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An existence result for quasilinear elliptic equations with variable exponents

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Abstract. We study the following elliptic equation involving weight and variable exponents

$$-\mathrm{div}(\phi(x,|\nabla u|)\nabla u)+|u|^{p(x)-2}u=\lambda|u|^{r(x)-2}u-h(x)|u|^{s(x)-2}u$$

in $\Omega\subset\mathbb{R}^N(N\geq 3)$, with Dirichlet boundary condition, where $\phi(x,t)$ is of type $|t|^{p(x)-2}$ with continuous function $p:\overline{\Omega}\to(1,\infty)$. Under appropriate conditions on ϕ, by means of variational methods and a variant of the mountain pass theorem, we show that for λ large enough there exist at least two nontrivial weak solutions for our problem. For this purpose we work on a generalized variable exponent Lebesgue-Sobolev space.

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1. Introduction

Equations involving variable exponent growth conditions constitued a real interest in the study of the partial differential equations in the last few decades (see [6, 7, 14, 16, 17]). This kind of problems can serve as models in the theory of electrorheological fluids or image restoration. In 1949 there was the first major discovery on the electrorheological fluids, konwn as the Winslow effect, and it describes the behavior of certain fluids that becomes solids or quasi-solids when subjected to an electric field. Electrorheological fluids (or smart fluids) have been used in robotics and space technology. The experimental research has been mainly in the United States, for instance in NASA laboratoires. In [19] more details about properties, modelling and applications of variable exponent spaces to these fluids was studied by V. Rădulescu and D. Repovš.

Let $\Omega\subset\mathbb{R}^N(N\geq 3)$ a bounded domain with smooth boundary. Let λ be a positive real parametre, p,r,s continuous functions on $\overline{\Omega}$ which satisfy the condition

$$2\leq p(x)<r(x)<s(x)<p^*(x), \tag{1}$$

where $p^*(x)=\frac{Np(x)}{N-p(x)}$ and $p(x)<N$ for all $x\in\overline{\Omega}$.

In this paper we study a nonlinear elliptic equations of $p(x)$-Laplace type

$$\begin{cases} -\mathrm{div}(\phi(x,|\nabla u|)\nabla u)+|u|^{p(x)-2}u=\lambda|u|^{r(x)-2}u-h(x)|u|^{s(x)-2}u & \text{in} \quad \Omega,\\ u=0 & \text{on} \quad \partial\Omega, \end{cases} \tag{2}$$

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where $\phi(x,t)$ is of type $|t|^{p(x)-2}$ with continuous function $p:\overline{\Omega}\to(1,\infty)$ and $h:\overline{\Omega}\to[0,\infty)$ is a continuous function which satisfies the following hypotheses:

$$\left(\frac{\lambda^{s(x)}}{h(x)^{r(x)}}\right)^{\frac{1}{s(x)-r(x)}} \in L^1(\Omega), \tag{3}$$

$$\left(\frac{\lambda^{s(x)-2}}{h(x)^{r(x)-2}}\right)^{\frac{1}{s(x)-r(x)}} \in L^{\frac{s(\cdot)}{s(\cdot)-2}}(\Omega). \tag{4}$$

Related to our problem, in [12] I. H. Kim and V. H. Kim studied the general case

$$-\mathrm{div}(\phi(x,|\nabla u|)\nabla u)=\lambda f(x,u) \text{ in } \Omega,$$

where $f:\Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function. The authors have established the existence of a nontrivial solution for the above problem and, under sufficient conditions on ϕ and f, they have proved the positivity of the infimum eigenvalue for this problem.

When $\phi(x,t)=|t|^{p(x)-2}$, the operator implicated in (2) is the $p(x)$-Laplace operator, defined by $\Delta_{p(x)}u=\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u)$. For the general case, problem 2 was studied by I. Stăncuţ in [21]. Problems involving the $p(x)$-Laplacian have been extensively studied in the last decades (see [1, 6, 7, 8, 11, 18]).

When $p(x)$ is a constant function we obtain the p-Laplacian (see [1, 5, 10]) which is less complicated then $p(x)$-Laplacian equation, because the $p(x)$-Laplace operator is nonhomogeneous.

Our purpose in this paper is to establish, under suitable conditions on ϕ, that for λ large enough there exist at least two nontrivial weak solutions. In order to prove this result, we use a special version of the mountain pass theorem (see [2] and [24, Theorem 1.15]) and a corresponding variational method.

Considering the presence of the $p(x)$-Laplace operator we introduce a variable exponent Lebesgue-Sobolev space setting for problems of type (2). Due to the presence of the continuous function $h(x)$ in the right side of our problem, we seek weak solution for (2) in a more generalized variable exponent Lebesgue-Sobolev space, namely in the weighted variable exponent Sobolev space.

This paper is organized as follows. In the next section we establish some basic properties of the variable exponent Lebesgue-Sobolev spaces, as otherwise a necessary conditions of ϕ. In section 3 we introduce the energy functional and we state the main result of this paper. Finally, the proof of the main result are developed in section 4.

2. Preliminaries

We start this section with some definitions and properties of the variable exponent Lebesgue-Sobolev spaces.

Let Ω be a bounded domain in $\mathbb{R}^N$. We define

$$C_+(\Omega)=\left\{p\in C(\overline{\Omega}):\min_{x\in\Omega}p(x)>1\right\}$$

and for any (Lebesgue) continuous function $p:\overline{\Omega}\to(1,\infty)$, denote

$$p^-=\inf_{x\in\Omega}p(x) \qquad \text{and} \qquad p^+=\sup_{x\in\Omega}p(x).$$

For any $p \in C_+(\overline{\Omega})$ we define the *variable exponent Lebesgue space*

$$L^{p(x)}(\Omega) = \left\{ u : \Omega \to \mathbb{R} \text{ a measurable function} : \int_\Omega |u|^{p(x)} dx < \infty \right\}.$$

Equipped with the *Luxemburg norm*

$$|u|_{p(x)} = \inf \left\{ \mu > 0 : \int_\Omega \left| \frac{u(x)}{\mu} \right| dx \leq 1 \right\},$$

$L^{p(x)}(\Omega)$ becomes a Banach space.

If $p(x) = p \equiv$ *constant* for every $x \in \Omega$, then the $L^{p(x)}(\Omega)$ space is reduced to the classic Lebesgue space $L^p(\Omega)$ and the Luxemburg norm becomes the standard norm in $L^p(\Omega)$, $\|u\|_{L^p} = \left(\int_\Omega |u(x)|^p dx\right)^{1/p}$.

For $1 < p^- \leq p^+ < \infty$, $L^{p(x)}(\Omega)$ is a reflexive uniformly convex Banach space, and for any measurable bounded exponent p, the $L^{p(x)}(\Omega)$ space is separable.

If p_1 and p_2 are two variable exponents such that $p_1(x) \leq p_2(x)$ almost everywhere in Ω, with $|\Omega| < \infty$, then there exists a continuous embedding

$$L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$$

whose norm does not exceed $|\Omega| + 1$.

We define the conjugate variable exponent $p' : \overline{\Omega} \to (1, \infty)$, satisfying $\dfrac{1}{p(x)} + \dfrac{1}{p'(x)} = 1$, for every $x \in \overline{\Omega}$.

We denote by $L^{p'(x)}(\Omega)$ the conjugate space of the $L^{p(x)}(\Omega)$.

If $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$ then the *Hölder type inequality* holds:

$$\left| \int_\Omega uv\, dx \right| \leq \left(\frac{1}{p^+} + \frac{1}{p^-} \right) |u|_{p(x)} |v|_{p'(x)} \leq 2 |u|_{p(x)} |v|_{p'(x)}. \tag{5}$$

The *modular* of the $L^{p(x)}(\Omega)$ space, defined by the mapping $\rho_{p(x)} : L^{p(x)}(\Omega) \to \mathbb{R}$,

$$\rho_{p(x)} = \int_\Omega |u(x)|^{p(x)} dx,$$

has an important role in manipulating the generalized Lebesgue spaces.

If $p(x) = p \equiv$ *constant* for every $x \in \Omega$, then the modular $\rho_{p(x)}(u)$ becomes $\|u\|^p_{L^p}$.

If $p(x) \not\equiv$ *constant* in Ω and $u, u_n \in L^{p(x)}(\Omega)$ then the following relations hold true:

$$|u|_{p(x)} < 1 \Rightarrow \; |u|^{p^+}_{p(x)} \leq \rho_{p(x)}(u) \leq |u|^{p^-}_{p(x)}, \tag{6}$$

$$|u|_{p(x)} > 1 \Rightarrow \; |u|^{p^-}_{p(x)} \leq \rho_{p(x)}(u) \leq |u|^{p^+}_{p(x)}, \tag{7}$$

$$|u|_{p(x)} = 1 \Rightarrow \; \rho_{p(x)}(u) = 1, \tag{8}$$

$$|u_n - u|_{p(x)} \to 0 \Leftrightarrow \; \rho_{p(x)}(u_n - u) \to 0. \tag{9}$$

For more details about these variable exponent Lebesgue spaces see into [13].

Let $r : \overline{\Omega} \to (1, \infty)$ and $h : \overline{\Omega} \to [0, \infty)$ be continuous functions, with $r^+ < \infty$. We define the *weighted Lebesgue space*

$$L^{r(\cdot)}_h(\Omega) = \left\{ u : \Omega \to \mathbb{R} \text{ a measurable function} \; ; \int_\Omega h(x) |u|^r(x) dx < \infty \right\}$$

endowed with the norm

$$|u|_{h,r(\cdot)} = \inf\left\{\mu > 0; \int_\Omega h(x)\left|\frac{u(x)}{\mu}\right|^{r(x)} dx \le 1\right\}.$$

In the particular case that $h(x)$ is constant on $\overline{\Omega}$, we note that we obtain the Luxemburg norm $|\cdot|_{r(\cdot)}$.

We define now the *variable exponent Sobolev space* as

$$W^{1,p(\cdot)}(\Omega) = \left\{u \in L^{p(\cdot)}(\Omega); |\nabla u| \in L^{p(\cdot)}(\Omega)\right\}$$

equipped with the norm

$$\|u\|_{p(\cdot)} = |u|_{p(\cdot)} + |\nabla u|_{p(\cdot)}$$

which is equivalent with the norm

$$\|u\| = \inf\left\{\mu > 0; \int_\Omega \left(\left|\frac{\nabla u(x)}{\mu}\right|^{p(x)} + \left|\frac{u(x)}{\mu}\right|^{p(x)}\right) dx \le 1\right\},$$

where $|\nabla u|_{p(\cdot)}$ is the Luxemburg norm of $|\nabla u|$.

We define $W_0^{1,p(\cdot)}(\Omega) = \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{p(\cdot)}}$ and we note that $W_0^{1,p(\cdot)}(\Omega)$ is a separable and reflexive Banach space.

For the density of $C_0^\infty(\Omega)$ in $W_0^{1,p(\cdot)}(\Omega)$ we consider $p \in C_+(\overline{\Omega})$ to be *logarithmic Hölder continuous*, that means, there exists $M > 0$ such that

$$|p(x) - p(y)| \le \frac{-M}{\log(|x-y|)}, \ \forall x, y \in \Omega,$$

with $|x - y| \le \frac{1}{2}$.

As well, we remark that if $s \in C_+(\overline{\Omega})$ and $s(x) < p^*(x)$ for all $x \in \overline{\Omega}$, then $W_0^{1,p(\cdot)}(\Omega)$ is compactly embedded in $L^{s(\cdot)}(\Omega)$, where $p^*(x) = \frac{Np(x)}{N-p(x)}$ and $p(x) < N$.

At last, we define the *modular* of the $W_0^{1,p(\cdot)}(\Omega)$ space by the mapping

$$\varrho_{p(\cdot)} : W_0^{1,p(\cdot)}(\Omega) \to \mathbb{R},$$

$$\varrho_{p(\cdot)}(u) = \int_\Omega \left(|\nabla u|^{p(x)} + |u|^{p(x)}\right) dx.$$

If $(u_n), u \in W_0^{1,p(\cdot)}(\Omega)$, then we have the following relations:

$$\|u\| > 1 \Rightarrow \|u\|^{p^-} \le \varrho_{p(\cdot)}(u) \le \|u\|^{p^+}, \tag{10}$$

$$\|u\| < 1 \Rightarrow \|u\|^{p^+} \le \varrho_{p(\cdot)}(u) \le \|u\|^{p^-}, \tag{11}$$

$$\|u_n - u\| \to 0 \Leftrightarrow \varrho_{p(\cdot)}(u_n - u) \to 0. \tag{12}$$

For more details about these spaces we refer [4, 13, 15, 19, 20, 22].

3. Existence results

In order to prove our main result, we state some necessary conditions on ϕ. We assume that

($h1$) $\phi : \Omega \times [0, \infty) \to [0, \infty)$ satisfies the following hypotheses: $\phi(\cdot, \omega)$ is a measurable function on Ω for all $\omega > 0$ and $\phi(x, \cdot)$ is locally absolutely continuous on $[0, \infty)$ for almost all $x \in \Omega$;

($h2$) Let $a \in L^{p'(x)}(\Omega)$ be a function and b a nonnegative constant such that

$$|\phi(x, |v|)v| \leq a(x) + b|v|^{p(x)-1}, \forall x \in \Omega, \forall v \in \mathbb{R}^N;$$

($h3$) There exists $c > 0$ a constant such that, for almost $x \in \Omega$, the following condition hold:

$$\phi(x, \omega) \geq c\omega^{p(x)-2}, \text{ for almost all } \omega > 0.$$

Throughout this paper, we seek weak solutions for problem (2) in a subspace of $W_0^{1,p(\cdot)}(\Omega)$, more exactly in the *weighted variable exponent Sobolev space* defined by

$$X = \left\{ u \in W_0^{1,p(\cdot)}(\Omega); \int_\Omega h(x)|u|^{s(x)}dx < \infty \right\}$$

endowed with the norm

$$\|u\|_X = |u|_{p(\cdot)} + |\nabla u|_{p(\cdot)} + |u|_{h,s(\cdot)}.$$

Definition 3.1. We call *weak solution* for problem (2) a function $u \in X$ which satisfies

$$\int_\Omega \Big(\phi(x, |\nabla u|)\nabla u \nabla \varphi + |u|^{p(x)-2}u\varphi\Big)dx = \lambda \int_\Omega |u|^{r(x)-2}u\varphi dx - \int_\Omega h(x)|u|^{s(x)-2}u\varphi dx,$$

for any $\varphi \in X$.

We define the *energy functional* $I : X \to \mathbb{R}$ by

$$I(u) = \int_\Omega \varPhi(x, |\nabla u|)dx + \int_\Omega \frac{1}{p(x)}|u|^{p(x)}dx - \lambda \int_\Omega \frac{1}{q(x)}|u|^{r(x)}dx + \int_\Omega \frac{h(x)}{r(x)}|u|^{s(x)}dx,$$

where

$$\Phi(x, t) = \int_0^t \phi(x, \omega)\omega d\omega.$$

Lemma 3.2. Suppose that hypotheses (H_1) and (H_2) are satisfied. Then $I \in C^1(X, \mathbb{R})$ and its Fréchet derivative is given by

$$\begin{aligned}\left\langle I'(u), \varphi \right\rangle &= \int_\Omega \Big(\phi(x, |\nabla u|)\nabla u \nabla \varphi + |u|^{p(x)-2}u\varphi\Big)dx \\ &\quad -\lambda \int_\Omega |u|^{r(x)-2}u\varphi dx + \int_\Omega h(x)|u|^{s(x)-2}u\varphi dx.\end{aligned}$$

The proof of Lemma (3.2) is based on standard arguments (see [12, Lemma 3.2] and [21]).

We note that the weak solutions of problem (2) are exactly the critical points of I.

Finnaly, we state our main result.

Theorem 3.3. There exists $\lambda_* > 0$ such that for $\lambda > \lambda_*$ problem (2) has at least two nontrivial weak solutions.

4. Proof of Theorem (3.3)

We divide the proof in two parts. In the first one we focus our attention on proving the existence of a nontrivial solution for problem (2), remaining as in the second part to find, for λ large enough, a second nontrivial weak solution for problem (2).

First part. We start by proving the following two results.

Lemma 4.1. The functional I is coercive on X.

Proof. Considering $k = r(x)$, $l = s(x)$, $a = \dfrac{\lambda}{r(x)}$ and $b = \dfrac{h(x)}{2s(x)}$ in the following inequality

$$a|t|^k - b|t|^l \leq ca\left(\frac{a}{b}\right)^{\frac{k}{l-k}}, \tag{13}$$

for any $t \in \mathbb{R}$, $a, b > 0$ and $0 < k < l$, where $c = c(k,l)$ is a positive constant, we obtain that

$$\frac{\lambda}{r(x)}|u|^{r(x)} - \frac{h(x)}{2s(x)}|u|^{s(x)} \leq c\left(\frac{1}{r(x)}\right)^{\frac{s(x)}{s(x)-r(x)}} (2s(x))^{\frac{r(x)}{s(x)-r(x)}} \left(\frac{\lambda^{s(x)}}{h(x)^{r(x)}}\right)^{\frac{1}{s(x)-r(x)}}.$$

Since $\left(\dfrac{1}{r(x)}\right)^{\frac{s(x)}{s(x)-r(x)}} (2s(x))^{\frac{r(x)}{s(x)-r(x)}}$ is a bounded expression, we find that for k_1 a positive constant we have

$$\frac{\lambda}{r(x)}|u|^{r(x)} - \frac{h(x)}{2s(x)}|u|^{s(x)} \leq k_1\left(\frac{\lambda^{s(x)}}{h(x)^{r(x)}}\right)^{\frac{1}{s(x)-r(x)}}.$$

By (3) we obtain that

$$\int_\Omega \left(\frac{\lambda}{r(x)}|u|^{r(x)} - \frac{h(x)}{2s(x)}|u|^{s(x)}\right)dx \leq k_2,$$

where k_2 is a positive constant.

Therefore, by (h_3), choosing $|u|_{p(\cdot)} < 1$, $|\nabla u|_{p(\cdot)} < 1$ and by (6) we obtain

$$\begin{aligned}
I(u) &\geq \int_\Omega \Phi(x, |\nabla u|)dx + \int_\Omega \frac{1}{p(x)}|u|^{p(x)}dx + \frac{1}{2}\int_\Omega \frac{h(x)}{s(x)}|u|^{s(x)} - k_2 \\
&\geq \frac{c}{p^+}|\nabla u|^{p^+}_{p(\cdot)} + \frac{1}{p^+}\int_\Omega |u|^{p(x)}dx + \frac{1}{2s^+}\int_\Omega h(x)|u|^{s(x)}dx - k_2 \\
&\geq \frac{1}{p^+}\left(|\nabla u|^{p^+}_{p(\cdot)} + |u|^{p^+}_{p(\cdot)}\right) + \frac{1}{2s^+}\int_\Omega h(x)|u|^{s(x)}dx - k_2 \\
&\geq k_3\left(|\nabla u|^{p^+}_{p(\cdot)} + |u|^{p^+}_{p(\cdot)} + \int_\Omega h(x)|u|^{s(x)}dx\right) - k_2,
\end{aligned} \tag{14}$$

where $k_3 = \dfrac{1}{2s^+}$.

Considering $v \in L^{s(\cdot)}$, $s \in C_+(\Omega)$ and $|v|_{s(\cdot)} < 1$ and taking into account (6) we obtain

$$\int_\Omega |v|^{s(x)}dx \geq |v|^{s^+}_{s(\cdot)}.$$

We choose $v(x) = h(x)^{\frac{1}{s(x)}} u(x)$, so we have

$$\int_\Omega h(x)|u|^{s(x)}dx \geq |v|_{s(\cdot)}^{s^+}.$$

Let $u \in X$ be such that $\|u\| > 1$, we have

$$\begin{aligned} I(u) &\geq k_3 \left(|\nabla u|_{p(\cdot)}^{p^+} + |u|_{p(\cdot)}^{p^+} + |u|_{h,s(\cdot)}^{s^+} \right) - k_2 \\ &\geq \left(|\nabla u|_{p(\cdot)} + |u|_{p(\cdot)} + |u|_{h,s(\cdot)} \right) - k_2 \\ &\geq k_4 \|u\|_X - k_2, \end{aligned}$$

where k_4 is a positive constant. We conclude that $I(u) \to +\infty$ as $\|u\|_X \to +\infty$, namely I is coercive on X.

□

Lemma 4.2. Suppose that (u_n) is a sequence in X such that $I(u_n)$ is bounded. Then there exists a subsequence of (u_n), noted again (u_n), such that

$$u_n \rightharpoonup u_0 \text{ in } X,$$

for some $u_0 \in X$ and

$$I(u_0) \leq \liminf_{n\to\infty} I(u_n).$$

Proof. Considering $v_n \in L^{s(\cdot)}(\Omega)$, $s \in C_+(\Omega)$, such that $|v_n|_{s(\cdot)} < 1$ and by (6) we have

$$\int_\Omega |v_n|^{s(x)}dx \leq |v_n|_{s(\cdot)}^{s^-}.$$

Choosing $v_n(x) = h(x)^{\frac{1}{s(x)}} u_n(x)$ we find the inequality

$$\int_\Omega h(x)|u_n|^{s(x)}dx \leq |u_n|_{h,s(\cdot)}^{s^-} < 1.$$

Analogously, for $|v_n|_{s(\cdot)} > 1$, taking into account relation (7) we obtain

$$1 < |u_n|_{h,s(\cdot)}^{s^-} \leq \int_\Omega h(x)|u_n|^{s(x)}dx.$$

Now, seeing that $\int_\Omega h(x)|u_n|^{s(x)}dx$ is bounded, we conclude that $|u_n|_{h,s(\cdot)}$ is bounded too.

However, by (14) we deduce that

$$I(u_n) \geq k_3 \left(\int_\Omega \phi(x, |\nabla u_n|)\nabla u_n dx + \int_\Omega |u_n|^{p(x)}dx + \int_\Omega h(x)|u_n|^{s(x)}dx \right) - k_2,$$

where $k_2, k_3 > 0$ are two constants. We know that $I(u_n)$ is bounded, therefore the above inequality yields to the boundedness of $\int_\Omega \phi(x, |\nabla u_n|)\nabla u_n dx$ and $\int_\Omega h(x)|u_n|^{s(x)}dx$.

Making use of the fact that $\int_\Omega \left(\phi(x, |\nabla u|)\nabla u + |u|^{p(x)} \right) dx$ is bounded and using relations (10) and (11), we obtain that $\|u_n\|$ is bounded. So, it follows that $\|u_n\|_X$ is bounded, that is, we have the existence of a subsequence of (u_n), labeled again (u_n), which converges weakly in X to some $u_0 \in X$.

Actually, there exists u_0 in X such that

$$u_n \rightharpoonup u_0 \text{ in } W_0^{1,p(\cdot)}(\Omega),$$

and

$$u_n \to u_0 \text{ in } L_h^{r(\cdot)}(\Omega).$$

Now, we define

$$G(x,u) = \frac{\lambda}{r(x)}|u|^{r(x)} - \frac{h(x)}{s(x)}|u|^{s(x)},$$

$$g(x,u) = G_u(x,u) = \lambda|u|^{r(x)-2}u - h(x)|u|^{s(x)-2}u$$

and we note that

$$g_u(x,u) = \lambda(r(x)-1)|u|^{r(x)-2} - h(x)(s(x)-1)|u|^{s(x)-2}.$$

We choose $a = \lambda(r(x)-1)$, $b = h(x)(s(x)-1)$, $k = r(x)-2$ and $l = s(x)-2$ and by (13) it follows that

$$g_u(x,u) \leq C\left(\frac{r(x)-1}{s(x)-1}\right)^{\frac{r(x)-2}{s(x)-r(x)}}(r(x)-1)\left(\frac{\lambda^{s(x)-2}}{h(x)^{r(x)-2}}\right)^{\frac{1}{s(x)-r(x)}}.$$

It is easy to see that $\left(\frac{r(x)-1}{s(x)-1}\right)^{\frac{r(x)-2}{s(x)-r(x)}}(r(x)-1)$ is a bounded expression, then the following inequality holds

$$g_u(x,u) \leq c_1\left(\frac{\lambda^{s(x)-2}}{h(x)^{r(x)-2}}\right)^{\frac{1}{s(x)-r(x)}}, \tag{15}$$

where $c_1 > 0$ is a constant.

By the definitions of I and G, we have

$$\begin{aligned} I(u_0) - I(u_n) \quad = \quad & \int_\Omega [\Phi(x,|\nabla u_0|) - \Phi(x,|\nabla u_n|)]dx + \int_\Omega \frac{1}{p(x)}\left(|u_0|^{p(x)} - |u_n|^{p(x)}\right)dx \\ & + \int_\Omega [G(x,u_n) - G(x,u_0)]dx. \end{aligned} \tag{16}$$

Integrating over $[0,1]$ the following inequalities

$$\begin{aligned} \int_0^\nu g_u(x,u_0+t(u_n-u_0))dt &= \frac{g(x,u_0+\nu(u_n-u_0)) - g(x,u_0)}{u_n-u_0} \\ &= \frac{G_u(x,u_0+\nu(u_n-u_0)) - G_u(x,u_0)}{u_n-u_0}, \end{aligned}$$

we obtain

$$\begin{aligned} \int_0^1\int_0^\nu g_u(x,u_0+t(u_n-u_0))dtd\nu &= \frac{\displaystyle\int_0^1 [G_u(x,u_0+\nu(u_n-u_0)) - G_u(x,u_0)]d\nu}{u_n-u_0} \\ &= \frac{G(x,u_n)-G(x,u_0)}{(u_n-u_0)^2} - \frac{g(x,u_0)}{u_n-u_0}. \end{aligned}$$

So, we deduce that

$$G(x,u_n)-G(x,u_0) = (u_n-u_0)^2 \int_0^1 \int_0^\nu g_u(x,u_0+t(u_n-u_0))dtd\nu+(u_n-u_0)g(x,u_0). \tag{17}$$

Taking into account relations (15), (16) and (17) we find that

$$\begin{aligned}
I(u_0)-I(u_n) &= \int_\Omega [\Phi(x,|\nabla u_0|)-\Phi(x,|\nabla u_n|)]dx + \int_\Omega \frac{1}{p(x)}\left(|u_0|^{p(x)}-|u_n|^{p(x)}\right)dx \\
&+ \int_\Omega (u_n-u_0)^2 \int_0^1 \int_0^\nu g_u(x,u_0+t(u_n-u_0))dtd\nu dx + \int_\Omega (u_n-u_0)g(x,u_0)dx \\
&\le \int_\Omega [\Phi(x,|\nabla u_0|)-\Phi(x,|\nabla u_n|)]dx + \int_\Omega \frac{1}{p(x)}\left(|u_0|^{p(x)}-|u_n|^{p(x)}\right)dx \\
&+ c_2 \int_\Omega (u_n-u_0)^2 \left(\frac{\lambda^{s(x)-2}}{h(x)^{r(x)-2}}\right)^{\frac{1}{s(x)-r(x)}} dx + \int_\Omega (u_n-u_0)g(x,u_0)dx,
\end{aligned} \tag{18}$$

where $c_2 > 0$ is a constant.

Now, our purpose is to prove that the last two integrals by the above inequality converge to 0 as $n \to \infty$.

Let define $J : X \to \mathbb{R}^N$,

$$J(v) = \int_\Omega g(x,u_0)vdx.$$

It is obvious that J is a linear functional. It remains to prove that J is also continuous.

$$\begin{aligned}
|J(v)| &\le \int_\Omega |g(x,u_0)v|dx = \int_\Omega \left|\lambda|u_0|^{r(x)-2}u_0 - h(x)|u_0|^{s(x)-2}u_0\right| \cdot |v|dx \\
&\le \lambda \int_\Omega |u_0|^{r(x)-1}|v|dx + \int_\Omega h(x)|u_0|^{s(x)-1}|v|dx.
\end{aligned} \tag{19}$$

We apply the Hölder-type inequality (5) and we obtain

$$\int_\Omega |u_0|^{r(x)-1}|v|dx \le 2\left||u_0|^{r(x)-1}\right|_{\frac{r(x)}{r(x)-1}} |v|.$$

By the continuous embedding $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow L^{r(\cdot)}(\Omega)$ we have that there exists C_0 a positive constant such that

$$|v|_{r(\cdot)} \le C_0 \|v\|,$$

for all $v \in W_0^{1,p(\cdot)}(\Omega)$. But, we have

$$\|v\| \le \|v\|_X .$$

Considering the last three inequalities we obtain

$$\int_\Omega |u_0|^{r(x)-1}|v|dx \le C_1 \|v\|_X , \tag{20}$$

with $C_1 > 0$ a constant. Again by (5) we find that

$$\begin{aligned}\int_\Omega h(x)|u_0|^{s(x)-1}|v|dx &= \int_\Omega \left(h(x)^{\frac{s(x)-1}{s(x)}}|u_0|^{s(x)-1}\right)\left(h(x)^{\frac{1}{s(x)}}|v|\right)dx\\ &\le 2\left|h(x)^{\frac{s(x)-1}{s(x)}}|u_0|^{s(x)-1}\right|_{\frac{s(\cdot)}{s(\cdot)-1}}\left|h(x)^{\frac{1}{s(x)}}|v|\right|_{s(\cdot)}\\ &\le C_2|v|_{h,s(\cdot)} \le \|v\|_X\,, \end{aligned}\tag{21}$$

where C_2 is a positive constant.

Hence, by (19)–(21) we have

$$|J(v)| \le C_3\,\|v\|_X\,,$$

for all $v \in X$ and C_3 a positive constant, that is, J is continuous. Taking into accont that (u_n) converges weakly to u_0 in X and by the fact that J is liniar and continuous we get to

$$J(u_n) \to J(u_0),$$

namely,

$$\lim_{n\to\infty}\int_\Omega g(x,u_0)(u_n-u_0)dx = 0. \tag{22}$$

We know that $u_n \rightharpoonup u_0$ in $W_0^{1,p(\cdot)}(\Omega)$. Whereas we have $s \in C_+(\Omega)$ and $s(x) < p^*(x)$, we obtain that $W_0^{1,p(\cdot)}(\Omega)$ is compactly embedded in $L^{s(\cdot)}$. Both results lead us to the strongly convergence $u_n \to u_0$ in $L^{s(\cdot)}$, that is

$$\int_\Omega |u_n-u_0|^{s(x)}dx \text{ converges strongly to } 0,$$

or

$$\int_\Omega (|u_n-u_0|^2)^{\frac{s(x)}{2}}dx \text{ converges strongly to } 0.$$

Therefore, we obtain that

$$(u_n-u_0)^2 \in L^{\frac{s(\cdot)}{2}}(\Omega). \tag{23}$$

By relations (4), (5) and (23) we obtain

$$\int_\Omega (u_n-u_0)^2\left(\frac{\lambda^{s(x)-2}}{h(x)^{r(x)-2}}\right)^{\frac{1}{s(x)-r(x)}}dx \le 2\left|\left(\frac{\lambda^{s(x)-2}}{h(x)^{r(x-2}}\right)^{\frac{1}{s(x)-r(x)}}\right|_{\frac{s(\cdot)}{s(\cdot)-2}}\left|(u_n-u_0)^2\right|_{\frac{s(\cdot)}{2}}.$$

But, combining

$$\rho_{\frac{s(\cdot)}{2}}\left((u_n-u_0)^2\right) = \int_\Omega (|u_n-u_0|^2)^{\frac{s(x)}{2}}dx = \int_\Omega |u_n-u_0|^{s(x)}dx \to 0$$

with relation (9) we obtain

$$\left|(u_n-u_0)^2\right|_{\frac{s(\cdot)}{2}} \to 0.$$

Thus, we have

$$\lim_{n\to\infty}\int_\Omega (u_n-u_0)^2\left(\frac{\lambda^{s(x)-2}}{h(x)^{r(x)-2}}\right)^{\frac{1}{s(x)-r(x)}}dx = 0. \tag{24}$$

Let now define $I_1 : W_0^{1,p(\cdot)}(\Omega) \to \mathbb{R}$,

$$I_1(u) = \int_\Omega \Phi(x,|\nabla u|)dx$$

and $I_2 : W_0^{1,p(\cdot)}(\Omega) \to \mathbb{R}$,

$$I_2(u) = \int_\Omega \frac{1}{p(x)} |u|^{p(x)} dx.$$

By sample computation, taking into account [12] and [21] it is easy to prove that I_1 and I_2 are convex functionals. Hence, it is obvious that $I_1 + I_2$ is also convex on $W_0^{1,p(\cdot)}(\Omega)$.

Now, our purpose is to prove that $I_1 + I_2$ is weakly lower semicontinuous on $W_0^{1,p(\cdot)}(\Omega)$ (see Corollary III.8 in [3]).

Let be $u \in W_0^{1,p(\cdot)}(\Omega)$ and $\epsilon > 0$ fixed. Let $v \in W_0^{1,p(\cdot)}(\Omega)$ be arbitrary. By relations (5), (h_3) and the fact that $I_1 + I_2$ is convex, we obtain

$$\begin{aligned}
I_1(v) + I_2(v) &= I_1(u) + I_2(u) + \left\langle I_1^{'}(u) + I_2^{'}(u), v - u \right\rangle \\
&= I_1(u) + I_2(u) + \int_\Omega \Phi(x, |\nabla u|) \nabla u \nabla (v - u) dx + \int_\Omega |u|^{p(x)-2} u (v - u) dx \\
&\geq I_1(u) + I_2(u) - c \int_\Omega |\nabla u|^{p(x)-1} \nabla (v - u) dx - \int_\Omega |u|^{p(x)-1} (v - u) dx \\
&\geq I_1(u) + I_2(u) \\
&- c_1 \left(\left| |\nabla u|^{p(x)-1} \right|_{\frac{p(\cdot)}{p(\cdot)-1}} |\nabla (v - u)|_{p(\cdot)} + \left| |u|^{p(x)-1} \right|_{\frac{p(\cdot)}{p(\cdot)-1}} |v - u|_{p(\cdot)} \right) \\
&\geq I_1(u) + I_2(u) \\
&- c_1 \left(\left| |\nabla u|^{p(x)-1} \right|_{\frac{p(\cdot)}{p(\cdot)-1}} + \left| |u|^{p(x)-1} \right|_{\frac{p(\cdot)}{p(\cdot)-1}} \right) \cdot \left(|\nabla (v - u)|_{p(\cdot)} + |v - u|_{p(\cdot)} \right) \\
&\geq I_1(u) + I_2(u) - c_2 \|v - u\| \\
&\geq I_1(u) + I_2(u) - \epsilon,
\end{aligned}$$

for all $v \in W_0^{1,p(\cdot)}(\Omega)$ with $\|v - u\| < \epsilon / c_2$, where c, c_1 and c_2 are positive constants. It follows that $I_1 + I_2$ is lower semicontinuous on $W_0^{1,p(\cdot)}(\Omega)$, hence $I_1 + I_2$ is weakly lower semicontinuous, that means

$$\liminf_{n \to \infty} (I_1 + I_2)(u_n) \geq (I_1 + I_2)(u_0)$$

or,

$$\int_\Omega \left(\Phi(x, |\nabla u_n|) + \frac{1}{p(x)} |u_n|^{p(x)} \right) dx \geq \int_\Omega \left(\Phi(x, |\nabla u_0|) + \frac{1}{p(x)} |u_0|^{p(x)} \right) dx.$$

Taking into account the above inequality, relations (22), (24) and passing to the limit in (18) we obtain that

$$\liminf_{n \to \infty} I(u_n) \geq I(u_0),$$

so, I is weakly lower semicontinuous on X and this completes our proof.

□

Now, using [23, Theorem 1.2], 4.1 and 4.2 we conclude that there exists $u \in X$ a global minimizer of I, therefore

$$I(u) = \inf_{v \in X} I(v).$$

It is clear that u is a weak solution for problem (2), it remains to prove that $u \not\equiv 0$ in X. It is sufficient to show that $\inf_X I < 0$ as long as λ is large enough.

We set

$$\tilde{\lambda} = \inf \left\{ r^+ \left(\int_\Omega \Phi(x, |\nabla u|) dx + \int_\Omega \frac{1}{p(x)} |u|^{p(x)} dx + \int_\Omega \frac{h(x)}{s(x)} |u|^{s(x)} dx \right); \right.$$
$$\left. u \in X, \int_\Omega |u|^{r(x)} dx = 1 \right\}.$$

By Hölder inequality (5) we obtain

$$\begin{aligned} \lambda &= \int_\Omega \frac{\lambda}{h(x)^{\frac{r(x)}{s(x)}}} h(x)^{\frac{r(x)}{s(x)}} |u|^{r(x)} dx \\ &\leq 2 \left| \frac{\lambda}{h(x)^{\frac{r(x)}{s(x)}}} \right|_{\frac{s(\cdot)}{s(\cdot)-r(\cdot)}} \left| h(x)^{\frac{r(x)}{s(x)}} |u|^{r(x)} \right|_{\frac{s(\cdot)}{r(\cdot)}}, \end{aligned} \tag{25}$$

for any $u \in X$ with $\displaystyle\int_\Omega |u|^{r(x)} dx = 1$.

By (6), (7), (25) and considering the case when $u \in X$ and

$$\left| h(x)^{\frac{r(x)}{s(x)}} |u|^{r(x)} \right|_{\frac{s(\cdot)}{r(\cdot)}} > 1,$$

we find that

$$\lambda \leq 2 \left[\int_\Omega \left(\frac{\lambda^{s(x)}}{h(x)^{r(x)}} \right)^{\frac{1}{s(x)-r(x)}} dx \right]^{\frac{1}{\left(\frac{s}{s-r}\right)^+}} \left(\int_\Omega h(x) |u|^{s(x)} dx \right)^{\frac{1}{\left(\frac{s}{r}\right)^-}}.$$

Therefore, we arrive at

$$\int_\Omega h(x) |u|^{s(x)} dx \geq \left(\frac{\lambda}{2} \right)^{\left(\frac{s}{r}\right)^-} \left[\int_\Omega \left(\frac{\lambda^{s(x)}}{h(x)^{r(x)}} \right)^{\frac{1}{s(x)-r(x)}} dx \right]^{\frac{-\left(\frac{s}{r}\right)^-}{\left(\frac{s}{s-r}\right)^\pm}}$$

and so

$$\tilde{\lambda} \geq r^+ \int_\Omega \frac{h(x)}{s(x)} |u|^{s(x)} dx \geq \frac{r^+}{s^+} \left(\frac{\lambda}{2} \right)^{\left(\frac{s}{r}\right)^-} \left[\int_\Omega \left(\frac{\lambda^{s(x)}}{h(x)^{r(x)}} \right)^{\frac{1}{s(x)-r(x)}} dx \right]^{\frac{-\left(\frac{s}{r}\right)^-}{\left(\frac{s}{s-r}\right)^\pm}}.$$

Then, we find that $\tilde{\lambda} > 0$. Let $\lambda > \tilde{\lambda}$. Consequently, there exists $\bar{u} \in X$ with $\displaystyle\int_\Omega |\bar{u}|^{r(x)} dx = 1$ such that the following inequality holds

$$\lambda \int_\Omega |\bar{u}|^{r(x)} dx = \lambda > r^+ \int_\Omega \Phi(x, |\nabla \bar{u}|) dx + r^+ \int_\Omega \frac{1}{p(x)} |\bar{u}|^{p(x)} dx + r^+ \int_\Omega \frac{h(x)}{s(x)} |\bar{u}|^{s(x)} dx.$$

Hence, we obtain

$$\begin{aligned} \lambda \int_\Omega \frac{1}{r(x)} |\bar{u}|^{r(x)} dx &\geq \frac{\lambda}{r^+} \int_\Omega |\bar{u}|^{r(x)} dx \\ &> \int_\Omega \Phi(x, |\nabla \bar{u}|) dx + \int_\Omega \frac{1}{p(x)} |\bar{u}|^{p(x)} dx + \int_\Omega \frac{h(x)}{s(x)} |\bar{u}|^{s(x)} dx. \end{aligned}$$

Now, it is obvious that

$$I(\bar{u}) = \int_\Omega \Phi(x, |\nabla \bar{u}|)dx + \int_\Omega \frac{1}{p(x)}|\bar{u}|^{p(x)}dx - \lambda \int_\Omega \frac{1}{r(x)}|\bar{u}|^{r(x)}dx + \int_\Omega \frac{h(x)}{s(x)}|\bar{u}|^{s(x)}dx < 0$$

and finally we obtain that $\inf_X I < 0$. Now, we can establish that there exists $\lambda_0 = \tilde{\lambda}$ such that, for any $\lambda > \lambda_0$, (2) has a nontrivial weak solution $\bar{u} \in X$, which satisfies $I(\bar{u}) < 0$. Whereas $I(\bar{u}) = I(|\bar{u}|)$, we can suppose that $\bar{u} \geq 0$ almost everywhere in Ω, that is, $\bar{u} \not\equiv 0$.

Second part. The goal of this section of our proof is to find a second nontrivial weak solution for problem (2). For this purpose, we fix $\lambda \geq 0$ and define

$$f(x,t) = \begin{cases} 0, & \text{if} \quad t < 0, \\ \lambda t^{r(x)-1} - h(x)t^{s(x)-1}, & \text{if} \quad 0 \leq t \leq \bar{u}(x), \\ \lambda \bar{u}(x)^{r(x)-1} - h(x)\bar{u}(x)^{s(x)-1}, & \text{if} \quad t > \bar{u}(x), \end{cases}$$

and

$$F(x,t) = \int_0^t f(x,\nu)d\nu.$$

Let be $J : X \to \mathbb{R}$ a functional defined by

$$J(u) = \int_\Omega \Phi(x, |\nabla u|)dx + \int_\Omega \frac{1}{p(x)}|u|^{p(x)}dx - \int_\Omega F(x,u)dx.$$

It is easy to show, by standard arguments, that $J \in C^1(X, \mathbb{R})$, with the Frèchet derivative given by

$$\left\langle J^{'}(u), v \right\rangle = \int_\Omega \left(\phi(x, |\nabla u|)\nabla u \nabla v + |u|^{p(x)-2}uv \right) dx - \int_\Omega f(x,u)v dx,$$

for any $u, v \in X$. Obviously, if u is a critical point of J then $u \geq 0$ almost everywhere in Ω.

Our main tool to find a critical point $\tilde{u} \in X$ of J such that $J(\tilde{u}) > 0$ is the mountain pass theorem. In the following, we intend to prove two lemmas.

Lemma 4.3. There exists $\delta \in (0, \|\bar{u}\|)$ and $\alpha > 0$ such that $J(u) \geq \alpha$, for any $u \in X$ which satisfies $\|u\| = \delta$.

Proof. Since $W_0^{1,p(\cdot)}(\Omega)$ is continuously embedded in $L^{r(\cdot)}(\Omega)$, there exists a constant $N > 1$ such that

$$|u|_{r(\cdot)} \leq N \cdot \|u\|, \text{ for all } u \in W_0^{1,p(\cdot)}(\Omega). \tag{26}$$

We fix $\delta \in (0,1)$ such that $\delta < \dfrac{1}{N}$. By the above inequality (26) we have

$$|u|_{r(\cdot)} < 1, \text{ for any } u \in W_0^{1,p(\cdot)}(\Omega) \text{ which satisfies } \|u\| = \delta.$$

By (6) and (26) we obtain that

$$\lambda \int_\Omega |u|^{r(x)}dx \leq N_1 \|u\|^{r^-}, \tag{27}$$

with $N_1 = \lambda N^{r^-}$.

By (h_3), considering $|\nabla u|_{p(\cdot)} < 1$, $|u|_{p(\cdot)} < 1$, we have

$$\begin{aligned}
J(u) &= \int_\Omega \Phi(x,|\nabla u|)dx + \int_\Omega \frac{1}{p(x)}|u|^{p(x)}dx - \int_{[u>\bar u]} F(x,u)dx - \int_{[u\le \bar u]} F(x,u)dx \\
&= \int_\Omega \Phi(x,|\nabla u|)dx + \int_\Omega \frac{1}{p(x)}|u|^{p(x)}dx - \lambda \int_{[u>\bar u]} \bar u^{r(x)-1}udx + \int_{[u>\bar u]} h(x)\bar u^{s(x)-1}udx \\
&- \lambda \int_{[u\le \bar u]} \frac{1}{r(x)} u^{r(x)}dx + \int_{[u\le \bar u]} \frac{h(x)}{s(x)} u^{r(x)}dx \\
&> \frac{c}{p^+}\int_\Omega |\nabla u|^{p(x)}dx + \frac{1}{p^+}\int_\Omega |u|^{p(x)}dx - \lambda \int_{[u>\bar u]} u^{r(x)}dx - \frac{\lambda}{r^-}\int_{[u\le \bar u]} u^{r(x)}dx \\
&> \frac{c}{p^+}\int_\Omega |\nabla u|^{p(x)}dx + \frac{1}{p^+}\int_\Omega |u|^{p(x)}dx - \lambda \int_\Omega u^{r(x)}dx \\
&> \frac{1}{p^+}\int_\Omega \left(|\nabla u|^{p(x)} + |u|^{p(x)}\right)dx - \lambda \int_\Omega u^{r(x)}dx,
\end{aligned} \tag{28}$$

for any $u \in X$.

Combining relations (11), (26) and (27) we get to

$$\begin{aligned}
J(u) &> \frac{1}{p^+}\|u\|^{p^+} - N_1\|u\|^{r^-} \\
&= \|u\|^{p^+}\left(\frac{1}{p^+} - N_1\|u\|^{r^- - p^+}\right),
\end{aligned}$$

for all $u \in X$ satisfying $\|u\| = \delta$.

Let be $\psi : [0,1] \to \mathbb{R}$ a function defined by

$$\psi(t) = \frac{1}{p^+} - N_1 t^{r^- - p^+}.$$

We can observe that ψ is a positive function in a neighborhood of the origin such that $\delta \in (0,1)$ is small enough and $\alpha = \delta^{p^+}\psi(\delta) > 0$. Then, this completes our proof. □

Lemma 4.4. The functional J is coercive on X.

Proof. For all $u \in X$, taking into account hypotheses (h_3), we have

$$\begin{aligned}
J(u) &= \int_\Omega \Phi(x,|\nabla u|)dx + \int_\Omega \frac{1}{p(x)}|u|^{p(x)}dx - \lambda \int_{[u>\bar u]} \bar u^{r(x)-1}udx + \int_{[u>\bar u]} h(x)\bar u^{s(x)-1}udx \\
&- \lambda \int_{[u\le \bar u]} \frac{1}{r(x)} u^{r(x)}dx + \int_{[u\le \bar u]} \frac{h(x)}{s(x)} u^{r(x)}dx \\
&> \frac{c}{p^+}\int_\Omega |\nabla u|^{p(x)}dx + \frac{1}{p^+}\int_\Omega |u|^{p(x)}dx - \lambda \int_{[u>\bar u]} \bar u^{r(x)}dx - \frac{\lambda}{r^-}\int_{[u\le \bar u]} \bar u^{r(x)}dx \\
&> \frac{1}{p^+}\int_\Omega \left(|\nabla u|^{p(x)} + |u|^{p(x)}\right)dx - \lambda \int_\Omega \bar u^{r(x)}dx \\
&= \frac{1}{p^+}\int_\Omega \left(|\nabla u|^{p(x)} + |u|^{p(x)}\right)dx - N_2,
\end{aligned} \tag{29}$$

where N_2 is a positive constant.

For $u \in X$ such that $\|u\| > 1$, by (10) and (29) we obtain that

$$J(u) > \frac{1}{p^+} \|u\|^{p^-} - N_2.$$

Therefore, J is coercive on X.

□

By means of 4.3 and the mountain pass theorem [24, Theorem 1.15] we can see that there is a sequence $(u_n) \in X$ such that

$$J(u_n) \to \tilde{c} > 0 \text{ and } J^{'}(u_n) \to 0, \tag{30}$$

with

$$\tilde{c} = \inf_{q \in Q} \max_{t \in [0,1]} J(q(t))$$

and

$$Q = \{q \in C([0,1], X); q(0) = 0, q(1) = \bar{u}\}.$$

By 4.4 and relation (30) we obtain that (u_n) is a bounded sequence, so there is $\tilde{u} \in X$ such that, up to a subsequence, (u_n) is weakly convergent to $\tilde{u}$ in X.

By standard arguments, which involve Sobolev embeddings, we can easily prove that

$$\lim_{n\to\infty} \left\langle J^{'}(u_n), v \right\rangle = \left\langle J^{'}(\tilde{u}), v \right\rangle,$$

for all $v \in C_0^\infty(\Omega)$. This together with the continuous embedding of X in $W_0^{1,p(\cdot)}(\Omega)$ and the density of $C_0^\infty(\Omega)$ in $W_0^{1,p(\cdot)}(\Omega)$ follows us to $J^{'}(\tilde{u}) = 0$, which means that $\tilde{u}$ is a critical point of J.

Consequently, our aim is to show that $\tilde{u}$ is also a critical point for I. For this purpose, we first prove that $\tilde{u} \leq \bar{u}$.

By [9, Theorem 7.6] we have that if $v \in X$ then $v^+ \in X$, where $v^+(x) = \max\{v(x), 0\}$. Thus, making use of hypotheses (h_3), we have

$$\begin{aligned}
0 &= \left\langle J^{'}(\tilde{u}), (\tilde{u}-\bar{u})^+ \right\rangle - \left\langle I^{'}(\bar{u}), (\tilde{u}-\bar{u})^+ \right\rangle \\
&= \int_\Omega \phi(x, |\nabla\tilde{u}|)\nabla\tilde{u}\nabla(\tilde{u}-\bar{u})dx + \int_\Omega |\tilde{u}|^{p(x)-2}\tilde{u}(\tilde{u}-\bar{u})dx - \int_\Omega f(x,\tilde{u})(\tilde{u}-\bar{u})dx \\
&\quad - \int_\Omega \phi(x, |\nabla\bar{u}|)\nabla\bar{u}\nabla(\tilde{u}-\bar{u})dx - \int_\Omega |\bar{u}|^{p(x)-2}\bar{u}(\tilde{u}-\bar{u})dx \\
&\quad +\lambda \int_\Omega |\bar{u}|^{r(x)-2}\bar{u}(\tilde{u}-\bar{u})dx - \int_\Omega h(x)|\bar{u}|^{s(x)-2}\bar{u}(\tilde{u}-\bar{u})dx \\
&= \int_\Omega \left(\phi(x, |\nabla\tilde{u}|)\nabla\tilde{u} - \phi(x, |\nabla\bar{u}|)\nabla\bar{u}\right)\nabla(\tilde{u}-\bar{u})^+dx \\
&\quad + \int_\Omega \left(|\tilde{u}|^{p(x)-2}\tilde{u} - |\bar{u}|^{p(x)-2}\bar{u}\right)(\tilde{u}-\bar{u})^+dx \\
&\quad - \int_\Omega \left(f(x,\tilde{u}) - \lambda|\bar{u}|^{r(x)-2}\bar{u} + h(x)|\bar{u}|^{s(x)-2}\bar{u}\right)(\tilde{u}-\bar{u})^+dx
\end{aligned}$$

$$
\begin{aligned}
&= \int_{[\tilde{u}>\bar{u}]} (\phi(x, |\nabla\tilde{u}|)\nabla\tilde{u} - \phi(x, |\nabla\bar{u}|)\nabla\bar{u})\,(\nabla\tilde{u} - \nabla\bar{u})dx \\
&\quad + \int_{[\tilde{u}>\bar{u}]} \left(|\tilde{u}|^{p(x)-2}\tilde{u} - |\bar{u}|^{p(x)-2}\bar{u}\right)(\tilde{u} - \bar{u})dx \\
&\geq c\int_{[\tilde{u}>\bar{u}]} \left(|\nabla\tilde{u}|^{p(x)-1} - |\nabla\bar{u}|^{p(x)-1}\right)(|\nabla\tilde{u}| - |\nabla\bar{u}|)dx \\
&\quad + \int_{[\tilde{u}>\bar{u}]} \left(|\tilde{u}|^{p(x)-1} - |\bar{u}|^{p(x)-1}\right)(|\tilde{u}| - |\bar{u}|)dx \geq 0.
\end{aligned}
$$

Hence, we find that $0 \leq \tilde{u} \leq \bar{u}$ in Ω. It follows that

$$f(x, \tilde{u}) = \lambda\tilde{u}^{r(x)-1} - h(x)\tilde{u}^{s(x)-1}$$

and

$$F(x, \tilde{u}) = \frac{\lambda}{r(x)}\tilde{u}^{r(x)} - \frac{h(x)}{s(x)}\tilde{u}^{s(x)}.$$

Therefore,

$$J(\tilde{u}) = I(\tilde{u})$$

and

$$J^{'}(\tilde{u}) = I^{'}(\tilde{u}).$$

Consequently, we have that

$$I(\tilde{u}) = J(\tilde{u}) > 0 = I(0) > I(\tilde{u})$$

and

$$0 = J^{'}(\tilde{u}) = I^{'}(\tilde{u}).$$

This means that $\tilde{u}$ is a weak solution for problem (2) so that $0 \leq \tilde{u} \leq \bar{u}$, with $\bar{u} \neq 0$ and $\tilde{u} \neq \bar{u}$. Thus, we conclude that problem (2) has at least two nontrivial weak solutions.

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