Annals of the University of Craiova, Mathematics and Computer Science Series
Volume 46(2), 2019, Pages 254–268
ISSN: 1223-6934

Separation axioms on topological effect algebras

R.A. Borzooei, M.R. Rakhshani, and G.R. Rezaei

Abstract. In this paper, by considering the notions of (para, semi)topological effect algebras, separation axioms on these topologies are investigated. Some conditions which under them, a (para, semi)topological effect algebra be a T_i -space ($i = 1, 2, 3, 4$) are found. Then compact Hausdorff (para, semi)topological algebras are studied. Finally, topological structure on quotient effect algebras are defined and separation axioms of it are investigated.

2010 Mathematics Subject Classification. 06B10, 03G10.
Key words and phrases. Effect algebra, (semi, para) topological effect algebras, T_i -spaces.

1. Introduction

Topology and algebra, the two fundamental domains of mathematics, play complementary roles. Topology studies continuity and convergence and provides a general framework to study the concept of a limit. Algebra considers all kinds of operations and provides a basis for algorithms and calculations. Many of the most important objects of mathematics represent a blend of algebraic and topological structures. For example, in applications, in higher level domains of mathematics, such as functional analysis, dynamical systems, representation theory, and among others, topology and algebra come in contact most naturally.

When an algebraic structure and a topology come naturally together, the rules that describe the relationship between topology and algebraic operation are almost always transparent, and natural-the operation has to be continuous or semicontinuous. Topological function spaces and linear topological spaces in general, topological groups and topological field, transformation groups, and topological lattices are objects of this kind. In recent decades, there has been much research on the structures associated with logical systems. For instance, BCK-algebras were introduced by Y. Imai and K. Iséki [11] as an algebraic formulation of Meredith's BCK-implicational calculus, and BL-algebras have been defined by Hájek [10] to investigate many-valued logic by algebraic means.

Furthermore, several mathematicians have endowed a number of algebraic structures associated with logical systems with a topology and have found some of their properties. For example, R.A. Borzooei et al. [2, 3, 4, 18, 21] who defined semi-topological and topological BL- algebras, Roudabri and Torkzadeh [22] used the left (right) stabilizers of a BCK- algebra and produced two bases for two different topologies, and M. Haveshki et al. [15] introduced the topology induced by uniformity on BL-algebras.

Received December 13, 2017. Accepted June 5, 2019.

Effect algebras have been introduced by D.J. Foulis and M.K. Bennet in 1994 for modelling unsharp measurement in a quantum mechanical system. In the last few years, the theory of effect algebras has enjoyed a rapid development. As an important tool of studying, the topological structures of effect algebras not only can help us to describe the convergence properties, but also can help us to characterize some algebra properties of effect algebras.

Now, in this paper we study the separation axioms on (para, semi)topological effect algebras and on quotient effect algebras.

2. Preliminary

In this section, we collect the relevant definitions and results from topology and effect algebra theory to make this paper self-contained and easy to read.

A set A with a family $\mathcal{T}$ of its subsets is called a *topological space*, denoted by $(A,\mathcal{T})$, if $A,\emptyset\in\mathcal{T}$, the intersection of any finite number of the members of $\mathcal{T}$ is in $\mathcal{T}$, and the arbitrary union of members of $\mathcal{T}$ is in $\mathcal{T}$. The members of $\mathcal{T}$ are called *open sets* of A, and the complement of an open set U, $A\backslash U$, is a *closed* set. If B is a subset of A, then $\overline{B}$, the smallest closed set containing B, is called the *closure* of B and B° the biggest open set contained in B, is named the *interior* of B. A subfamily $\{U_\alpha\}$ of $\mathcal{T}$ is named a *base* of $\mathcal{T}$ if for each $x\in U\in\mathcal{T}$, there is an $\alpha\in I$ such that $x\in U_\alpha\subseteq U$, or equivalently, each U in $\mathcal{T}$ is the union of members of $\{U_\alpha\}$. A subfamily $\{U_\beta\}$ of $\mathcal{T}$ is a *subbase* for $\mathcal{T}$ if the family of finite intersections of members of $\{U_\beta\}$ forms a basis of $\mathcal{T}$. A subset P of A is a *neighborhood* of $x\in A$, if there exists an open set U such that $x\in U\subseteq P$. Let $\mathcal{T}_x$ denote the all neighborhoods of x in A, then a subfamily $\mathcal{V}_x$ of $\mathcal{T}_x$ is a *fundamental system* of neighborhoods of x, if for each U_x in $\mathcal{T}_x$, there exists a V_x in $\mathcal{V}_x$ such that $V_x\subseteq U_x$. Let $(A,\mathcal{T})$ and $(B,\mathcal{V})$ be two topological spaces, a mapping f of A into B is *continuous* if $f^{-1}(U)\in\mathcal{T}$ for any $U\in\mathcal{V}$. The mapping f is called *homeomorphism* if f is bijective, and f and f^{-1} are continuous, or equivalently, if f is bijective, continuous and open (closed). A topological space $(A,\mathcal{T})$ is *compact* if each open covering of A is reducible to a finite open covering and *locally compact* if for each $x\in A$ there exists an open neighborhood U of x and a compact subset K such that $x\in U\subseteq K$. Let $(A,\mathcal{T})$ be a topological space. Then $(A,\mathcal{T})$ is a:
(*i*) T_0-*space* if, for each $x,y\in A$ and $x\neq y$, there is at least one in an open neighborhood excluding the other.
(*ii*) T_1-*space* if, for each $x,y\in A$ and $x\neq y$, each has an open neighborhood not containing the other.
(*iii*) T_2-*space* if, for each $x,y\in A$ and $x\neq y$, both have disjoint open neighborhoods U, V such that $x\in U$ and $y\in V$. A T_2-space is also known as a *Hausdorff space.*
(*iv*) *regular space* if, for any closed subset C of $(A,\mathcal{T})$ and $x\in A$ such that $x\notin C$, there exist disjoint open sets U and V such that $x\in U$ and $C\subseteq V$.
(*v*) *normal space* if, C and D are two disjoint closed subsets of A, then there exist two disjoint open subsets U and V such that $C\subseteq U$ and $D\subseteq V$.
(*v*) T_3-*space* if it is T_1 and *regular* space.
(*vi*) T_4-*space* if it is T_1 and *normal* space (See, [5]).

Proposition 2.1. (*i*) *A topological space $(A,\mathcal{T})$ is a T_3-space if and only if for each $x\in U\in\mathcal{T}$, there exists an open set H such that $x\in H\subseteq\overline{H}\subseteq U$.*
(*ii*) *A topological space $(A,\mathcal{T})$ is a T_4-space if and only if for each closed set S and*

each open set U contains S, there exists an open set H such that $S \subseteq H \subseteq \overline{H} \subseteq U$ (See, [20]*).*

An *effect algebra* is algebraic structure $(E, \oplus, 0, 1)$ where 0,1 are distinct elements of E and $\oplus$ is a partial binary operation on E that satisfies the following conditions, for any $a, b, c \in E$:
$(E1)$(Commutative law) If $a \oplus b$ is defined, then $b \oplus a$ is defined and $a \oplus b = b \oplus a$.
$(E2)$(Associative law) If $a \oplus b$ and $(a \oplus b) \oplus c$ are defined, then $b \oplus c$ and $a \oplus (b \oplus c)$ are defined and $(a \oplus b) \oplus c = a \oplus (b \oplus c)$.
$(E3)$(Orthosupplementation law) For each $a \in E$ there exist a unique $b \in E$ such that $a \oplus b$ is defined and $a \oplus b = 1$.
$(E4)$(Zero-Unit law) If $a \oplus 1$ is defined, then $a = 0$.
For each $a \in E$, we denote the unique b in condition $(E3)$ by $a^{'}$ and call it the orthosupplement of a. The sense is that if a presents a proposition, then $a^{'}$ corresponds to the negation. The binary relation $\leq$ which is define by $a \leq b$, if and only if there exists $c \in E$ such that $a \oplus c = b$ is a totally relation. Such an element c is unique, and therefore we can introduce a dual operation $\ominus$ in E by $a \ominus b = c$ if and only if $a = c \oplus b$. Let A and B be subsets of a effect algebra E. Then $A \oplus B$ denotes the set $\{a \oplus b : a \in A, b \in B\}$, and $A^{'}$ denotes the set $\{a^{'} : a \in A\}$, and for $a \in E$, $a \oplus A$ denotes the set $\{a \oplus b : b \in A\}$. If $a \oplus b$ is defined, then we say that a and b are *orthogonal* and write $a \perp b$ (See, [8, 9, 12]).

Proposition 2.2. [12] *The following properties hold for any effect algebra, for any $a, b \in E$:*
(i) $a^{''} = a$,
(ii) $1^{'} = 0$ *and* $0^{'} = 1$,
(iii) $0 \leq a \leq 1$,
(iv) $a \oplus 0 = a$,
(v) $a \oplus b = 0 \Rightarrow a = b = 0$,
(vi) $a \leq a \oplus b$,
(vii) $a \leq b \Rightarrow b^{'} \leq a^{'}$,
(viii) $b \ominus a = (a \oplus b^{'})^{'}$,
(ix) $a \oplus b^{'} = (b \ominus a)^{'}$,
(x) $a = a \ominus 0$,
(xi) $a \ominus a = 0$,
(xii) $a^{'} = 1 \ominus a$ *and* $a = 1 \ominus a^{'}$.

Let $(E, \oplus, 0, 1)$ be an effect algebra. A nonempty subset I of E is said to be an ideal of E, if for all $a, b \in E$, $a \in I$ and $b \leq a$ implies $b \in I$ and $a \ominus b \in I$ and $b \in I$ implies $a \in I$. Equivalently, if $a \oplus b$ is defined, then $a \oplus b \in I \Leftrightarrow a, b \in I$ (See, [19]).

A binary relation $\sim$ on effect algebra E is called a *congruence relation* if:
$(C1)$ $\sim$ is an equivalence relation,
$(C2)$ $a \sim a_1, b \sim b_1, a \perp b$ and $a_1 \perp b_1$ then $a \oplus b \sim a_1 \oplus b_1$.
$(C3)$ if $a \sim b$ and $b \perp c$, then there exists $d \in E$ such that $d \sim c$ and $a \perp d$.

Note: Condition $(C3)$ is equivalent to the conditions:

$(C4)$ If $a \sim b$ and $a \oplus a_1 \sim b \oplus b_1$, then $a_1 \sim b_1$.

$(C5)$ If $a \sim b \oplus c$, then there are a_1, a_2 such that $a = a_1 \oplus a_2$ and $a_1 \sim b, a_2 \sim c$.
Condition $C(4)$ is equivalent to condition
$(C6)$ if $a \sim b$, then $a^{'} \sim b^{'}$ (See, [14]).
If $\sim$ is a congruence relation, then quotient $E/\sim$ is an effect algebra (See, [9]). If I is an ideal of E, we define a binary relation $\sim_I$ on E by $a \sim_I b$ if and only if there are $i, j \in I$, such that $i \leq a, j \leq b$ and $a \ominus i = b \ominus j$. Equivalently, $a \sim_I b$ if and only if there is $k \in E$, such that $k \leq a, b$, and $a \ominus k, b \ominus k \in I$.

Let $(E, \oplus, 0, 1)$ be an effect algebra and I be an ideal of E. We say that I is a *Riesz ideal* of E if for any $i \in I$ and $a, b \in E$ if $a \perp b$ and $i \leq a \oplus b$, then there exist $a_1, b_1 \in I$ such that $a_1 \leq a, b_1 \leq b$ and $i \leq a_1 \oplus b_1$ (See, [9]).

Theorem 2.3. [13] *Let I be an ideal in effect algebra E. Then $\sim_I$ is a congruence relation on E if and only if I is a Riesz ideal of E.*

Note: From now one, in this paper we let $(E, \oplus, 0, 1)$ or E is an effect algebra.

Definition 2.1. [21] Let $\mathcal{T}$ be a topology on effect algebra E. Then $(E, \mathcal{T})$ is called:
(i) *semitopological* effect algebra if $\oplus$ is semicontinuous, or equivalently, if for any $a, x \in E$ and any neighborhood U of $a \oplus x$, there exists an open neighborhood V of x such that $a \oplus V \subseteq U$,
(ii) *paratopological* effect algebra if the operation $\oplus$ is continuous, or equivalently, if for any x, y in E and any open set W of $x \oplus y$ there exist two open neighborhoods U and V of x and y, respectively, such that $U \oplus V \subseteq W$,
(iii) *topological* effect algebra, if the operations $\oplus$ and $'$ are continuous.

Example 2.1. [21] (i) Let $E = [0, 1]$ be effect algebra with the interval topology $\mathcal{T}$ of $\mathbb{R}$. Then E is topological effect algebra.
(ii) $C_4 = \{0, 1/4, 2/4, 3/4, 1\}$ with topology $\mathcal{T} = \{\emptyset, \{0\}, \{0, 1/4\}, C_4\}$ is a paratopological effect algebra.

Notation: [21] For $a \in E$, we define the maps $T_a, L_a, R_a, ' : E \to E$ as follows:

$$T_a(x) = a \oplus x \ \ , \ \ L_a(x) = a \ominus x \ \ , \ \ R_a(x) = x \ominus a \ \ , \ \ '(x) = x^{'} .$$

Theorem 2.4. [21] *Let $(E, \mathcal{T})$ be a topological effect algebra. Then the operation $\ominus$ is continuous.*

3. Separation axioms on effect algebras

In this section, we get some conditions on (semi, para)topological effect algebras which causes it becomes a T_i-space. Recall that a topological space $(X, \mathcal{T})$ is called discrete if every element admits a neighborhood consisting of that element only.

Proposition 3.1. *If $\{0\}$ is an open set in a topological effect algebra E, then E is discrete.*

Proof. Since E ia an effect algebra, by Proposition 2.2(xi), $x \ominus x = 0 \in \{0\}$, for all $x \in E$. Because $\{0\}$ is open, by continuity of $\ominus$ in E, there exist two neighborhoods U and V of x, such that $U \ominus V = \{0\}$. Let $W = U \cap V$. Then $W \ominus W \subseteq U \ominus V = \{0\}$ and so $W \ominus W = \{0\}$. It follows from $x \in W$, that $W = \{x\}$, which means that E is discrete. □

Theorem 3.2. *Suppose that $\mathcal{T}$ is a topology on effect algebra E.*
(i) If for any $a \in E$, T_a is an open map and there exists $U \in \mathcal{T}$ containing 0, then $(E, \mathcal{T})$ is a T_0-space.
(ii) If for any $a \in E$, $R_a(L_a)$ is an open map and there exists $U \in \mathcal{T}$ containing 1, then $(E, \mathcal{T})$ is a T_0-space.

Proof. (i) Let for any $a \in E$, T_a be an open map and $x, y \in E$ such that $x \neq y$. Suppose U be an open set containing 0. Then $U \oplus x$ and $U \oplus y$ are open sets containing x and y, respectively. We claim that $x \notin U \oplus y$ or $y \notin U \oplus x$. Let $x \in U \oplus y$ and $y \in U \oplus x$. Then, for some $a \in U$, $x = a \oplus y$. By definition of order relation $\leq$ in E, we get that $y \leq x$. Similarly, $y \in U \oplus x$, follows that $x \leq y$. Hence, $x = y$, which is a contradiction.
(ii) Let for any $a \in E$, $R_a(L_a)$ be an open map and $x, y \in E$ such that $x \neq y$. Let U be an open set containing 1. By Proposition 2.2(xii), $1 \ominus x^{'} = x$ and $1 \ominus y^{'} = y$. Hence, $R_{x'}(U) = U \ominus x^{'}$ and $R_{y'}(U) = U \ominus y^{'}$ are open sets containing x and y, respectively. We claim that $y \notin U \ominus x^{'}$ or $x \notin U \ominus y^{'}$. Let $y \in U \ominus x^{'}$ and $x \in U \ominus y^{'}$. Then, for some $a \in U$, $y = a \ominus x^{'}$ and so by Proposition 2.2(viii) $y^{'} = (a \ominus x^{'})^{'} = x^{'} \oplus a^{'}$. Thus $x^{'} \leq y^{'}$ and so by Proposition 2.2(vii), $y \leq x$. Similarly, for $x \in U \ominus y^{'}$, we get $x \leq y$. Hence $x = y$, that is a contradiction. Therefore, $(E, \mathcal{T})$ is a T_0-space. □

Theorem 3.3. *Let effect algebra E has a proper ideal I (that is, distinct from E and $\{0\}$). Then,*
(i) there is a nontrivial topology $\mathcal{T}$ on E such that $(E, \mathcal{T})$ is a paratopological effect algebra and T_0-space.
(ii) I is a closed and open set.

Proof. (i) Let I be a proper ideal in E and $\mathcal{T} = \{U \subseteq E : \forall x \in U, I \oplus x \subseteq U\}$. It is easy to prove that $\mathcal{T}$ is a topology on E. We show that for any $a \in E$, $a \oplus I \in \mathcal{T}$. Let $a \in E$ and $y \in a \oplus I$. Then

$$I \oplus y \subseteq I \oplus (I \oplus a) = (I \oplus I) \oplus a \subseteq I \oplus a.$$

Hence, $\mathcal{T}$ is a nontrivial topology on E. If $x, y \in E$ and $x \oplus y \in U \in \mathcal{T}$, then $(I \oplus x) \oplus y \subseteq U$. On the other hand $x \oplus I$ and $y \oplus I$ are two open neighborhoods of x and y, respectively, such that

$$(x \oplus I) \oplus (y \oplus I) = x \oplus y \oplus I \oplus I \subseteq x \oplus y \oplus I \subseteq U.$$

Hence, $(E, \mathcal{T})$ is a paratopological effect algebra and so T_a is a continuous map. Now, we prove that for any $a \in E, T_a$ is an open map. Let $a \in E$ and $U \in \mathcal{T}$. If $y \in a \oplus U$, then there exists $x \in U$ such that $y = a \oplus x$. Thus, by $(E1)$ and $(E2)$ $I \oplus y = I \oplus (a \oplus x) = a \oplus (I \oplus x) \subseteq a \oplus U$. Hence, $a \oplus U$ is an open set in E. This proves that T_a is an open map. Therefore, Theorem 3.2(i), $(E, \mathcal{T})$ is a T_0-space.
(ii) For $x \in I$, $x \oplus I \subseteq I$. Then $I \in \mathcal{T}$. If $x \in \overline{I}$, then $(I \oplus x) \cap I \neq \emptyset$. Hence, there exists $a \in I$ such that $a \oplus x \in I$. Now since by Proposition 2.2(vi), $x \leq x \oplus a$ and I is an ideal, we get $x \in I$. Then, I is a closed set. □

Example 3.1. Let $E = \{0, a, b, 1\}$, be a poset where $0 < a, b < 1$. Consider the following tables :

$\oplus$	0	a	b	1
0	0	a	b	1
a	a	a	1	-
b	b	1	b	-
1	1	-	-	-

$'$	0	a	b	1
	1	b	a	0

Then, $(E, \oplus, 0, 1)$ is an effect algebra. Since $\{0, a\}$ is an ideal of E, by Theorem 3.3,

$$\begin{aligned}\mathcal{T} =& \{U \subseteq E : \forall x \in U, I \oplus x \subseteq U\}\\ =& \{\emptyset, E, \{0, a\}, \{1, b\}, \{a\}, \{1\}, \{a, b, 1\}, \{0, a, 1\}, \{a, 1\}\}\end{aligned}$$

is a topology on E such that $(E, \mathcal{T})$ is a paratopological effect algebra and T_0-space.

Theorem 3.4. *Let $\mathcal{T}$ be a topology on effect algebra E such that for any $a \in E$, L_a be an open map. Then $(E, \mathcal{T})$ is a T_0-space if and only if for any $0 \neq x \in E$, there exists an open neighborhood U of 0 such that $x \notin U$.*

Proof. Let for any $0 \neq x \in E$, there exists an open neighborhood U of 0 such that $x \notin U$. We prove that $(E, \mathcal{T})$ is a T_0-space. Let $x \neq y$. By definition of $\ominus$, $x \ominus y$ or $y \ominus x$ is defined. W.O.L.G, suppose that $x \ominus y$ is defined and $x \ominus y \neq 0$. Then there is an open neighborhood U of 0 such that $x \ominus y \notin U$. We claim $y \notin x \ominus U$ or $x \notin y \ominus U$. Let $y \in x \ominus U$ and $x \in y \ominus U$. If $y \in x \ominus U$, then for some $a \in U$, $y = x \ominus a$ or $x = y \oplus a$ and so $\leq$, $y \leq x$. Similarly, $x \in y \ominus U$ follows $x \leq y$ and so $x = y$, which is a contradiction. Therefore, $(E, \mathcal{T})$ is a T_0-space. □

Theorem 3.5. *Let $(E, \mathcal{T})$ be a topological effect algebra. If for any $x \in E \setminus \{0\}$ there exists an open set U containing x such that $0 \notin U$, then $(E, \mathcal{T})$ is a T_1-space.*

Proof. Suppose $x \neq y$. By definition of $\ominus$, $x \ominus y$ or $y \ominus x$ is defined. W.O.L.G let $x \ominus y$ is defined and $x \ominus y \neq 0$. Then, there exists an open neighborhood U of $x \ominus y$ such that $0 \notin U$. On the other hand, since $(E, \mathcal{T})$ is a topological effect algebra, there exist open neighborhoods V and W of x and y, respectively, such that $V \ominus y \subseteq U$ and $x \ominus W \subseteq U$. We claim that $x \notin W$ and $y \notin V$. Because if $x \in W$ or $y \in V$, then by Proposition 2.2(xi), $x \ominus x = 0$ or $y \ominus y = 0$ and so $0 \in U$, which is a contrudiction. Therefore, $(E, \mathcal{T})$ is a T_1-space. □

Example 3.2. Let $E = [0, 1]$ be topological effect algebra as Example 2.1(i). For any $x \in E \setminus \{0\}$ and $x < 1$, we consider $\delta = min\{x - 0, 1 - x\}$. Then $x \in N_\delta(x)$ and $0 \notin N_\delta(x)$. Hence by Proposition 3.5 $(E, \mathcal{T})$ is a T_1 space.

Theorem 3.6. *Let $(E, \mathcal{T})$ be a semitopological effect algebra and $'$ be a continuous map. Then, $(E, \mathcal{T})$ is a T_1-space if and only if for any $x \neq 1$ there are open neighborhoods U and V of x and 1, respectively, such that $1 \notin U$ and $x \notin V$.*

Proof. If $(E, \mathcal{T})$ is a T_1-space, then the proof is clear. Conversely, let for any $x \neq 1$, there are open neighborhoods U and V of x and 1, respectively, such that $1 \notin U$ and $x \notin V$. We show that $(E, \mathcal{T})$ is a T_1-space. Let $x, y \in E$ and $x \neq y$. Hence $x \oplus y' \neq 1$ (if $x \oplus y' = 1$, by $(E3)$ $y' = x'$ and so $x'' = y''$ and by Proposition 2.2(i) $x = y$, which is contradiction). Let U be an open neighborhood of $x \oplus y'$ such that $1 \notin U$. Since $(E, \mathcal{T})$ is a semitopological effect algebra, there are two open neighborhoods V and

W of x and y', respectively, such that $V \oplus y' \subseteq U$ and $x \oplus W \subseteq U$. We claim that $x \notin W'$ and $y \notin V$. If $x \in W'$ or $y \in V$, then by $(E3)$, $1 \in U$, which is a contradiction. Moreover it is easily check that the map $' : E \longrightarrow E$ is homomorphism. Hence, W' is an open set and so $(E, \mathcal{T})$ is a T_1-space. □

Theorem 3.7. *Let $(E, \mathcal{T})$ be a topological effect algebra. Then $(E, \mathcal{T})$ is a T_0-space if and only if $(E, \mathcal{T})$ is a T_1-space.*

Proof. Let $(E, \mathcal{T})$ be a T_0-space and $x \neq y$ for $x, y \in E$. Then $x \ominus y$ or $y \ominus x$ is defined. W.O.L.G, suppose $x \ominus y$ is defined and $x \ominus y \neq 0$. Since E is a T_0-space, there exists an open set U such that $0 \in U$ and $x \ominus y \notin U$ or $0 \notin U$ and $x \ominus y \in U$. First, we suppose $0 \in U$ and $x \ominus y \notin U$. Since $(E, \mathcal{T})$ is a topological effect algebra, there exist open neighborhoods W and V of x and y, respectively, such that $V \ominus y \subseteq U$ and $x \ominus W \subseteq U$. We claim that $x \notin V$ and $y \notin W$. If $x \in V$ or $y \in W$, then $x \ominus y \in U$, which is a contradiction.

Now, suppose $0 \notin U$ and $x \ominus y \in U$. Since $\ominus$ is continuous, there exist open neighbourhoods V and W of x and y, respectively, such that $V \ominus y \subseteq U$ and $x \ominus W \subseteq U$. We claim that $y \notin V$ and $x \notin W$. If $y \in V$ or $x \in W$, then by Proposition 2.2(xi), $y \ominus y = x \ominus x = 0 \in U$, which is a contradiction. The proof of convers, ia clear. □

Theorem 3.8. *Let $(E, \mathcal{T})$ be a topological effect algebra. Then $(E, \mathcal{T})$ is a T_2-space if and only if $(E, \mathcal{T})$ is a T_0-space.*

Proof. Let $(E, \mathcal{T})$ be a T_0-space. Since E is a topological effect algebra, by Theorem 3.7, $(E, \mathcal{T})$ is a T_1-space. Let $x \neq y$, for $x, y \in E$. Then, $x \ominus y$ or $y \ominus x$ is defined and $x \ominus y \neq 0$ or $y \ominus x \neq 0$. W.O.L.G, suppose $x \ominus y \neq 0$. Since $(E, \mathcal{T})$ is a T_1-space, there exists an open set U such that $x \ominus y \in U$ and $0 \notin U$. By continuity of operation $\ominus$, there exist open sets W and V of x and y, respectively, such that $W \ominus V \subseteq U$. We claim that $W \cap V = \emptyset$. If $z \in W \cap V$, then by Proposition 2.2(xi), $0 = z \ominus z \in W \ominus V \subseteq U$, which is a contradiction. Hence, $(E, \mathcal{T})$ is a T_2-space. The converse is clear. □

Theorem 3.9. *Let $(E, \mathcal{T})$ be a topological effect algebra. Then $\{0\}$ is closed in E if and only if E is Hausdorff space.*

Proof. Assume that $\{0\}$ is closed in E and $x, y \in E$ with $x \neq y$. Then $x \ominus y$ or $y \ominus x$ is defined. W.L.O.G, suppose $x \ominus y$ is defined and $x \ominus y \neq 0$. Since $E \setminus \{0\}$ is open and operation $\ominus$ is continuous, there exist neighborhood U and V of x and y, respectively such that $U \ominus V \subseteq E \setminus \{0\}$. We claim $U \cap V = \emptyset$. If $z \in U \cap V$ then, $0 = z \ominus z \in U \ominus V \subseteq E \setminus \{0\}$, which is contradiction. Therefore $(E, \mathcal{T})$ is Hausdorff. Conversely, let $(E, \mathcal{T})$ be Hausdorff. We claim that $E \setminus \{0\}$ is open. If $x \in E \setminus \{0\}$, then $x \neq 0$ and so there exist distinct neighborhoods U and V of x and 0, respectively. Hence $0 \notin U$ and so $U \subseteq E \setminus \{0\}$, which implies that $E \setminus \{0\}$ is open. Therefore, $\{0\}$ is closed. □

Corollary 3.10. *Let $(E, \mathcal{T})$ be a topological effect algebra. Then $\{0\}$ is closed in E if and only if E is a $T_0(T_1)$-space.*

Proof. By Theorems 3.7, 3.8 and 3.9, the proof is clear. □

Proposition 3.11. *Let $\{0\}$ be a closed set in topological effect algebra $(E, \mathcal{T})$ and $\mathcal{N}_0$ be a fundamental system of open neighborhoods of 0. Then $\bigcap \mathcal{N}_0 = \{0\}$.*

Proof. Let $\{0\}$ be a closed set. By Theorem 3.9, $(E, \mathcal{T})$ is a Hausdorff space. Since $(E, \mathcal{T})$ is a Hausdorff space, for $0 \neq x \in E$, there exists an open neighborhood U of 0 such that $x \notin U$. Hence $x \notin \bigcap \mathcal{N}_0$. □

Proposition 3.12. *Let I be an ideal of topological effect algebra E. If 0 is an interior point of I, then I is open.*

Proof. Let $x \in I$. Since $x \ominus x = 0 \in I$ and 0 is an interior point of I, there exists a neighborhood U of 0 such that $U \subseteq I$. By continuity of $\ominus$, there exist neighborhoods U and H of x such that $G \ominus H \subseteq U \subseteq I$. On the other hand, for every $y \in G$, $y \ominus x \in G \ominus H \subseteq I$. Now, since I is an ideal and $x \in I$, we get $y \in I$ and so $x \in G \subseteq I$. Hence I is open. □

Theorem 3.13. *Let E be an effect algebra and $\Lambda = \{I_i\}_{i=1}^{\infty}$ be a family of ideals in E such that for any $i, j \in N$*

$$i \leq j \Leftrightarrow I_j \subseteq I_i$$

Now, let:

$$\mathcal{T}_\Lambda = \{U \subseteq E : \forall x \in U, \exists I_j \in \Lambda \ \ s.t. \ \ x \oplus I_j \subseteq U\}$$

Then $\mathcal{T}_\Lambda$ is a topology on E. Moreover if $\cap\Lambda = \{0\}$, then $(E, \mathcal{T}_\Lambda)$ is Hausdorff space.

Proof. It is clear that $\mathcal{T}_\Lambda$ is a topology on E. Now, let $\cap\Lambda = \{0\}$ and $x, y \in E$ such that $x \neq y$. Then $x \ominus y \neq \{0\}$ and so $x \ominus y \notin \cap\Lambda$. Hence there exists $I_i \in \Lambda$ such that $x \ominus y \notin I_i$. On the other hand $x \in x \oplus I_i$ and $y \in y \oplus I_i$. We claim that $(x \oplus I_i) \cap (y \oplus I_i) = \emptyset$. Let $z \in x \oplus I_i$ and $z \in y \oplus I_i$. Hence there exist $t_1, t_2 \in I_i$ such that $z = x \oplus t_1$ and $z = y \oplus t_2$. Then $x \oplus t_1 = y \oplus t_2$ and so $x \ominus y \in I_i$ which, is a contradiction. Therefore, $(E, \mathcal{T}_\Lambda)$ is a Hausdorff space. □

Recall that a topological space $(X, \mathcal{T})$ is an *Uryshon space* if for each $x \neq y \in X$, there exist two open neighborhoods U and V of x and y, respectively, such that $\overline{U} \cap \overline{V} = \emptyset$. An Uryshon space is also known as a $T_{5/2}$ space.

Proposition 3.14. *Let $(E, \mathcal{T})$ be a topological effect algebra. Then $(E, \mathcal{T})$ is an Uryshon space if and only if for each $x \neq 0$, there exist two open neighborhoods U and V of x and 0, respectively, such that $\overline{U} \cap \overline{V} = \emptyset$.*

Proof. If $(E, \mathcal{T})$ is an Uryshon space, then the proof is clear. Conversely, let for each $x \neq 0$, there exist two open neighborhoods U and V of x and 0, respectively, such that $\overline{U} \cap \overline{V} = \emptyset$. Let $x \neq y$, then $x \ominus y$ or $y \ominus x$ is defined. W.O.L.G, let $x \ominus y$ is defined, $x \ominus y \neq 0$ and U and V be two open neighborhoods of $x \ominus y$ and 0, respectively, such that $\overline{U} \cap \overline{V} = \emptyset$. Since $(E, \mathcal{T})$ is a topological effect algebra, by Theorem 2.4, $\ominus$ is continuous and so there are two open neighborhoods W_1 and W_2 of x and y, respectively, such that $W_1 \ominus W_2 \subseteq U$. We prove that $\overline{W}_1$ and $\overline{W}_2$ are disjoint. Let $z \in \overline{W}_1 \cap \overline{W}_2$. Since the operation $\ominus$ is continuous, we get

$$\ominus(\overline{W_1}, \overline{W_2}) \subseteq \overline{\ominus(W_1, W_2)}$$

Hence, $0 \in \overline{W_1} \ominus \overline{W_2} \subseteq \overline{W_1 \ominus W_2} \subseteq \overline{U}$, which is a contradiction. Therefore, $(E, \mathcal{T}_\Lambda)$ is an Uryshon space. □

Example 3.3. $E = [0, 1]$ be effect algebra as Example 2.1. We claim that $(E, \mathcal{T})$ is an Uryshon space. Let $x = a$ and $a \neq 1$. Then $U = (a - \varepsilon, a + \varepsilon)$ and $V = [0, a - 2\varepsilon)$ are two open neighborhoods of a and 0, respectively, such that $\overline{U} \cap \overline{V} = \emptyset$. Now, let

$x = 1$. Then $W = [0, 1/4)$ and $K = (1/2, 1]$ are two open neighborhoods of 0 and 1, respectively, such that $\overline{W} \cap \overline{K} = \emptyset$. Therefore, by Proposition 3.14, $(E, \mathcal{T})$ is an Uryshon space.

Theorem 3.15. *Let effect algebra E has a proper ideal I. Then:*
(i) $(E, \mathcal{T})$ is a T_3-space if and only if for each $x \in U \in \mathcal{T}$, the set $I \oplus x$ is a closed set,
(ii) $(E, \mathcal{T})$ is a T_4-space if and only if for each closed set S, the set $\bigcup_{x \in S} I \oplus x$ is a closed set.

Proof. (i) Let $(E, \mathcal{T})$ be a T_3-space and $x \in U \in \mathcal{T}$. Since $I \oplus x$ is an open neighborhood of x, by Proposition 2.1(i), there exists an open set H such that $x \in H \subseteq \overline{H} \subseteq I \oplus x$. As $x \in H$ and H is an open set, $I \oplus x$ contained in H. Hence, $I \oplus x = \overline{H}$, which implies that $I \oplus x$ is closed. Conversely, let $x \in U \in \mathcal{T}$. Since $I \oplus x$ is a closed set, $x \in I \oplus x = \overline{I \oplus x} \subseteq U$. By Proposition 2.1$(i)$, $(E, \mathcal{T})$ is a T_3-space.
(ii) Let $(E, \mathcal{T})$ be a T_4- space and S be a closed set in E. Since $\bigcup_{x \in S} I \oplus x$ is an open set contains S, by Proposition 2.1(ii), there exists an open set H such that $S \subseteq H \subseteq \overline{H} \subseteq \bigcup_{x \in S} I \oplus x$. As $S \subseteq H$, and H is an open set, the set $\bigcup_{x \in S} I \oplus x$ is contained in H. Hence, $\bigcup_{x \in S} I \oplus x = \overline{H}$. Conversely, let S be a closed set in E and U be an open set contains S. Then, $\bigcup_{x \in S} I \oplus x$ is a closed and open set such that $S \subseteq \bigcup_{x \in S} I \oplus x = \overline{\bigcup_{x \in S} I \oplus x} \subseteq U$. Therefore, $(E, \mathcal{T})$ is a T_4-space. □

4. Compact Hausdorff effect algebras

In this section, we study the locally compact (para)topological Hausdorff effect algebras. We are going to get some conditions under which a (para)topological effect algebra becomes a T_4-*spaces*. Also we bring some conditions that imply an effect algebra becomes a paratopological effect algebra. Recall that, a topological space X is called a *locally compact space* if for every $x \in X$, there exists a neighbourhood U of the point x such that $\overline{U}$ is a compact subspace of X. Since the compact space $\overline{U}$ is a T_1-*space*, the set $\{x\}$ is closed in $\overline{U}$ and this implies $\{x\}$ is closed in X, i.e., that every locally compact space is a T_1-*space*.

Proposition 4.1. *Let $(E, \mathcal{T})$ be a locally compact paratopological Hausdorff effect algebra and C be a compact subset of open set U. If for any $a \in E$, T_a is an open map, then there exists an open neighborhood V of 0 such that $\overline{C \oplus V}$ is compact and contained in U.*

Proof. Let $U \in \mathcal{T}$ and C be a compact subset of U. Suppose that $x \in C$. Since $\oplus$ is continuous, there are two open neighborhoods V_x and W_x of 0 such that $x \oplus V_x \subseteq U$ and $W_x \oplus W_x \subseteq V_x$. As for any $a \in A$, T_a is an open map, $\{x \oplus W_x, x \in C\}$ is an open cover of C. Since C is compact, this cover have a finite subcover such that

$\{x_i \oplus W_{x_i} : x_i \in C, i = 1, 2, ..., n\}$. Put $W = \bigcap_{i=1}^{n} W_{x_i}$, then we have

$$C \oplus W \subseteq C \oplus W_{x_i} \subseteq (\bigcup_{i=1}^{n}(x_i \oplus W_{x_i})) \oplus W_{x_i} \subseteq \bigcup_{i=1}^{n}(x_i \oplus W_{x_i}) \oplus W_{x_i} \subseteq \bigcup_{i=1}^{n} x_i \oplus V_{x_i} \subseteq U.$$

Since $(E, \mathcal{T})$ is a locally compact space, there exists an open set V by closure compact containing 0 such that $\overline{V} \subseteq W$. Because of continuity $\oplus$, the mapping T_a is continuous. Therefore, $C \oplus \overline{V}$ is a compact subset of E. As $(E, \mathcal{T})$ is a Hausdorff space, $C \oplus \overline{V}$ is a closed subset of E. Then, $\overline{C \oplus V} \subseteq \overline{C \oplus \overline{V}} = C \oplus \overline{V}$. On the other hand

$$C \oplus \overline{V} = \bigcup_{x \in C} T_x(\overline{V}) \subseteq \bigcup_{x \in C} \overline{T_x(V)} = \overline{C \oplus V}.$$

Hence, $\overline{C \oplus V}$ is a compact subset contained in U. □

Theorem 4.2. *Let $(E, \mathcal{T})$ be a locally compact Hausdorff paratopological effect algebra and for any $a \in E$, T_a be an open map. Then, $(E, \mathcal{T})$ is a T_4-space.*

Proof. Let F be a closed set and $V \in \mathcal{T}$ containing F. By Proposition 4.1, there exists an open neighborhood U of 0 such that $F \subseteq U \oplus F \subseteq \overline{U \oplus F} \subseteq V$. By Proposition, 2.1 it is clear that $(E, \mathcal{T})$ is a T_4-space. □

Proposition 4.3. *Let $(E, \mathcal{T})$ be a semitopological effect algebra and locally compact Hausdorff space such that for each $b \in E$, T_b is an open map and the operation $\oplus$ is continuous at $(0, b)$. Then, for any $x, y \in E$, there exist open neighborhoods U and V of x and y, respectively, such that $\overline{U \oplus V}$ is compact.*

Proof. Let $x, y \in E$. Since E is a locally compact Hausdorff space, there exists open neighborhood W of y such that $\overline{W}$ is compact. As by Proposition 2.2(iv), $0 \oplus y = y \in W$, there exist open neighborhoods U_0 and V of 0 and y, respectively, such that $U_0 \oplus V \subseteq W$. Put $U = x \oplus U_0$. Then $x \in U \in \mathcal{T}$ and so by $(E2)$,

$$U \oplus V = (x \oplus U_0) \oplus V = x \oplus (U_0 \oplus V) \subseteq x \oplus W.$$

Since $x \oplus \overline{W}$ is compact, then $\overline{x \oplus W} \subseteq \overline{x \oplus \overline{W}} = x \oplus \overline{W}$. On the other hand

$$x \oplus \overline{W} = T_x(\overline{W}) \subseteq \overline{T_x(W)} = \overline{x \oplus W}.$$

Then $\overline{x \oplus W} = x \oplus \overline{W}$. Therefore, $\overline{U \oplus V}$ is compact. □

Theorem 4.4. *Let E be an effect algebra and $\mathcal{T}$ be a topology on E and*
(i) for any $x, y \in E$, there are two open sets U and V of x and y, respectively, such that $\overline{U \oplus V}$ is compact,
(ii) for any $x, y \in E$ and any open set W of $x \oplus y$, if $z \in E \setminus W$, there exist open sets U and V of x and y, respectively, such that $z \notin \overline{U \oplus V}$.
Then $(E, \mathcal{T})$ is a paratopological effect algebra.

Proof. Let $x, y \in E$ and W be an open neighbourhood of $x \oplus y$. By (i) we can consider open sets U_x and V_y in E such that $x \in U_x, y \in V_y$ and $\overline{U_x \oplus V_y}$ is compact. Let $\mathcal{B}_x$ be the family of all open neighbourhoods of x contained in U_x and $\mathcal{B}_y$ be the family of all open neighbourhoods of y contained in V_y. For subsets $U \in \mathcal{B}_x$ and $V \in \mathcal{B}y$ of E, we put $F_{U,V} = (E \setminus W) \cap \overline{U \oplus V}$. Clearly, $F_{U,V}$ is closed. Now, suppose that $\eta = \{F_{U,V} : U \in \mathcal{B}_x, V \in \mathcal{B}_y\}$. Obviously, each $P \in \eta$ is compact. We show that at least one element of η is empty. Assume that all elements of η are non-empty. Since the elements of η are closed compact sets, by finite intersection property, there is $z \in \cap\eta$. On the other hand, by (ii) there are $U \in \mathcal{B}_x$ and $V \in \mathcal{B}_y$ such that $z \notin \overline{U \oplus V}$. Then $z \notin F_{U,V} \in \eta$, which is a contradiction. Hence, for some $F_{U,V} \in \eta$, $U \oplus V \subseteq W$. □

Lemma 4.5. [1] *Let X and Y be two locally compact Hausdorff spaces, f be a separately continuous mapping of $X \times Y$ to a regular space Z and $(x, y) \in X \times Y$. Let W be an open set of $f(x, y)$ and U be an open set of x. Then there exist non-empty open sets U_1 in X and V in Y such that $U_1 \subseteq U$, $y \in V$ and $f(U_1 \times V) \subseteq W$.*

Theorem 4.6. *Let $(E, \mathcal{T})$ be a locally compact T_3-space. If for any $a \in E$, T_a is an open map and $\oplus$ is continuous at $(0, a)$, then $(E, \mathcal{T})$ is a paratopological effect algebra.*

Proof. First, we prove that $(E, \mathcal{T})$ is a semitopological effect algebra. Let $x \oplus y \in U \in \mathcal{T}$. Since $\oplus$ is continuous at $(0, x \oplus y)$, there is an open neighborhood V of 0 such that $V \oplus x \oplus y \subseteq U$. Assuming $W = V \oplus x$ is an open neighborhood of x and $W \oplus y \subseteq U$. This shows that E is a semitopological effect algebra. Now, we prove that $(E, \mathcal{T})$ satisfies in conditions (i) and (ii) of Theorem 4.4. By Proposition 4.3, $(E, \mathcal{T})$ satisfies condition (i). Let $x \oplus y \in W \in \mathcal{T}$ and $z \in E \setminus W$. Since E is locally compact, we can assume that $z \notin \overline{W}$. As $0 \oplus z = z \in E \setminus \overline{W}$ and $\oplus$ is continuous at $(0, z)$, there is an open neighborhood G of 0 such that $(G \oplus z) \cap \overline{W} = \emptyset$. Since $G \oplus x$ is an open neighborhood of x, Then,by Lemma 4.6, there are two non-empty open sets U_0 and V such that $y \in V$, $U_0 \subseteq G \oplus x$ and $U_0 \oplus V \subseteq W$. Since, $\emptyset \neq U_0 \subseteq G \oplus x$, there exists $g \in G$ such that $g \oplus x \in U_0$. By continuity T_a, there exists $U \in \mathcal{T}$ containing 0 such that $g \oplus U \subseteq U_0$. Hence $(g \oplus U) \oplus V \subseteq U_0 \oplus V \subseteq W$ and so

$$g \oplus (\overline{U \oplus V}) = T_g(\overline{U \oplus V}) \subseteq \overline{g \oplus (U \oplus V)} \subseteq \overline{W}.$$

We claim that $z \notin \overline{U \oplus V}$. Let $z \in \overline{U \oplus V}$, then $g \oplus z \in g \oplus (\overline{U \oplus V})$ implies that $g \oplus z \in \overline{W} \cap (G \oplus z)$, which is a contradiction. Hence (ii) holds. Therefore, $(E, \mathcal{T})$ is a paratopological effect algebra. □

5. Separation axioms on quotient effect algebras

In this section, we first introduce the concept of topological quotient effect algebra by ideals of an effect algebra. Then, we study the relationship between separation axioms and topological quotient effect algebras. We are going to get some conditions under which a quotient topological effect algebra becomes a T_1-*space* or Hausdorff or regular.

Let E be an effect algebra, I be an ideal of E and $E/I = \{[x] : x \in E\}$, where $[x] = \{y : x \sim_I y\}$. If I is Riesz ideal of E, then E/I is an effect algebra which is called a quotient effect algebra and map $\pi_I : E \to E/I$ is defined by $x \mapsto [x]$. Let $(E, \mathcal{T})$ be a topological effect algebra, then

$$\tilde{\mathcal{T}} = \{U \subseteq E/I : \pi_I^{-1}(U) \in \mathcal{T}\}$$

is a topology on E/I. In the other words, subset U of E/I is open if and only if $\pi_I^{-1}(U)$ is an open subset of E (See, [9]).

Proposition 5.1. *Let I be an ideal of effect algebra E. Then for any $F \subseteq E$,*

$$(\pi_I^{-1} \circ \pi_I)(F) = \bigcup_{x \in F} [x]$$

Proof. Let $F \subseteq E$ and $z \in (\pi_I^{-1} \circ \pi_I)(F)$. Then, there exists $x \in F$ such that $\pi_I(z) = \pi_I(x)$. Since $\pi_I(x) = [x]$, thus $\pi_I(z) = [x]$ and this means that $z \in [x]$. Hence $(\pi_I^{-1} \circ \pi_I)(F) \subseteq \bigcup\limits_{x \in F} [x]$.

Conversely, let $z \in \bigcup\limits_{x \in F} [x]$. Then there exists $x_0 \in F$ such that $z \in [x_0]$ and so $\pi_I(z) = \pi_I(x)$. Thus $z \in (\pi_I^{-1} \circ \pi_I)(F)$ and so $(\pi_I^{-1} \circ \pi_I)(F) = \bigcup\limits_{x \in F} [x]$. □

Proposition 5.2. *Let I be a Riesz ideal of effect algebra E. Then for any $x, y \in E$, $[x \oplus y] = [x]\tilde{\oplus}[y]$ and $[x]^{'} = [x^{'}]$.*

Proof. Let I be a Riesz ideal of E and $x, y \in E$. If $z \in [x \oplus y]$, then $z \sim_I x \oplus y$. Since I is a Riesz ideal, there exist $z_1, z_2 \in E$ such that $z = z_1 \oplus z_2$ and $z_1 \sim_I x, z_2 \sim_I y$. Hence $z_1 \in [x], z_2 \in [y]$ and $z_1 \oplus z_2 \in [x]\tilde{\oplus}[y]$. Thus $z \in [x]\tilde{\oplus}[y]$ and so $[x \oplus y] \subseteq [x]\tilde{\oplus}[y]$. Similarly, we have $[x]\tilde{\oplus}[y] \subseteq [x \oplus y]$ and so proof is completed. Now let $a \in [x^{'}]$. Then $a \sim_I x^{'}$ and so $a^{'} \sim_I x$. Hence $a^{'} \in [x]$ and so $a \in [x]^{'}$ and this implies that $[x^{'}] \subseteq [x]^{'}$. By vice versa relations above, we have $[x]^{'} \subseteq [x^{'}]$. Then $[x]^{'} = [x^{'}]$.

□

Definition 5.1. [7] An equivalence relation $\sim$ on E is called *open* if for any open set $U \subseteq E$, $[U] = \{y : y \sim u, u \in U\}$ is open set in E, too.

Lemma 5.3. *Let E be an effect algebra and I be an ideal of E. Then $\sim_I$ is an open relation.*

Proof. Let $U \subseteq E$ be an open set. Then

$[U] = \{y : y \sim_I u; u \in U\} = \{y : \exists i, j \in I; y \ominus i = u \ominus j\} = \{y : y = (u \ominus j) \oplus i\} = (U \ominus j) \oplus i$

Since operations $\ominus$ and $\oplus$ are continuous, $(U \ominus j)$ and $(U \ominus j) \oplus i$ are open and this means that $[U]$ is open in E. Hence $\sim_I$ is an open relation. □

Lemma 5.4. *Let I be a Riesz ideal of effect algebra E and $x, y \in E$. If $[x]_I \leq [y]_I$, then for any $b \in [y]_I$, there exists $a \in [x]_I$ such that $a \leq b$.*

Proof. Let I be a Riesz ideal of E and $[x]_I \leq [y]_I$. Then there exists $[z]_I \in E/I$ such that $[x]_I \tilde{\oplus} [z]_I = [y]_I$ or $[x \oplus z]_I = [y]_I$. Now let $b \in [y]_I$. Then $b \in [x \oplus z]_I$ and this means that $b \sim_I x \oplus z$. By $(C5)$, there exist $b_1, b_2 \in E$ such that $b = b_1 \oplus b_2$ and $b_1 \sim_I x, b_2 \sim_I z$. Hence $b_1 \in [x]_I$ and $b_1 \leq b$. We consider $b_1 = a$ and the proof is completed. □

Lemma 5.5. [7] *Let X be a topological space and $\sim$ is an equivalence relation on X. Then the map $\pi : X \to X/\sim$ is open if and only if for any open set $U \subseteq X$, $[U]$ is open.*

Theorem 5.6. *Let $(E, \mathcal{T})$ be a topological effect algebra, I be a Riesz ideal of E and $x, y \in E$. Then*
(i) If $x \leq y$, then $[x] \leq [y]$, for any $x, y \in E$.
(ii) $(E/I, \tilde{\mathcal{T}})$ is a topological effect algebra.

Proof. (*i*) Let $x \leq y$ for $x, y \in E$. Then there exists, $z \in E$ such that $x \oplus z = y$. Hence $[x \oplus z] = [y]$. Since I is a Riesz ideal, $[x]\tilde{\oplus}[z] = [y]$ and so $[x] \leq [y]$.
(*ii*) Suppose that $\tilde{\oplus} : E/I \times E/I \to E/I$ is defined by $([x], [y]) \mapsto [x]\tilde{\oplus}[y]$ and $\tilde{\prime} : E/I \to E/I$ is defined by $[x] \to [x^{'}]$. It is enough to show $\tilde{\oplus}$ and $\tilde{\prime}$ are continuous. Let U be an open set in E/I. Then $\pi^{-1}(U)$ is open in E, thus there is an open set $V \subseteq E$ such that $V \oplus V \subseteq \pi^{-1}(U)$. Hence $(\pi \times \pi)(V \times V) = \pi(V) \times \pi(V)$ is open in $E/I \times E/I$ and this means that $\tilde{\oplus}(\pi(V) \times \pi(V)) = \pi \circ (\oplus(V \times V)) \subseteq U$. We have shown that there exist an open set $\pi(V) \times \pi(V) \subseteq E/I \times E/I$ such that $\tilde{\oplus}(\pi(V) \times \pi(V)) \subseteq U \subseteq E/I$ and this means that $\tilde{\oplus}$ is continuous. Now, we show that $\tilde{\prime} : E/I \to E/I$ which is defined by $[x] \to [x^{'}]$ is continuous. Let U be an open set in E/I. Then $\pi^{-1}(U)$ is open in E. Hence $(\prime)^{-1}(\pi^{-1}(U))$ is an open set in E and so $\pi((\prime)^{-1}(\pi^{-1}(U)))$ is open in E/I. On the other hand, $(\tilde{\prime})^{-1} = \pi \circ (\prime)^{-1} \circ \pi^{-1}$. Thus $(\tilde{\prime})^{-1}(U)$ is an open relation. Therefore $\tilde{\prime}$, is continuous and so $(E/I, \tilde{\mathcal{T}})$ is a topological effect algebra. □

Recall that, a net in a topological space X is an arbitrary function from a non-empty directed set to the space X. Nets will be denoted by the symbol $S = \{x_\sigma : \sigma \in \Sigma\}$, where x_σ is the point of X assigned to the element σ of the directed set Σ. A point x is called a limit of a net $S = \{x_\sigma : \sigma \in \Sigma\}$ if for every neighbourhood U of x there exists $\sigma_0 \in \Sigma$ such that $x_\sigma \in U$ for every $\sigma \geq \sigma_0$. In this case, we say then the net S is converges to x.(See, [7])

Proposition 5.7. *Let I be a Riesz ideal of effect algebra E. If $\pi_I(0)$ be an open subset of E/I, then π_I is closed.*

Proof. Let $K \subseteq E$ and $[y] \in \overline{\pi_I(K)}$. Then there exists a net $\{x_\sigma\}_{\sigma\in\Sigma}$ such that $\{[x_\sigma] : \sigma \in \Sigma\}$ converges to $[y]$. Since $(E, \mathcal{T})$ is an effect algebra, the net $\{[y \ominus x_\sigma] : \sigma \in \Sigma\}$ or $[x_\sigma \ominus y] : \sigma \in \Sigma\}$ is defined and by Theorem 5.6, it converges to $[0]$. W.O.L.G, suppose $[y \ominus x_\sigma]_{\sigma\in\Sigma}$ converges to $[0]$. Because $\pi_I(0)$ is closed subset of E/I, there exists $\sigma \in \Sigma$ such that $[y \ominus x_\sigma] \in \pi_I(0)$ and $[y \ominus x_\sigma]_I = [0]$. Hence $[x_\sigma] = [y]$ and so $[y] \in \pi_I(K)$. Thus $\overline{\pi_I(K)} \subseteq \pi_I(K) \subseteq \pi_I(\overline{K})$. Therefore, π_I is closed. □

Proposition 5.8. *Let E be a topological effect algebra and I be an ideal of E. Then I is open on E if and only if $(E/I, \tilde{\mathcal{T}})$ is discrete.*

Proof. Let I be an open set on E. Then for any $x \in E, [x]$ is an open set on E. On the other hand for any $U \in \mathcal{T}$, $\pi_I^{-1} \circ \pi_I(U) = \cup_{x\in U}[x]$ is an open set on E/I. Hence for any $x \in E, \pi_I(x) = [x]$ is an open set on E/I and this means that $(E/I, \tilde{\mathcal{T}})$ is discrete. The proof of converse is clear. □

Proposition 5.9. *Let $(E, \mathcal{T})$ be a topological effect algebra and I be an ideal of E. Then I is closed in E if and only if $(E/I, \tilde{\mathcal{T}})$ is T_1-space.*

Proof. Let I be a closed subset of E. Then for any $x \in E$, $\pi_I^{-1} \circ \pi_I(x)$ is a closed subset of I. Hence $\pi_I(x) = \{[x]\}$ is closed in E/I and this means that $(E/I, \tilde{\mathcal{T}})$ is T_1-space. Conversely, let $(E/I, \tilde{\mathcal{T}})$ be a T_1-space. Then $\pi_I^{-1}([I]) = I$ is closed in E. □

Corollary 5.10. *Let $(E, \mathcal{T})$ be an effect algebra and I be an ideal of E. If I is an open neighborhood of 0, then $(E/I, \tilde{\mathcal{T}})$ is a T_1-space.*

Proof. Let $x \in \bar{I}$ and $\{x_\sigma : \sigma \in \Sigma\}$ be a net in I that converges to x. Since $(E, \mathcal{T})$ is a topological effect algebra, hence $\{x \ominus x_\sigma : \sigma \in \Sigma\}$ converges to 0. On the other hand, I is an open neighborhood of 0. Then there exist $\sigma \in \Sigma$ such that $x \ominus x_\sigma \in I$. Since I is an ideal and $x_\sigma \in I$, thus $x \in I$. Therefore, I is closed and by Proposition 5.9, $(E/I, \tilde{\mathcal{T}})$ is a T_1-space. □

Theorem 5.11. *Let $(E, \mathcal{T})$ be a topological effect algebra and I be a Riesz ideal of E. Then $(E/I, \tilde{\mathcal{T}})$ is a Hausdorff topological effect algebra if and only if I is closed in E.*

Proof. Let E/I be Hausdorff. Then $\{I\} \subseteq E/I$ is closed. On the other hand π is continuous and so $I = \pi^{-1}(\{I\})$ is closed. Conversely, let I be closed (See, [16]). Since $\sim_I$ is an open relation, it is enough to prove that $\mathcal{M} = \{(x, y) : x \sim_I y\} \subseteq E \times E$ is closed. Because I is an ideal of E, thus

$$\begin{aligned}\mathcal{M} = \{(x, y) : x \sim_I y\} &= \{(x, y) : x \ominus k, y \ominus k \in I \ for\ some\ k \in E\} \\ &= \{(x, y) : x \in k \oplus I, y \in k \oplus I \ for\ some\ k \in E\} \\ &= \{(k \oplus I, k \oplus I) : k \in E\} \\ &= (k \oplus I) \times (k \oplus I)\end{aligned}$$

Then I, $k \oplus I$ and cartesian product $(k \oplus I) \times (k \oplus I)$ are closed and this means that $\mathcal{M}$ is closed. □

Theorem 5.12. *Let I be an ideal of effect algebra E. If I is an open neighborhood of 0 and $\tilde{\mathcal{T}}$ is closed under finite intersection, then $(E/I, \tilde{\mathcal{T}})$ is regular and T_1-space.*

Proof. By Theorem 5.6(i) and Proposition 5.9, $(E/I, \tilde{\mathcal{T}})$ is a topological effect algebra and a T_1-space. Let S be a closed subset of E/I and $[x]_I \notin S$. If $[y]_I \in S$, then $[x]_I \ominus [y]_I \neq [0]_I$ or $[y]_I \ominus [x]_I \neq [0]_I$. Let $[x]_I \ominus [y]_I \neq [0]_I$. Since $(E/I, \tilde{\mathcal{T}})$ is a T_1-space, there exists $U \in \tilde{\mathcal{T}}$ such that $[0]_I \notin U$ and $[x]_I \ominus [y]_I \in U$. On the other hand, $(E/I, \tilde{\mathcal{T}})$ is a topological space and so $\ominus$ is continuous. Hence there exist V_y and $W_y \in \tilde{\mathcal{T}}$ such that $V_y \ominus W_y \subseteq U$, $[x]_I \in V_y$ and $[y]_I \in W_y$. We claim that $V_y \cap W_y = \emptyset$. If $V_y \cap W_y \neq \emptyset$, there exist $[z]_I \in V_y \cap W_y$, such that

$$[0]_I = [z]_I \ominus [z]_I \in V_y \ominus W_y \subseteq U$$

which is a contradiction. If $V = \bigcap_{[y]_I \in S} V_y$ and $W = \bigcup_{[y]_I \in S} W_y$, then $[x]_I \in V \in \tilde{\mathcal{T}}, S \subseteq W \in \tilde{\mathcal{T}}$ and $V \cap W = \emptyset$. Therefore, $(E/I, \tilde{\mathcal{T}})$ is a *regular* space. □

6. Conclusion

In this paper, separation axioms on topological effect algebras were investigated and the conditions that is a T_i-space, were obtained. Also, these conditions were perused in the case of locally compact (para)topological Hausdorff effect algebra. Finally, quotient topological effect algebras were introduced and existence of separation axioms was expressed and proved. Next researches can study many of other concepts of topology on effect algebras.

References

[1] A. Arhangel'skii, M. Tkachenko, *Topological groups and related structures*, Atlantis press, 2008.

[2] M. Aaly Kologani, N. Kouhestani, R.A. Borzooei, On topological Semi-hoops, *Quasigroups and Related Systems* **25** (2017), 165–179.

[3] R.A. Borzooei, N. Kohestani, G.R. Rezaei Metrizability on (Semi)Topological BL-algebras, *Soft Computing* **16** (2012),no. 10, 1681–1690.

[4] R.A. Borzooei, G.R. Rezaei, N. Kouhestani, On (semi)topological BL-algebra, *Iranian Journal Mathematical Sciences Informatics* **6** (2001), no. 1, 59–77.

[5] N. Bourbaki, *Elements of Mathematics General topology*, Addison-Wesley Publishing Company, 1966.

[6] C.C. Chang, Algebraic analysis of many-valued logics, *Trans. Amer. Math. Soc.* **88** (1958), 467–490.

[7] R. Engelking, *General Topology*, Berlin: Heldermann, 1989.

[8] S. Gudder, *Examples, Problems, and Results in Effect Algebras* **35** (1996), no. 11, 2365–2376.

[9] S. Gudder, S. Pulmannova, Quotients of partial abelian monoids, *Algebra Universalis* **38** (1997), no. 4, 395–421.

[10] P. Hájek, *Metamathematics of Fuzzy Logic*, Kluwer Academic Publishers, 1998.

[11] Y. Imai, K. Iséki, On axiom system of propositional calculi XIV, *Proc. Japan Academy* **42** (1966), 19–22.

[12] D.J. Foulis, M.K. Bennett, Effect algebras and unsharp quantumlogics, *Foundations of Physics* **24** (1994), no. 10, 1331–1352.

[13] G. Jenca, Notes on R_1-Ideals in Generalized Effect Algebras, *Algebra Universalis* **43** (2000), 302–319.

[14] G. Jenca, S. Pulmannova, Ideals and quotients in lattice ordered effect algebras, *Soft Computing* **5** (2001), no. 5, 376-380.

[15] M. Haveshki, E. Eslami, A. Borumand Saeid, A topology induced by uniformity on BL-algebras, *Math. Log. Quart* **53** (2007), no. 2, 162–169.

[16] T. Hosain, *Introduction to topological groups*, W.B.Sunders Company, 1966.

[17] C.S. Hoo, Topological MV -algebras, *Topology Appl.* **81** (1997), 103–121.

[18] N. Kouhestani, R.A. Borzooei, Topological Residuated Lattices, *International Journal of Industrial Mathematics* **9** (2017), no. 3, 183–194.

[19] Z. Ma, Note on ideals of effect algebras, *Information Sciences* **179** (2009), no. 5, 505–507.

[20] J.R. Monkres, *Topology (Second Edition)*, Prentice-Hall, Saddle River NJ, 2000.

[21] M.R. Rakhshani, R.A. Borzooei, G.R. Rezaei, On Topological Effect Algebras, *Italian Journal of pure and Applied Mathematics* **39** (2018), 312–325.

[22] T. Roudabri, L. Torkzadeh, A topology on BCK-algebras via left and right stabilizers, *Iranian Journal of Mathematical Sciences and Informatics* **4** (2009), no. 2, 1–8.

(R.A. Borzooei) Department of Mathematics, Shahid Beheshti University, Tehran, Iran
E-mail address: borzooei@sbu.ac.ir

(M.R. Rakhshani) Department of Mathematics, Zahedan Branch, Islamic Azad University, Zahedan, Iran
E-mail address: rezarakhshani@yahoo.com

(G.R. Rezaei) Department of Mathematics, Sistan and Baluchestan University, Zahedan, Iran
E-mail address: grezaei@math.usb.ac.ir