_1(x) \vee x = x$,
(S5) $S_i(\sim x) = \sim S_{n-i}(x)$, *for* $i = 1, \ldots, n-1$,
(S6) $S_1(x) \vee \neg S_1(x) = 1$, *with* $\neg x = x \to 0$.

In [13], Orłowska and Rewitzky introduced the notion of SHn–frame as a structure $(X \leq, N, \{s_i\}_{i=1}^{n-1})$, where X is a non–empty set, $\leq$ is a quasi–order on X and N and s_i (for $i = 1, \ldots, n-1$) are functions on X satisfying, for any $x, y \in X$,

(N1) $x \leq y$ implies $N(y) \leq N(x)$,
(N2) $N(N(x)) = x$,

(s1) $N(s_i(x)) = s_{n-i}(N(x))$, $i = 1, \cdots, n-1$,
(s2) $s_j(s_i(x)) = s_j(x)$, $i, j = 1, \cdots, n-1$,
(s3) $s_1(x) \leq x$,
(s4) $x \leq s_{n-1}(x)$,
(s5) $s_i(x) \leq s_j(x)$ for $i \leq j$,
(s6) $x \leq y$ implies $s_i(x) \leq s_i(y)$ and $s_i(y) \leq s_i(x)$, $i = 1, \cdots, n-1$,
(s7) $s_i(y) \leq y$ and $y \leq s_i(y)$ imply $s_i(y) = y$, $i = 1, \cdots, n-1$,
(s8) $x \leq s_i(x)$ or $s_{i+1}(x) \leq x$, $i = 1, \cdots, n-2$, $n \geq 3$.

Let T be a binary relation on a set X and let A be a subset of X. In what follows we will denote by $[T]A$ the set $\{x \in X : \text{ for all } y, x\,T\,y \text{ implies } y \in A\}$.

Recall that, the complex algebra of an SHn–frame $(X \leq, N, \{s_i\}_{i=1}^{n-1})$ is

$$(\mathcal{C}(X), \vee^c, \wedge^c, \rightarrow^c, \sim^c, \{S_i^c\}_{i=1}^{n-1}, 0^c, 1^c)$$

where $\mathcal{C}(X) = \{A \subseteq X : [\leq]A = A\}$, $0^c = \emptyset$, $1^c = X$, $A \vee^c B = A \cup B$, $A \wedge^c B = A \cap B$, $A \rightarrow^c B = [\leq]((X \setminus A) \cup B)$, $\sim^c A = \{x \in X : N(x) \notin A\}$ and $S_i^c(A) = \{x \in X : s_i(x) \in A\}$, for all $A, B \in \mathcal{C}(X)$.

On the other hand, the canonical frame of an SHn–algebra $(W, \vee, \wedge, \rightarrow, \sim, \{S_i\}_{i=1}^{n-1}, 0, 1)$ is

$$(\mathcal{X}(W), \leq^c, N^c, \{s_i^c\}_{i=1}^{n-1})$$

where $\mathcal{X}(W)$ is the set of all prime filters of W, $\leq^c$ is $\subseteq$ and for every $F \in \mathcal{X}(W)$, $N^c(F) = \{a \in W : \sim a \notin F\}$, $s_i^c(F) = \{a \in W : S_i(a) \in F\}$, $i = 1, \cdots, n-1$.

These results allowed them to obtain a discrete duality for SHn–algebras by defining the embeddings as follows:
(E1) $h : W \rightarrow \mathcal{C}(\mathcal{X}(W))$, defined by $h(a) = \{F \in \mathcal{X}(W) : a \in F\}$, for any $a \in W$,
(E2) $k : X \rightarrow \mathcal{X}(\mathcal{C}(X))$, defined by $k(x) = \{A \in \mathcal{C}(X) : x \in A\}$, for any $x \in X$.

3. Tense SHn–algebras

In this section we shall recall the definition and basic results on tense SHn–algebras from [5].

Definition 3.1. *A tense SHn–algebra is an algebra $(W, \vee, \wedge, \rightarrow, \sim, \{S_i\}_{i=1}^{n-1}, G, H, 0, 1)$, where the reduct $(W, \vee, \wedge, \rightarrow, \sim, \{S_i\}_{i=1}^{n-1}, 0, 1)$ is an SHn–algebra and G, H are unary operators on W verifying the following conditions:*
(T1) $G(1) = 1$, $H(1) = 1$,
(T2) $G(x \wedge y) = G(x) \wedge G(y)$, $H(x \wedge y) = H(x) \wedge H(y)$,
(T3) $x \leq G(\sim H(\sim x))$, $x \leq H(\sim G(\sim x))$,
(T4) $S_i(G(x)) = G(S_i(x))$, $S_i(H(x)) = H(S_i(x))$, *for* $i = 1, \ldots, n-1$.

Remark 3.1. (i) *From* (T2) *it follows that G and H are increasing.*
(ii) *If $(W, \vee, \wedge, \rightarrow, \sim, \{S_i\}_{i=1}^{n-1}, G, H, 0, 1)$ is a tense SHn–algebra in which satisfies the identity $(x \wedge \sim x) \vee (y \vee \sim y) = y \vee \sim y$, then $(W, \vee, \wedge, \sim, \{S_i\}_{i=1}^{n-1}, G, H, 0, 1)$ is a tense Łukasiewicz–Moisil algebra.*

Lemma 3.1. *Let G, H be two unary operations on an SHn–algebra W such that $G(1) = 1$, $H(1) = 1$. Then condition* (T2) *is equivalent to the following one:*

$$(T2)' \;\; G(x \rightarrow y) \leq G(x) \rightarrow G(y), \;\; H(x \rightarrow y) \leq H(x) \rightarrow H(y).$$

Proof. We will only prove the equivalence between (T2) and $(T2)'$ in the case of G. From (T2) and (i) in Remark 3.1, we have that $G(x) \wedge G(x \rightarrow y) = G(x \wedge (x \rightarrow y)) = G(x \wedge y) \leq G(y)$. Therefore, $G(x \rightarrow y) \leq G(x) \rightarrow G(y)$. Conversely, let

$x, y \in W$ be such that $x \leq y$. Then, $x \to y = 1$ and so, from $(T2)'$ and the hypothesis, we obtain that $1 = G(x \to y) \leq G(x) \to G(y)$. Hence, $G(x) \leq G(y)$ from which we get that G is increasing. This last assertion and $(T2)'$ we infer that $G(x) \leq G(y \to (x \wedge y)) \leq G(y) \to G(x \wedge y)$. Thus, $G(x) \wedge G(y) \leq G(x \wedge y)$. From this statement and taking into account that G is increasing we conclude that $G(x) \wedge G(y) = G(x \wedge y)$. $\square$

Thus, if in Definition 3.1 we replace the axiom $(T2)$ by $(T2)'$, we obtain an equivalent definition of tense SHn–algebras.

4. A discrete duality for SHn–algebras

In this section, we describe a discrete duality for tense SHn–algebras bearing in mind the results indicated in Section 2 for SHn–algebras. To this end, we introduce the following

Definition 4.1. *A tense SHn–frame is a structure $(X, \leq, N, \{s_i\}_{i=1}^{n-1}, R, Q)$ where $(X, \leq, N, \{s_i\}_{i=1}^{n-1})$ is an SHn–frame, R, Q are binary relations on X and the following conditions are satisfied:*

(K1) $(\leq \circ R \circ \leq) \subseteq R$,
(K2) $(\leq \circ Q \circ \leq) \subseteq Q$,
(K3) $x \, R \, N(y)$ *if and only if* $y \, Q \, N(x)$,
(K4) $x \, R_T \, y$ *implies* $s_i(x) \, R_T \, s_i(y)$ *for* $T = G$ *and* $T = H$,
(K5) $s_i(z) \, R_T \, y$ *implies that there is* $x \in X$ *such that* $z \, R_T \, x$ *and* $s_i(x) \leq y$ *for* $T = G$ *and* $T = H$.

In what follows, tense SHn–frames will be denoted simply by X when no confusion may arise.

Definition 4.2. *A canonical frame of a tense SHn–algebra $(W, \vee, \wedge, \to, \sim, \{S_i\}_{i=1}^{n-1}, G, H, 0, 1)$ is a structure $(\mathcal{X}(W), \leq^c, N^c, \{s_i^c\}_{i=1}^{n-1}, R^c, Q^c)$, where $(\mathcal{X}(W), \leq^c, N^c, \{s_i^c\}_{i=1}^{n-1})$ is the canonical frame of the reduct $(W, \vee, \wedge, \to, \sim, \{S_i\}_{i=1}^{n-1}, 0, 1)$ and the following conditions are verified for $P, F \in \mathcal{X}(W)$:*

(F1) PR^cF *if and only if* $G^{-1}(P) \subseteq F$,
(F2) PQ^cF *if and only if* $H^{-1}(P) \subseteq F$.

Lemma 4.1. *The canonical frame of a tense SHn–algebra is a tense SHn–frame.*

Proof. Taking into account the results established in [13], we only have to prove $(K1) - (K5)$.
(K1): Let $(P, F) \in (\leq^c \circ R^c \circ \leq^c)$. Then there exist $T, S \in \mathcal{X}(W)$ such that $P \subseteq T$, TR^cS and $S \subseteq F$. From the last two assertions we have that $G^{-1}(T) \subseteq F$. Therefore, since $P \subseteq T$ we infer that $P \, R^c \, F$.
(K2): It is proved in a similar way to (K1).
(K3): Let $F \, R^c N^c(P)$ and $a \in H^{-1}(P)$. Suppose that $\sim a \in F$. On the other hand, from (T3) we have that $\sim a \leq G(\sim H(a))$ and so, we get that $G(\sim H(a)) \in F$. From this last assertion and the fact that $G^{-1}(F) \subseteq N^c(P)$, we obtain $\sim H(a) \in N^c(P)$. Hence, $H(a) \notin P$ which is a contradiction. Therefore, $a \in N^c(F)$ from which we conclude that $PQ^cN^c(F)$. The converse is proved similarly.
(K4): It is a direct consequence of (F1), (F2) and $(T4)$.
(K5): Let $G^{-1}(s_i^c(F)) \subseteq P$ and considering $E = \{z \in W : G(z) \in F\}$, then we have that $\bigwedge I \not\leq \bigvee J$, for all finite subsets $I \subseteq E$, $J \subseteq S_i(W \setminus P)$. Indeed: Suppose that there is $I \subseteq E$, $J \subseteq S_i(W \setminus P)$ finite subsets such that $\bigwedge I \leq \bigvee J$. From

this last assertion and (T2) we infer that $G(\bigwedge I) \in F$. Since, G is increasing we obtain that $G(\bigvee J) \in F$. On the other hand, it is straightforward to prove that $\bigvee J \in S_i(W \setminus P)$. From this last assertion, there is $a \in W \setminus P$ such that $S_i(a) = \bigvee J$. Hence, $G(S_i(a)) \in F$. Then, from (T4) we have deduced that $a \in P$, which is a contradiction. Therefore E is separated (see [6, p. 185]) from $S_i(W \setminus P)$, then from [6, p. 186], there is $Z \in \mathcal{X}(W)$ such that $E \subseteq Z$ and $Z \cap S_i(W \setminus P) = \emptyset$. This last assertion allows us to conclude that $s_i^c(Z) \subseteq P$ and $F\, R^c\, Z$. Similarly, it is proved (K5) for $T = H$. □

Definition 4.3. *The complex algebra of a tense SHn–frame $(X, \leq, N, \{s_i\}_{i=1}^{n-1}, R, Q)$ is $(\mathcal{C}(X), \vee^c, \wedge^c, \to^c, \sim^c, \{S_i^c\}_{i=1}^{n-1}, G^c, H^c, 0^c, 1^c,)$, where the reduct $(\mathcal{C}(X), \vee^c, \wedge^c, \to^c, \sim^c, \{S_i^c\}_{i=1}^{n-1}, 0^c, 1^c)$ is the complex algebra of the SHn–frame $(X, \leq, N, \{s_i\}_{i=1}^{n-1})$, $G^c(A) = [R]A$ and $H^c(A) = [Q]A$, for all $A \in \mathcal{C}(X)$.*

Lemma 4.2. *The complex algebra of a tense SHn–frame is a tense SHn–algebra.*

Proof. From [13], $\mathcal{C}(X)$ is closed under the lattice operations, $\sim^c$, $\to^c$ and $\{S_i^c\}_{i=1}^{n-1}$. Now, we show that it is also closed under G^c i.e., $G^cA = [\leq]G^cA$. From the reflexivity of $\leq$, we have that $[\leq]G^cA \subseteq G^cA$. Assume that $x \in G^cA$. Let $y \in X$ be such that $x \leq y$ and take any $z \in X$ verifying yRz. Hence, from the reflexivity of $\leq$ and (K1) we infer that xRz. So, $z \in A$ and therefore, $x \in [\leq]G^cA$. Thus, $G^cA \subseteq [\leq]G^cA$. Similarly, it is proved that $H^cA = [\leq]H^cA$. On the other hand, clearly (T1) and (T2) are verified. Therefore, it only remains to prove (T3) and (T4).
(T3): Let $x \in A$ and suppose that $x \notin G^c(\sim^c H^c(\sim^c A))$. Then there is y such that xRy and $y \notin \sim^c H^c(\sim^c A)$. From this last statement, $y \in N(H^c(\sim^c A))$ and so, $y = N(z)$ for some $z \in H^c(\sim^c A)$. Hence, $xRN(z)$ and from (K3) we get that $zQN(x)$. This assertion and the fact that $z \in H^c(\sim^c A)$ enable us to infer that $N(x) \notin N(A)$, which is a contradiction. So, $A \subseteq G^c(\sim^c H^c(\sim^c A))$. Analogously, it is proved that $A \subseteq H^c(\sim^c G^c(\sim^c A))$.
(T4): Let $s_i(y) \in [R]A$ and yRx. Then, from (K4) we have that $s_i(y)\, R\, s_i(x)$. From this last assertion we infer that $s_i(x) \in A$. Therefore, $S_i^c([R]A) \subseteq [R]S_i^c(A)$. On the other hand, let $z \in [R](S_i^c(A))$ and $s_i(z)\, R\, y$. Then by virtue of (K5), there is $x \in X$ such that $z\, R\, x$ and $s_i(x) \leq y$. This last assertion allows us to conclude that $y \in A$. Similarly, it is proved that $S_i^c[Q](A) = [Q]S_i^c(A)$.

□

Theorem 4.1. *Every tense SHn–algebra is embeddable into the complex algebra of its canonical frame.*

Proof. Let us consider the function $h : W \to \mathcal{C}(\mathcal{X}(W))$ defined by $h(a) = \{P \in \mathcal{X}(W) : a \in P\}$, for all $a \in W$. Let $F \in h(G(a))$; then $G(a) \in F$. Suppose that $P \in \mathcal{X}(W)$ verifies that FR^cP. Then from (F1), $G^{-1}(F) \subseteq P$ and so, $a \in P$. Therefore, $F \in G^c(h(a))$ from which we infer that $h(G(a)) \subseteq G^c(h(a))$. Conversely, assume that $F \in G^c(h(a))$. Then for every $P \in \mathcal{X}(W)$, FR^cP implies that $P \in h(a)$. Suppose that $G(a) \notin F$. Then $G^{-1}(F)$ is a filter and $a \notin G^{-1}(F)$. Hence, there is $T \in \mathcal{X}(W)$ such that $a \notin T$ and $G^{-1}(F) \subseteq T$. This last assertion and (F1) allow us to conclude that FR^cT. From this statement we have that $T \in h(a)$ and so, $a \in T$, which is a contradiction. Therefore, $h(G(a)) = G^c(h(a))$. Similarly, it is shown that $h(H(a)) = H^c(h(a))$. Thus, by virtue of the results established in [13] the proof is completed. □

Lemma 4.3 will show that the order–embedding $k : X \to \mathcal{X}(\mathcal{C}(X))$ defined by $k(x) = \{A \in \mathcal{C}(X) : x \in A\}$ for every $x \in X$ preserves the relations R and Q.

Lemma 4.3. *Let* $(X, \leq, N, \{s_i\}_{i=1}^{n-1}, R, Q)$ *be a tense* SHn*–frame and let* $x, y \in X$*. Then*
(i) xRy *if and only if* $k(x)R^c k(y)$,
(ii) xQy *if and only if* $k(x)Q^c k(y)$.

Proof. We will only prove (i). Assume that xRy and suppose that $A \in \mathcal{C}(X)$ verifies $G^c(A) \in k(x)$. Then it is easy to see that $y \in A$ and so, $k(x)R^c k(y)$. Conversely, let $x, y \in X$ be such that $k(x)R^c k(y)$. Then $G^{c^{-1}}(k(x)) \subseteq k(y)$. On the other hand, note that $[\leq](X \setminus (y]) \in \mathcal{C}(X)$ and $y \notin [\leq](X \setminus (y])$. Thus, $[\leq](X \setminus (y]) \notin k(y)$ and so, $[\leq](X \setminus (y]) \notin G^{c^{-1}}(k(x))$. Therefore, $[R]([\leq](X \setminus (y])) \notin k(x)$ from which we infer that $x \notin [R]([\leq](X \setminus (y]))$. Then there is z such that xRz and $z \notin [\leq](X \setminus (y])$. From this last assertion there is w such that $z \leq w$ and $w \leq y$, which allow us to infer that $z \leq y$. Hence, by virtue of the reflexivity of $\leq$ and (K1), xRy as required. □

Lemma 4.3 and the results indicated in [13] enable us to conclude

Theorem 4.2. *Every tense* SHn*–frame is embeddable into the canonical frame of its complex algebra.*

Theorems 4.1 and 4.2 enable us to obtain a discrete duality for tense SHn–algebras.

5. A propositional calculus based on tense SHn–algebras

In this section, we will describe a propositional calculus that has tense SHn–algebras as the algebraic counterpart. The terminology and symbols used here coincide in general with those used in [14].

Let $\mathcal{L} = (A^0, For[V])$ be a formalized language of zero order, where in the alphabet $A^0 = (V, L_0, L_1, L_2, U)$ the set

- V of propositional variables is enumerable,
- L_0 is empty,
- L_1 contains $n+2$ elements denoted by $\sim$, S_i (for $i = 1, \cdots, n-1$), G and H called negation sign, modal operators signs and tense operators signs, respectively,
- L_2 contains three elements denoted by $\vee$, $\wedge$, $\rightarrow$, called disjunction sign, conjunction sign and implication sign, respectively,
- U contains two elements denoted by (,).

For any α, β in the set $For[V]$ of all formulas over A^0, instead of $\alpha \rightarrow\sim (\alpha \rightarrow \alpha)$, $(\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha)$, $\sim G \sim \alpha$ and $\sim H \sim \alpha$ we will write for brevity $\neg\alpha$, $\alpha \leftrightarrow \beta$, $F\alpha$ and $P\alpha$, respectively.

We assume that the set $\mathcal{A}_l$ of logical axioms consists of all formulas of the following form, where α, β, γ are any formulas in $For[V]$:

(M0) the axioms of the SHn–logic, i.e., the axioms (A1)-(A15) indicated in Section 2,
(M1) $G(\alpha \rightarrow \beta) \rightarrow (G\alpha \rightarrow G\beta),\ H(\alpha \rightarrow \beta) \rightarrow (H\alpha \rightarrow H\beta)$,
(M2) $\alpha \rightarrow GP\alpha,\ \alpha \rightarrow HF\alpha$,
(M3) $S_iG\alpha \leftrightarrow GS_i\alpha,\ S_iH\alpha \leftrightarrow HS_i\alpha$.

The consequence operation $C_{\mathcal{L}}$ in $\mathcal{L}$ is determined by $\mathcal{A}_l$ and by the following rules of inference:

$$\text{(R1)}\ \frac{\alpha,\ \alpha \rightarrow \beta}{\beta} \qquad \text{(R2)}\ \frac{\alpha \rightarrow \beta}{\sim \beta \rightarrow \sim \alpha} \qquad \text{(R3)}\ \frac{\alpha \rightarrow \beta}{S_1\alpha \rightarrow S_1\beta} \qquad \text{(R4)}\ \frac{\alpha}{H\alpha} \qquad \text{(R5)}\ \frac{\alpha}{G\alpha}.$$

The system $\mathcal{TSH}n = (\mathcal{L}, C_{\mathcal{L}})$ thus obtained will be called the tense propositional SHn–calculus. We will denote by $\mathcal{T}$ the set of all formulas derivable in $\mathcal{TSH}n$. If α belongs to $\mathcal{T}$ we will write $\vdash \alpha$.

Let $\approx$ be the binary relation on $For[V]$ defined by

$$\alpha \approx \beta \text{ if and only if } \vdash \alpha \leftrightarrow \beta.$$

Then it is easy to check that $\approx$ is a congruence relation on $(For[V], \vee, \wedge, \rightarrow, \sim, \{S_i\}_{i=1}^{n-1}, G, H)$ and $\mathcal{T}$ determines an equivalence class which we will denote by 1. Moreover, taking into account [7], p. 300 it is straightforward to prove

Theorem 5.1. *$(For[V]/\approx, \vee, \wedge, \rightarrow, \sim, \{S_i\}_{i=1}^{n-1}, G, H, 0, 1)$ is a tense SHn–algebra, being $0 = \sim 1$.*

Definition 5.1. *A tense SHn–model based on a tense SHn–frame $K = (X, \leq, N, \{s_i\}_{i=1}^{n-1}, R, Q)$ is a system $M = (K, m)$ such that $m : V \rightarrow \mathcal{P}(X)$ is a meaning function that assigns subsets of states to propositional variables, i.e. satisfies the following condition:*

(her) *$x \leq y$ and $x \in m(p)$ imply $y \in m(p)$.*

Definition 5.2. *A tense SHn–model $M = ((X, \leq, N, \{s_i\}_{i=1}^{n-1}, R, Q); m)$ satisfies a formula α at the state x and we write $M \models_x \alpha$, if the following conditions are satisfied:*

- *$M \models_x p$ if and only if $x \in m(p)$ for $p \in V$,*
- *$M \models_x \alpha \vee \beta$ if and only if $M \models_x \alpha$ or $M \models_x \beta$,*
- *$M \models_x \alpha \wedge \beta$ if and only if $M \models_x \alpha$ and $M \models_x \beta$,*
- *$M \models_x \sim \alpha$ if and only if $M \not\models_{N(x)} \alpha$,*
- *$M \models_x \alpha \rightarrow \beta$ if and only if for all y, if $x \leq y$ and $M \models_y \alpha$ then $M \models_y \beta$,*
- *$M \models_x \neg\alpha$ if and only if for all y, if $x \leq y$ then $M \not\models_y \alpha$,*
- *$M \models_x S_i\alpha$ if and only if $M \models_{s_i(x)} \alpha$,*
- *$M \models_x G\alpha$ if and only if for all y, if xRy then $M \models_y \alpha$,*
- *$M \models_x H\alpha$ if and only if for all y, if xQy then $M \models_y \alpha$.*

A formula α is *true in a tense SHn–model M* (denoted by $M \models \alpha$) if and only if for every $x \in W$, $M \models_x \alpha$. The formula α is *true in a tense SHn–frame K* (denoted by $K \models \alpha$) if and only if it is true in every tense SHn–model based on K. The formula α is *valid* if and only if it is true in every tense SHn–frame.

Proposition 5.1. *Given a tense SHn–model $M = (K, m)$, the meaning function m can be extended to all formulae by $m(\alpha) = \{x \in X : M \models_x \alpha\}$. For every tense SHn–model M and for every formula α, this extension has the property*

(her) *if $x \leq y$ and $x \in m(\alpha)$ then $y \in m(\alpha)$.*

Proof. The proof is by induction with respect to complexity of α. By way of an example we show (her) for formulas of the form $G\alpha$. Let (1) $x \leq y$ and (2) $M \models_x G(\alpha)$. Suppose that yRz, then by (1),(2) and (K1), we have $M \models_z \alpha$. ☐

Theorem 5.2. (Completeness Theorem) *Let α be a formula in $\mathcal{TSH}n$. Then the following conditions are equivalent:*

(i) *α is derivable in $\mathcal{TSH}n$,*
(ii) *α is valid.*

Proof. (i) $\Rightarrow$ (ii): We proceed by induction on the complexity of the formula α. For example, we shall prove that the axioms (M2) and (M3) are valid. Let $K = (X, \leq, N, \{s_i\}_{i=1}^{n-1}, R, Q)$ be a tense SHn–frame and M a tense SHn–model based on K.

(M2) $\alpha \rightarrow HF\alpha$ is valid. Indeed:

(1) Let $y \in X$ be such that $x \leq y$, [hip.]
(2) $M \models_y \alpha$, [hip.]
(3) Let $z \in X$ be such that $y\,Q\,z$, [hip.]

Suppose that

(4) $M \models_{N(z)} G \sim \alpha$, [hip.]

(5) $N(z)\,R\,N(y)$, [(3),(K3)]
(6) $M \models_{N(y)} \sim \alpha$, [(4),(6)]
(7) $M \not\models_x \alpha$. [(6)]

(7) contradicts (2). Then

(8) $M \not\models_{N(z)} G \sim \alpha$, [(4),(7)]
(9) $M \models_z \sim G \sim \alpha$, [(8)]
(10) $M \models_y H \sim G \sim \alpha$, [(3),(9)]
(11) $M \models_x \alpha \to H \sim G \sim \alpha$. [(1),(2),(10)]

In a similar way we can prove that $\alpha \to GP\alpha$ is valid.

(M3) $S_iG\alpha \leftrightarrow GS_i\alpha$ is valid. Indeed:

(1) Let $y \in X$ be such that $x \leq y$, [hip.]
(2) $M \models_y S_iG\alpha$, [hip.]
(3) Let $z \in X$ be such that $y\,R\,z$, [hip.]
(4) $s_i(y)\,R\,s_i(z)$, [(3),(K4)]
(5) $M \models_{s_i(z)} \alpha$, [(2),(4)]
(6) $M \models_z S_i\alpha$, [(5)]
(7) $M \models_y GS_i\alpha$, [(3),(6)]
(8) $M \models_x S_iG\alpha \to GS_i\alpha$, [(2),(7)]

On the other hand

(9) Let $y \in X$ be such that $x \leq y$, [hip.]
(10) $M \models_y GS_i\alpha$, [hip.]
(11) Let $z \in X$ be such that $s_i(y)\,R\,z$, [hip.]
(12) there is $w \in X$ such that $y\,R\,w$ and $s_i(w) \leq z$, [(11),(K5)]
(13) $M \models_{s_i(w)} \alpha$, [(10),(12)]
(14) $M \models_z \alpha$. [(12),(13),(her)]
(15) $M \models_y S_iG\alpha$, [(11),(14)]
(16) $M \models_x GS_i\alpha \to S_iG\alpha$. [(9),(10),(15)]

Therefore, $S_iG\alpha \leftrightarrow GS_i\alpha$ is valid. Analogously we can prove that $S_iH\alpha \leftrightarrow HS_i\alpha$ is valid.

(ii) $\Rightarrow$ (i): Assume that α is not derivable, i.e. $[\alpha]_\approx \neq 1$. We apply Theorem 4.1 to the tense SHn–algebra $For[V]/\approx$, hence there exists a tense SHn–frame $\mathcal{X}(For[V]/\approx)$ and an injective morphism of tense SHn–algebras $h : For[V]/\approx \to \mathcal{C}(\mathcal{X}(For[V]/\approx))$. Let us consider the function $m : \mathcal{TSH}n \to \mathcal{C}(\mathcal{X}(For[V]/\approx))$ defined by $m(\alpha) = h([\alpha]_\approx)$ for all $\alpha \in For[V]$. It is straightforward to prove that m is an meaning function. Since h is injective, $m(\alpha) = h([\alpha]_\approx) \neq \mathcal{X}(For[V]/\approx)$, i.e. $(\mathcal{X}(For[V]/\approx), m) \not\models_{x_o} \alpha$ for some $x_o \in \mathcal{X}(For[V]/\approx)$. Thus α is not valid. □

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