Problem No. 11024, The American Mathematical Monthly, No. 6 (June-July), 2003, p. 543

Let $g : (0,+\infty) \to (0,+\infty)$ be a continuous function such that

$$\lim_{x\to+\infty} \frac{g(x)}{x^{1+\alpha}} = +\infty\,, \tag{1}$$

for some $\alpha > 0$. Let $f : \mathbb{R} \to (0,+\infty)$ be a twice differentiable function. Assume that there exists $a > 0$ and $x_0 \in \mathbb{R}$ such that

$$f''(x) + f'(x) > ag(f(x)), \qquad \text{for all } x \geq x_0\,. \tag{2}$$

Prove that $\lim\limits_{x\to+\infty} f(x)$ exists, is finite and compute its value.

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Solution. If $x_1 > x_0$ is a critical point of f then, by (2), $f''(x_1) > 0$, so x_1 is a relative minimum point of f. This implies that $f'(x)$ does not change sign if x is sufficiently large. Consequently, we can assume that f is monotone on $(x_0,+\infty)$, hence $\ell := \lim_{x\to+\infty} f(x)$ exists.

The difficult part of the proof is to show that ℓ is finite. This will be deduced after applying in a decisive manner our superlinear growth assumption (1). Arguing by contradiction, let us assume that $\ell = +\infty$. In particular, it follows that f is monotone increasing on $(x_0,+\infty)$. Define the function

$$u(x) = e^{x/2} f(x), \qquad x \geq x_0.$$

Then u is increasing and, for any $x \geq x_0$,

$$u''(x) = \frac{1}{4}u(x) + e^{x/2}\left(f''(x) + f'(x)\right) > \frac{1}{4}u(x) + ae^{x/2}g(f(x)). \tag{3}$$

Our hypothesis (1) and the assumption $\ell = +\infty$ yield some $x_1 > x_0$ such that

$$g(f(x)) \geq f^{1+\alpha}(x), \qquad \forall x \geq x_1. \tag{4}$$

So, by (3) and (4),

$$u''(x) > \frac{1}{4}u(x) + Cu(x)f^{\alpha}(x) > Cu(x)f^{\alpha}(x), \qquad \forall x \geq x_1, \tag{5}$$

for some $C > 0$. In particular, since $\ell = +\infty$, there exists $x_2 > x_1$ such that

$$u''(x) > u(x), \qquad \forall x \geq x_2. \tag{6}$$

We claim a little more, namely that there exists $C_0 > 0$ such that

$$u''(x) > C_0 u^{1+\alpha/2}(x), \qquad \forall x \geq x_2. \tag{7}$$

Indeed, let us first choose $0 < \delta < \min\{e^{-x_2}u(x_2), e^{-x_2}u'(x_2)\}$. We prove that

$$u(x) > \delta e^{x}, \qquad \forall x \geq x_2. \tag{8}$$

For this purpose, consider the function $v(x) = u(x) - \delta e^x$. Arguing by contradiction and using $v(x_2) > 0$ and $v'(x_2) > 0$, we deduce the existence of a relative maximum point $x_3 > x_2$ of v. So, $v(x_3) > 0$, $v'(x_3) = 0$ and $v''(x_3) \leq 0$. Hence $\delta e^{x_3} = u'(x_3) < u(x_3)$. But, by (6), $u''(x_3) > u(x_3)$, which yields $v''(x_3) > 0$, a contradiction. This concludes the proof of (8).

Returning to (5) and using (8) we find

$$u''(x) > Cu^{1+\alpha/2}(x)u^{\alpha/2}(x)e^{-\alpha x/2} > C_0u^{1+\alpha/2}(x), \qquad \forall x > x_2,$$

where $C_0 = C\delta^{\alpha/2}$. This proves our claim (7). So

$$u'(x)u''(x) > C_0u^{1+\alpha/2}(x)u'(x), \qquad \forall x > x_2.$$

Hence

$$\left(\frac{1}{2}u'^2(x) - C_1u^{2+\beta}(x)\right)' > 0, \qquad \forall x > x_2,$$

where $C_1 = 2C_0/(4+\alpha)$ and $\beta = \alpha/2 > 0$. Therefore

$$u'^2(x) \geq C_2 + C_3u^{2+\beta}(x), \qquad \forall x > x_2,$$

for some positive constants C_2 and C_3. So, since u is unbounded, there exists $x_3 > x_2$ and $C_4 > 0$ such that

$$u'(x) \geq C_4u^{1+\gamma}(x), \qquad \forall x > x_3,$$

where $\gamma = \beta/2 > 0$.

Applying the mean value theorem we find

$$u^{-\gamma}(x_3) - u^{-\gamma}(x) = \gamma(x - x_3)u^{-\gamma-1}(\xi_x)u'(\xi_x) \geq C_4\gamma(x - x_3), \qquad \forall x > x_3,$$

where $\xi_x \in (x_3, x)$. Taking $x \to +\infty$ in the above inequality we obtain a contradiction since the left hand-side converges to $u^{-\gamma}(x_3)$ (because $\ell = +\infty$) while the right hand-side diverges to $+\infty$. This contradiction shows that $\ell = \lim_{x\to+\infty} f(x)$ must be finite.

We prove in what follows that $\ell = 0$. Arguing by contradiction, let us assume that $\ell > 0$. We first observe that relation (2) yields, by integration,

$$f'(x) - f'(x_0) + f(x) - f(x_0) \geq a\int_{x_0}^{x} g(f(t))dt. \tag{9}$$

Since ℓ is finite, it follows by (9) that $\lim_{x\to+\infty} f'(x) = +\infty$. But this contradicts the fact that $\lim_{x\to+\infty} f(x)$ is finite.

Remark. The result stated in our problem does not remain true if g has a linear growth at $+\infty$, so if (1) fails. Indeed, it is enough to choose $f(x) = e^x$ and g the identity map. We also remark that "ℓ is finite" does not follow if the growth hypothesis (1) is replaced by the weaker one $\lim_{x\to\infty} g(x)/x = +\infty$. Indeed, if $g(x) = x\ln(1+x)$ and $f(x) = e^{x^2}$, then $\ell = +\infty$.