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# Numerical analysis of a class of variational inequalities with integral term

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Abstract. We study the numerical approximation of a class of abstract evolutionary variational inequalities with Volterra-type integral term. We introduce a fully discrete scheme for which we show the existence of a unique solution and derive error estimates. Then, we apply the abstract results to the analysis and numerical approximations of a Signorini frictionless contact problem in viscoelasticity with long-term memory.

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## 1. Introduction

The aim of the present paper is to provide the numerical analysis of an abstract problem which includes as special cases the mathematical models derived in [1] and [2] in the study of frictionless contact problems for viscoelastic materials with long term memory. To this end, we suppose in what follows that $V$ is a real Hilbert space endowed with the inner product $(\cdot,\cdot)_V$ and the associated norm $\|\cdot\|_V$. We also denote by $\mathcal{L}(V)$ the space of linear and continuous operators from $V$ to $V$ with norm $\|\cdot\|_{\mathcal{L}(V)}$. Let $T>0$. If $(X,\|\cdot\|_X)$ represents a Banach space we denote by $C([0,T];X)$ the space of continuous functions from $[0,T]$ to $X$, with norm

$$\|x\|_{C([0,T];X)} = \max_{t\in[0,T]} \|x(t)\|_X.$$

We shall use the space $C([0,T];X)$ both in the cases $X=V$ and $X=\mathcal{L}(V)$.

The abstract problem we are interested in is the following.

Problem **P**. *Find* $u:[0,T]\to V$ *such that*

$$\begin{aligned} u(t)\in U, \quad (Au(t), v-u(t))_V + \Big(\int_0^t B(t-s)u(s)ds, v-u(t)\Big)_V \\ \geq (f(t), v-u(t))_V, \qquad \forall\, v\in U, \quad t\in[0,T]. \end{aligned} \tag{1}$$

In the study of this problem we suppose that:

$$U \text{ is a nonempty closed convex subset of } V. \tag{2}$$

$$\left\{ \begin{array}{l} A: V\to V \text{ is a strongly monotone Lipschitz continuous operator, i.e.} \\ \text{(a) there exists } m>0 \text{ such that} \\ \qquad (Au_1 - Au_2, u_1-u_2)_V \geq m\|u_1-u_2\|_V^2, \quad \forall\, u_1,u_2\in V; \\ \text{(b) there exists } L_A>0 \text{ such that} \\ \qquad \|Au_1-Au_2\|_V \leq L_A\|u_1-u_2\|_V, \quad \forall\, u_1,u_2\in V. \end{array} \right. \tag{3}$$

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$$B \in C([0,T]; \mathcal{L}(V)). \tag{4}$$

$$f \in C([0,T]; V). \tag{5}$$

The well-posedness of problem **P** is a consequence of the following existence and uniqueness result.

**Theorem 1.1.** *Under the assumptions* (2)–(5)*, there exists a unique solution* $u \in C([0,T]; V)$ *to the problem* **P**.

The proof of Theorem 1.1 may be found in [3]. It is based on arguments of time dependent variational inequalities and the Banach fixed point theorem.

The present paper is structured as follows. In Section 2 we introduce a fully discrete scheme for the approximation of the solution of problem **P**, for which we prove an existence and uniqueness result and derive error estimates under suitable regularity of the solution. Then, in Section 3, we present the frictionless viscoelastic problem with Signorini's contact conditions, which leads to an evolutionary variational inequality of the form (1). We apply to this contact problem the results obtained in Section 2, by using the finite element method to discretize the spatial domain.

Everywhere in this paper $c$ will denote a positive constant whose value may change from line to line; it may depend on the problem data but does not depend neither on the time variable $t$, nor on the discretization parameters $h$ and $k$, to be introduced later. In addition, in Section 2 , we denote by $\tilde{c}$ a generic positive constant which may depend on the solution $u$ of problem **P**.

## 2. Fully discrete approximation

In this section we consider a fully discrete approximation of the problem **P**. Let $V^h \subset V$ be a finite-dimensional space (which can be constructed by the finite element method) and let $U^h \subset U$ be a non-empty closed convex subset of $V^h$. The parameter $h > 0$ has the intuitive meaning that the lower it is, the better approximation of $V$ represents $V^h$. We consider also a partition of the time interval $[0,T]:\ 0 = t_0 < t_1 < \cdots < t_N = T$. We denote the time step-size $k_n = t_n - t_{n-1}$ for $n = 1,\ldots,N$ and let $k = \max_{1\le n\le N} k_n$ be the maximal step-size. For reasons that will be explained later, we use a partition of the time interval such that

$$k < \frac{2m}{\|B_0\|_{\mathcal{L}(V)}}. \tag{6}$$

Here $B_0$ denotes the operator $B(0) \in \mathcal{L}(V)$. More general, for a function $w \in C([0,T]; X)$ we use the notation $w_j = w(t_j)$ and, for each time step $t_n$ with $1 \le n \le N$, we denote $B^{n,j} = B(t_n - t_j)$, for $0 \le j \le n$. Notice that, in particular, $B^{n,n} = B_0$. No summation is considered over the repeated indices $n$ and $j$.

The fully discrete scheme is the following, where $u_n^{hk} \in U^h$ is the approximation of the element $u(t_n) \in U$.

Problem $\mathbf{P}^{hk}$. *Find* $\{u_n^{hk}\}_{n=0}^N \subset U^h$*, such that for all* $v^h \in U^h$

$$(Au_0^{hk}, v^h - u_0^{hk})_V \ge (f_0, v^h - u_0^{hk})_V, \tag{7}$$

$$(Au_n^{hk}, v^h - u_n^{hk})_V + (\sum_{j=0}^{n} B^{n,j}\alpha_j^n u_j^{hk}, v^h - u_n^{hk})_V \tag{8}$$

$$\ge (f_n, v^h - u_n^{hk})_V, \quad n = 1,\ldots,N.$$

Here, for each time step $t_n$, the constants $\alpha_j^n > 0$ $(0 \le j \le n)$ denote weights of a quadrature formula of $n+1$ points in $[0, t_n]$. We restrict our analysis to the case of quadrature formulae that are exact for continuous and piecewise linear functions in $[t_0, t_n]$. As a consequence, we have

$$\alpha_n^n \le k/2. \tag{9}$$

In practice, we have used the composed trapezoidal formula where the subintervals are $[t_j, t_{j+1}]$, $0 \le j \le n-1$, i.e.,

$$\alpha_0^n = \frac{1}{2}(t_1 - t_0), \quad \alpha_n^n = \frac{1}{2}(t_n - t_{n-1}), \quad \alpha_j^n = \frac{1}{2}(t_{j+1} - t_{j-1}). \tag{10}$$

We also define the numerical error of the quadrature formula by $I_0 = 0$,

$$I_n = \Big\| \int_0^{t_n} B(t_n - s)u(s)ds - \sum_{j=0}^{n} \alpha_j^n B^{n,j} u_j \Big\|_V, \quad n = 1, 2, \ldots, N.$$

We assume that the quadrature formula is such that $I_n$ verifies

$$\lim_{k \to 0} I_n = 0, \quad n = 1, 2, \ldots, N. \tag{11}$$

This condition is fullfilled by the composed trapezoidal formula (10). Moreover, if $B$ and $u$ are Lipschitz continuous on $[0, T]$, it is verified that

$$I_n \le ck(\|u\|_{C([0,T];V)} + \|B\|_{C([0,T];\mathcal{L}(V))}). \tag{12}$$

The well posedness of the fully discrete problem $\mathbf{P}^{hk}$ is provided by the following result.

**Theorem 2.1.** *Assume* (2)–(5), (6) *and* (9). *Then there exists a unique solution* $\{u_n^{hk}\}_{n=0}^N \subset U^h$ *to the problem* $\mathbf{P}^{hk}$.

**Proof.** We proceed by induction. For $n = 0$ the classical theory of elliptic variational inequalities guarantees the existence and uniqueness of $u_0^{hk} \in U^h$ satisfying (7). We assume the existence and uniqueness of $u_j^{hk} \in U^h$ verifying (8) for every $j$, $0 \le j < n$, and we prove it for the $n$-th time step. If $\alpha_n^n = 0$ we can apply the same arguments on elliptic variational inequalities to prove the existence and uniqueness of $u_n^{hk}$. Otherwise, the procedure will be based on a fixed point strategy. Indeed, we consider the auxiliar problem of finding $u_{\eta^h}^{hk} \in U^h$ such that

$$(Au_{\eta^h}^{hk}, v^h - u_{\eta^h}^{hk})_V \ge (\sum_{j=0}^{n-1} B^{n,j} \alpha_j^n u_j^{hk}, u_{\eta^h}^{hk} - v^h)_V \tag{13}$$

$$+(f_n, v^h - u_{\eta^h}^{hk})_V + (\eta^h, u_{\eta^h}^{hk} - v^h)_V, \quad \forall v^h \in U^h.$$

where $\eta^h \in V^h$ is given and arbitrary. Using again classical results we conclude that there exists a unique solution $u_{\eta^h}^{hk} \in U^h$ to problem (13). Now, we define the operator $\Lambda^{hk} : V^h \longrightarrow V^h$ by $\Lambda^{hk}\eta^h = \alpha_n^n B_0 u_{\eta^h}^{hk}$. We can easily prove that $\Lambda^{hk}$ is contractive. Indeed, given $\eta_1^h$, $\eta_2^h \in V^h$, we have

$$\|\Lambda^{hk}\eta_1^h - \Lambda^{hk}\eta_2^h\|_V \le \alpha_n^n \|B_0\|_{\mathcal{L}(V)} \|u_{\eta_1^h}^{hk} - u_{\eta_2^h}^{hk}\|_V.$$

After some calculus on (13) and using (9), from the previous inequality we obtain

$$\|\Lambda^{hk}\eta_1^h - \Lambda^{hk}\eta_2^h\|_V \le \frac{k\|B_0\|_{\mathcal{L}(V)}}{2m} \|\eta_1^h - \eta_2^h\|_V.$$

Then, from (6), we get that $\Lambda^{hk}$ is contractive and, as a consequence, it has a unique fixed point that we denote $\eta^h_*$. It is easily seen that the element $u^{hk}_{\eta^h_*}$, which represents the solution to problem (13) for $\eta^h = \eta^h_*$, is a solution to (8). The uniqueness of the fixed point of the operator $\Lambda^{hk}$ provides the uniqueness of the solution of (8). □

In the sequel, we denote by $e_n = \|u_n - u_n^{hk}\|_V$, for $0 \le n \le N$, the numerical error in the approximation, for which we are interested in obtaining estimates.

Taking $v = u_0^{hk}$ in (1) at $t = 0$ and using (7), after some calculations we find

$$\|u_0 - u_0^{hk}\|_V \le c\,(\|v^h - u_0\|_V + R_0(v^h)^{\frac{1}{2}}), \quad \forall v^h \in U^h, \tag{14}$$

where $c$ depends on $A$ and $R_0(v^h) = (Au_0 - f_0, v^h - u_0)$. Similarly, taking $v = u_n^{hk}$ in (1) at $t = t_n$ and using (8), after some calculations we find

$$e_n \le c\left(\|v^h - u_n\|_V + I_n + \sum_{j=0}^{n} \alpha_j^n e_j + R_n(v^h)^{\frac{1}{2}}\right), \quad \forall\, v^h \in U^h, \tag{15}$$

where $c$ depends on $A$ and $B$ and

$$R_n(v^h) = (Au_n + \int_0^{t_n} B(t_n - s)u(s)ds - f_n, v^h - u_n)_V. \tag{16}$$

We denote $g_n = \|v^h - u_n\|_V + I_n + R_n(v^h)^{\frac{1}{2}}$, $0 \le n \le N$. From (14) and (15) we obtain

$$\begin{cases} e_n \le c\left(g_n + \displaystyle\sum_{j=0}^{n} \alpha_j^n e_j\right), & 1 \le n \le N, \\ e_0 \le c g_0. \end{cases} \tag{17}$$

Using a discrete version of the Gronwall's lemma (see for example [6]) we conclude from (17) that

$$e_n \le c \sum_{j=0}^{n} \alpha_j g_j,$$

where $\alpha_j = \max\limits_{0 \le n \le N} \alpha_j^n$. To resume, we have proved the following theorem.

**Theorem 2.2.** *Assume the hypothesis of the Theorem* 2.1 *and* (11). *Let* $u \in C([0,T];V)$ *and* $\{u_n^{hk}\}_{n=0}^N \subset U^h$ *be the solutions to the problems* $\mathbf{P}$ *and* $\mathbf{P}^{hk}$, *respectively. Then there exists a positive constant c which depends on the operators A and B such that the following error bound is satisfied:*

$$\max_{0 \le n \le N} \|u_n - u_n^{hk}\|_V \le c \max_{0 \le n \le N} \left\{ I_n + \|v^h - u_n\|_V + R_n(v^h)^{\frac{1}{2}} \right\}, \ \forall\, v^h \in U^h. \tag{18}$$

Now we study the convergence of the fully discrete solution towards the solution of the continuous problem.

**Proposition 2.3.** *Assume the hypothesis of the Theorem* 2.1 *and, moreover, assume that there exists a subspace* $\mathcal{V}$ *dense in* $V$ *and* $\alpha > 0$ *such that*

$$\inf_{v^h \in U^h} \|v - v^h\|_V \le ch^\alpha, \qquad \forall v \in \mathcal{V}, \tag{19}$$

*where c may depend on v. Then,*

$$\lim_{h \to 0} \inf_{v^h \in U^h} \|v^h - u_n\|_V = 0, \quad 0 \le n \le N. \tag{20}$$

**Proof.** Let $\delta > 0$, $0 < \varepsilon < \delta$ and let $n \in \mathbb{N}, 0 \le n \le N$. There exists $\upsilon_{\varepsilon,n} \in \mathcal{V}$ such that $\|\upsilon_{\varepsilon,n} - u_n\|_V \le \varepsilon$ and, using (19), it follows that

$$\inf_{v^h \in U^h} \|v^h - u_n\|_V \le \inf_{v^h \in U^h} \|v^h - \upsilon_{\varepsilon,n}\|_V + \|\upsilon_{\varepsilon,n} - u_n\|_V \le ch^\alpha + \varepsilon.$$

Notice that here $c$ may depend on $\upsilon_{\varepsilon,n}$. The limit (20) follows from the previous inequality, since taking $ch^\alpha < \delta - \varepsilon$ we obtain

$$\inf_{v^h \in U^h} \|v^h - u_n\|_V < \delta. \qquad \square$$

**Corolary 2.4.** *If the assumptions of Theorem* 2.2 *and Proposition* 2.3 *hold, then the fully discrete solution* $\{u_n^{hk}\}_{n=0}^N$ *converges to the exact solution* $u$ *as* $h$ *and* $k$ *tend to zero:*

$$\lim_{h,k\to 0} \left\{ \max_{0\le n\le N} \|u_n - u_n^{hk}\|_V \right\} = 0. \tag{21}$$

**Proof.** We use (16) to obtain

$$R_n(v^h) \le \tilde{c}\, \|v^h - u_n\|_V \tag{22}$$

where $\tilde{c}$ depends on $u$ and on $A$, $B$ and $f$. Equality (21) is now a consequence of (11), (18), (20) and (22). $\square$

## 3. A frictionless contact problem with a rigid foundation

In this section we apply the results presented in Sections 1 and 2 to a frictionless contact problem with Signorini conditions in the form with a zero gap function. The physical setting is the following. A viscoelastic body occupies a regular domain $\Omega$ of $\mathbb{R}^d$ $(d = 1, 2, 3)$ with the boundary $\Gamma$ partitioned into three disjoint measurable parts $\Gamma_1$, $\Gamma_2$ and $\Gamma_3$ such that $meas\,(\Gamma_1) > 0$. The time interval of interest is $[0, T]$ with $T > 0$. The body is clamped on $\Gamma_1$ and so the displacement field vanishes there. Surface tractions of density $\boldsymbol{\varphi}_2$ act on $\Gamma_2$ and volume forces of density $\boldsymbol{\varphi}_0$ act in $\Omega$. We assume that the acceleration of the system is negligible and the body is in frictionless contact on $\Gamma_3$ with a rigid obstacle, the so-called foundation. We use Signorini's conditions to model the contact and a viscoelastic constitutive law with long-term memory to model the material's behavior.

Under these assumptions, the classical formulation of the mechanical problem of frictionless contact of the viscoelastic body is the following:

*Find the displacement field* $\boldsymbol{u} : \Omega \times [0, T] \to \mathbb{R}^d$ *and the stress field* $\boldsymbol{\sigma} : \Omega \times [0, T] \to S_d$ *such that*

$$\boldsymbol{\sigma}(t) = \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \int_0^t \mathcal{B}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}(s))ds \qquad \text{in} \quad \Omega \times (0, T)\,, \tag{23}$$

$$\operatorname{Div} \boldsymbol{\sigma} + \boldsymbol{\varphi}_0 = \boldsymbol{0} \qquad \text{in} \quad \Omega \times (0, T)\,, \tag{24}$$

$$\boldsymbol{u} = \boldsymbol{0} \qquad \text{in} \quad \Gamma_1 \times (0, T)\,, \tag{25}$$

$$\boldsymbol{\sigma}(t)\boldsymbol{\nu} = \boldsymbol{\varphi}_2(t) \qquad \text{in} \quad \Gamma_2 \times (0, T)\,, \tag{26}$$

$$u_\nu \le 0, \quad \sigma_\nu \le 0, \quad \sigma_\nu u_\nu = 0, \quad \boldsymbol{\sigma}_\tau = \boldsymbol{0} \qquad \text{in} \quad \Gamma_3 \times (0, T)\,. \tag{27}$$

Here $S_d$ represents the space of second-order symmetric tensors on $\mathbb{R}^d$ and $\boldsymbol{\nu}$ denotes the unit outward normal to $\Gamma$. The relation (23) is the viscoelastic constitutive law in which $\mathcal{A}$ and $\mathcal{B}(t)$ are given fourth-order tensors, called the *elasticity* and the *relaxation* tensor, respectively, see e.g. [5]. As usual, $\boldsymbol{\varepsilon}(\boldsymbol{u})$ is the infinitesimal strain tensor. Relation (24) represents the equilibrium equation in which Div denotes the divergence operator, relations (25) and (26) are the displacement and traction boundary conditions, respectively. Finally, conditions (27) are the Signorini frictionless contact conditions where $u_\nu$ denotes the normal displacement, $\sigma_\nu$ is the normal stress and $\boldsymbol{\sigma}_\tau$ represents the tangential stress.

We denote in the sequel by "$\cdot$" and $|\cdot|$ the inner product and the Euclidean norm on the spaces $\mathbb{R}^d$ and $S_d$ and we introduce the spaces

$$\begin{aligned}
&V = \{\boldsymbol{v} = (v_i) \in (H^1(\Omega))^d : \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_1\},\\
&Q = \{\boldsymbol{\tau} = (\tau_{ij}) \in (L^2(\Omega))^{d\times d} : \tau_{ij} = \tau_{ji},\ 1 \le i,j \le d\},\\
&Q_1 = \{\boldsymbol{\tau} \in Q : \mathrm{Div}\,\boldsymbol{\tau} \in (L^2(\Omega))^d\}.
\end{aligned}$$

The spaces $Q$ and $Q_1$ are real Hilbert spaces with their canonical inner products denoted $(\cdot,\cdot)_Q$ and $(\cdot,\cdot)_{Q_1}$, respectively. Over the space $V$, we use the inner product

$$(\boldsymbol{u},\boldsymbol{v})_V = (\boldsymbol{\varepsilon}(\boldsymbol{u}),\boldsymbol{\varepsilon}(\boldsymbol{v}))_Q, \qquad \forall\, \boldsymbol{u},\boldsymbol{v} \in V \tag{28}$$

and the associated norm $\|\cdot\|_V$. Since *meas* $(\Gamma_1) > 0$ it follows from Korn's inequality that $(V,(\cdot,\cdot)_V)$ is a real Hilbert space.

For all $\boldsymbol{v} \in V$ we denote by $v_\nu$ and $\boldsymbol{v}_\tau$ the normal and tangential components of $\boldsymbol{v}$ on $\Gamma$ given by

$$v_\nu = \boldsymbol{v}\cdot\boldsymbol{\nu}, \qquad \boldsymbol{v}_\tau = \boldsymbol{v} - v_\nu\boldsymbol{\nu},$$

and recall that the normal and tangential stress in (27) are given by formulas

$$\sigma_\nu = (\boldsymbol{\sigma}\boldsymbol{\nu})\cdot\boldsymbol{\nu}, \qquad \boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu\boldsymbol{\nu}.$$

Finally, we consider the space of fourth-order tensor fields

$$\mathbf{Q}_\infty = \{\ \mathcal{E} = (\mathcal{E}_{ijkl}) \mid \mathcal{E}_{ijkl} = \mathcal{E}_{jikl} = \mathcal{E}_{klij} \in L^\infty(\Omega), \quad 1 \le i,j,k,l \le d\},$$

which is a real Banach space with the norm

$$\|\mathcal{E}\|_{\mathbf{Q}_\infty} = \max_{0\le i,j,k,l\le d} \|\mathcal{E}_{ijkl}\|_{L^\infty(\Omega)}.$$

In the study of the mechanical problem (23)–(27) we assume that the elasticity and relaxation tensors satisfy

$$\mathcal{A} \in \mathbf{Q}_\infty, \tag{29}$$

$$\exists\, m > 0 \text{ such that } \ \mathcal{A}\boldsymbol{\xi}\cdot\boldsymbol{\xi} \ge m|\boldsymbol{\xi}|^2, \ \ \forall \boldsymbol{\xi} \in S_d, \ \text{a.e. } \boldsymbol{x} \in \Omega, \tag{30}$$

$$\mathcal{B} \in C([0,T];\mathbf{Q}_\infty). \tag{31}$$

We also assume that the force and traction densities satisfy

$$\boldsymbol{\varphi}_0 \in C([0,T];(L^2(\Omega))^d), \qquad \boldsymbol{\varphi}_2 \in C([0,T];(L^2(\Gamma_2))^d), \tag{32}$$

and we denote by $\boldsymbol{f}(t)$ the element of $V$ given by

$$(\boldsymbol{f}(t),\boldsymbol{v})_V = \int_\Omega \boldsymbol{\varphi}_0(t)\cdot\boldsymbol{v}\,dx + \int_{\Gamma_2} \boldsymbol{\varphi}_2(t)\cdot\boldsymbol{v}\,da \tag{33}$$

for all $\boldsymbol{v} \in V$ and $t \in [0,T]$. We note that conditions (32) imply

$$\boldsymbol{f} \in C([0,T];V). \tag{34}$$

Finally, let $U$ denote the set of admissible displacement fields defined by

$$U = \{\ \boldsymbol{v} \in V \mid v_\nu \le 0 \ \text{ a.e. on } \Gamma_3\ \}. \tag{35}$$

Proceeding in a standard way, we obtain the following variational formulation of the contact problem (23)–(27).

PROBLEM $\mathbf{P}_1$. *Find the displacements field* $\boldsymbol{u} : [0,T] \to V$ *such that*

$$\boldsymbol{u}(t) \in U, \quad (\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)))_Q \tag{36}$$
$$+(\int_0^t \mathcal{B}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}(s))ds, \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)))_Q$$
$$\geq (\boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}(t))_V, \quad \forall \boldsymbol{v} \in U,\ t \in [0,T].$$

In the study of problem $\mathbf{P}_1$ we have the following result.

**Theorem 3.1.** *Assume that* (29)–(32) *hold. Then the problem* $\mathbf{P}_1$ *has a unique solution* $\boldsymbol{u} \in C([0,T];V)$.

**Proof.** Let $A : V \to V$ and $B : [0,T] \to \mathcal{L}(V)$ be the operators defined as follows for all $\boldsymbol{v}, \boldsymbol{w} \in V$ and $t \in [0,T]$:

$$(A\boldsymbol{v}, \boldsymbol{w})_V = (\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}), \boldsymbol{\varepsilon}(\boldsymbol{w}))_Q, \quad (B(t)\boldsymbol{v}, \boldsymbol{w})_V = (\mathcal{B}(t)\boldsymbol{\varepsilon}(\boldsymbol{v}), \boldsymbol{\varepsilon}(\boldsymbol{w}))_Q.$$

Using (29)–(31) it follows that $A$ and $B$ satisfy conditions (3) and (4). Moreover, the set $U$ defined by (35) is a nonempty closed convex subset of $V$. Thus, keeping in mind (34), by Theorem 1.1, we conclude that problem $\mathbf{P}_1$ has a unique solution wich satisfies $\boldsymbol{u} \in C([0,T];V)$. □

Let now $\boldsymbol{u} \in C([0,T];V)$ be the solution of the problem $\mathbf{P}_1$ and let $\boldsymbol{\sigma}$ be the stress field defined by (23). Using (36) and (32) it can be shown that $\operatorname{Div} \boldsymbol{\sigma} \in C([0,T];(L^2(\Omega))^d)$ and therefore $\boldsymbol{\sigma} \in C([0,T];Q_1)$. A pair of functions $(\boldsymbol{u}, \boldsymbol{\sigma})$ which satisfies (23) and (36) is called a *weak solution* of the problem (23)–(27). We conclude that the problem (23)–(27) has a unique weak solution which represents a result already obtained in [1].

Now, we describe briefly how to construct the finite dimensional space $V^h$ via the finite element method. Details can be found in [1, 4]. For simplicity, we assume that $\Omega$ is polygonal. Let $\mathcal{T}^h$ be a regular finite element partition of $\Omega$ composed by $d-$simplex in such a way that if one $(d–1)$-face of an element lies on the boundary, the $(d–1)$-face belongs entirely to one of the subsets $\Gamma_1$, $\Gamma_2$ and $\Gamma_3$. Let $h$ be the maximal diameter of the elements, and let $V^h \subset V$ be the finite element space consisting of continuous piecewise polynomials of degree less than or equal to $l$, corresponding to the partition $\mathcal{T}^h$. We also define the discrete set of admissible displacements by

$$U^h = \{\ \boldsymbol{v}^h \in V^h \mid \boldsymbol{v}^h_\nu \leq 0 \text{ a.e. on } \Gamma_3\ \}.$$

We apply now the analysis made in Section 2 to obtain that the fully discrete approximation for the problem $\mathbf{P}_1$, denoted $\mathbf{P}_1^{hk}$, has a unique solution $\{\boldsymbol{u}_n^{hk}\}_{n=0}^N \subset U^h$ and the estimate (18) holds. Moreover, the solution of $\mathbf{P}_1^{hk}$ converges to the solution of $\mathbf{P}_1$.

Assume now that $\boldsymbol{u} : [0,T] \to V$ and $\mathcal{B} : [0,T] \to \mathbf{Q}_\infty$ are Lipschitz continuous functions and, moreover,

$$\sigma_\nu \in C([0,T];L^2(\Gamma_3)), \quad \boldsymbol{u} \in C([0,T];(H^{l+1}(\Omega))^d), \tag{37}$$
$$u_\nu \in C([0,T];H^{l+1}(\Gamma_3)),$$

where $l \geq 1$. In this case, using the composed trapezoidal formula (10) and (12), it can be proved from (18) that

$$\max_{0\leq n\leq N} \|\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}\|_V \leq ck\Big(\|\boldsymbol{u}\|_{C([0,T];V)} + \|\mathcal{B}\|_{C([0,T];\mathbf{Q}_\infty)}\Big) \tag{38}$$

$$+ch^{(l+1)/2} \max_{0\leq n\leq N} \{\|\boldsymbol{u}_n\|_{(H^{l+1}(\Omega))^d} + \|(u_n)_\nu\|^{1/2}_{H^{l+1}(\Gamma_3)} \|(\sigma_n)_\nu\|^{1/2}_{L^2(\Gamma_3)}\},$$

where $c$ depends on $\mathcal{A}$ and $\mathcal{B}$. An usual choice is $l = 1$, so we conclude from (38) that the error in the fully discrete approximation is of order $O(h + k)$.

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