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Stability analysis for a class of implicit fractional differential equations with instantaneous impulses and Riemann–Liouville boundary conditions

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Abstract. In this article some conditions are established for the existence and uniqueness regarding our proposed model, Implicit fractional differential equation with instantaneous impulses and Riemann–Liouville fractional integral boundary condition in view of Schafer's fixed point theorem. The paper also discusses different types of Ulam's stability, i.e. Ulam–Hyers–stability, generalized Ulam–Hyers–stability, Ulam–Hyers–Rassias stability and generalized Ulam–Hyers–Rassias stability for the proposed model. An example is given to illustrate our main result.

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1. Introduction

Fractional differential equations are the generalization of integer order differential equations. Fractional differential equations can describe many practical phenomena in various fields of applied science and engineering such as economics, medicine, astrophysics, chemical engineering, statistical physics, optics etc. For detail study see the monographs [1, 6, 7, 18, 21]. The aforementioned equations involving the RiemannLiouville and Caputo fractional derivative have been paid more attention. They were studied by many mathematicians from different aspects, using various techniques [2, 3, 8, 9, 19]. On the other hand, the theory of impulsive differential equations arises normally from a wide assortment of utilization for example air ship control, inspection prepare in operations research, drug organization and trash hold hypothesis in biology. The impulsive differential is an adequate apparatus for mathematical simulation of numerous real processes and phenomena studied under the umbrella of theory of optimal control, economics, physics, chemistry, biology, engineering, population growth, medicine such as can be seen in [22]. Due to its significant applications, this theory received great repute and remarkable attention from the researchers.

Wang et. al [23] studied the existence and uniqueness of solutions to a class of nonlocal Cauchy problem of the form

$$\begin{cases} {}^cD^p_{0,t}u(t) = g(t, u(t)), \quad t \in [0, T] \quad t \neq t_m, \\ \Delta u(t) = I_m(u(t), \qquad m = 1, 2, \ldots, k, \\ u(0) = u_0. \end{cases} \tag{1.1}$$

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The notation ${}^cD^p_{0,t}$ is used for Caputo fractional derivative of order $p \in (0,1)$, the function $g : J \times \Re \to \Re$ is continuous and $u_0 \in \Re$.

In many applications such as numerical analysis, mathematical biology, business mathematics, economics etc, we come across the situation where finding the exact solution is quite difficult task. In such a case Ulam–type stability concept is very beneficial and effective. This concept has been introduced in the mid of 19^{th} century and now it is a well explored area for further research [12, 14, 15, 16, 17]. For recent advances in the area we recommend [4, 5, 10, 11, 24, 25, 26].

In this article the existence, uniqueness and Ulam's type stability are investigated for the implicit fractional differential equation with instantaneous impulses and Riemann–Liouville fractional integral boundary conditions having the following form

$$\begin{cases} {}^cD^\beta_{0,t}u(t) = y(t, u(t), {}^cD^\beta u(t)), \quad t \neq t_m \in I, \quad 0 < \beta \leq 1, \\ \Delta u(t_m) = I_m(u(t_m)), \quad m = 1, 2, \dots, q-1, \\ \eta_1 u(0) + \xi_1 I^\beta u(t) \mid_{t=0} = \nu_1, \\ \eta_2 u(T) + \xi_2 I^\beta u(t) \mid_{t=T} = \nu_2. \end{cases} \tag{1.2}$$

where ${}^cD^\beta_{0,t}$ is a generalization of classical Caputo derivative of order β with lower bound at 0, $u : I \times \Re \times \Re \to \Re$ is a continuous function and $I = [0,T], I_0 = [0,t_1], I_1 = (t_1,t_2], I_m = (t_{q-1,t_q}], I_q = (t_q,T]$. Furthermore, $I_m : \Re \to \Re$ is a nonlinear mapping and $u(t^+_m)$ and $u(t^+_m)$ represent the right-sided and left-sided limits respectively at $t = t_m$ for $m = 1, 2, \dots, q-1$.

2. Background materials and preliminaries

This section recalled some preliminaries which are used throughout this paper.

Definition 2.1. [18] The Caputo fractional derivative of order $\beta \in \Re_+$, for a function $\zeta : [0,T] \to \Re$ is defined as

$$({}^cD^\beta_{0,t}\zeta)(t) = \frac{1}{\Gamma(n-\beta)} \int_0^t (t-s)^{n-\beta-1}\zeta^{(n)}(s)ds, \quad n = [\beta]+1,$$

where $[\beta]$ denotes the integer part of the real number β.

Definition 2.2. [18] An arbitrary order fractional integral of a function $\zeta \in L^1([0,T], \Re_+)$ of order $\beta \in \Re_+$ is defined as

$$I^\beta_{0,t}\zeta(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1}\zeta(s)ds, \quad t > 0,$$

where Γ is the Euler gamma function defined by $\Gamma(\beta) = \int_0^\infty p^{\beta-1}e^{-p}dp$.

Lemma 2.3. [18] *For a non-negative value of β, we have*

$$I^\beta_{0,t}[{}^cD^\beta_{0,t}\zeta(t)] = \zeta(t) - \sum_{m=0}^{n-1} \frac{\zeta^m(0)}{m!}t^m, \quad n = [\beta]+1.$$

Lemma 2.4. [18] *For $\beta > 0$, the Caputo fractional differential equation* ${}^cD^{\beta}_{0,t}\zeta(t) = 0$ *has a solution of the following form*

$$\zeta(t) = a_0 + a_1t + a_2t^2 + \cdots + a_{n-1}t^{n-1},$$

where $a_i \in \Re$, $i = 0, 1, \ldots, n-1$ and $n = [\beta] + 1$.

Lemma 2.5. [18] *For $\beta > 0$, we have*

$$I^{\beta}_{0,t}[{}^cD^{\beta}_{0,t}\zeta(t)] = \zeta(t) + a_0 + a_1t + a_2t^2 + \cdots + a_{n-1}t^{n-1},$$

where $a_i \in \Re$, $i = 0, 1, \ldots, n-1$ and $n = [\beta] + 1$.

Let $I = [0, T]$ and $C(I, \Re)$ be the space of all continuous functions from I to $\Re$. Let $\mathbb{B} = PC(I, \Re)$ represents the space of piecewise continuous functions. Obviously $\mathbb{B} = PC(I, \Re)$ is a Banach space with the norm

$$||u||_{PC} = \sup_{t\in I}\{|u(t)|\}.$$

Now, we introducing the concept of Ulam type stabilities for the problem (1.2). Let $\Re^c_- = [0, +\infty)$, $\epsilon > 0$, $\mu \geq 0$ and $\psi \in PC(I, \Re^+)$ be nondecreasing. we focus on the following inequalities:

$$\begin{cases} |{}^cD^{\beta}_{0,t}x(t) - f(t, x(t), {}^cD^{\beta}_{0,t}x(t))| \leq \epsilon, \quad t \neq t_m \in I, \quad m = 0, 1, 2, \ldots, q, \beta \in (0, 1], \\ |\Delta w(t_m) - I_m(w(t_m))| \leq \epsilon, \qquad m = 1, 2, \ldots, q. \end{cases} \tag{2.1}$$

$$\begin{cases} |{}^cD^{\beta}_{0,t}x(t) - f(t, x(t), {}^cD^{\beta}_{0,t}x(t))| \leq \varphi(t), \quad t \neq t_m \in I, \quad m = 0, 1, 2, \ldots, q, \quad \beta \in (0, 1], \\ |\Delta w(t_m) - I_m(w(t_m))| \leq \psi, \qquad m = 1, 2, \ldots, q. \end{cases} \tag{2.2}$$

and

$$\begin{cases} |{}^cD^{\beta}_{0,t}x(t) - f(t, x(t), {}^cD^{\beta}_{0,t}x(t))| \leq \epsilon\varphi(t), \quad t \neq t_m \in I, \quad m = 0, 1, 2, \ldots, q, \quad \beta \in (0, 1], \\ |\Delta w(t_m) - I_m(w(t_m))| \leq \epsilon\psi, \qquad m = 1, 2, \ldots, q. \end{cases} \tag{2.3}$$

Definition 2.6. The problem (1.1) is said to be Hyers-Ulam stble on I if there exists a real number $N_{F,m} > 0$ such that,for every $\epsilon > 0$ and for every solution $y \in PC^n(I, \Re)$ of (2.1), there exists a solution $x_0 \in PC^n(I, \Re)$ of (1.1) with

$$|y(t) - x_0(t)| < N_{F,m}\epsilon, \quad t \in I.$$

Definition 2.7. The problem (1.1) is said to be generalized Hyers-Ulam stble on I if there exists a function $g_{F,m} \in C(\Re^+, \Re^+)$ with $g_{F,m}(0) = 0$ such that,for every $\epsilon > 0$ and for every solution $y \in PC^n(I, \Re)$ of (2.1), there exists a solution $x_0 \in PC^n(I, \Re)$ of (1.1) with

$$|y(t) - x_0(t)| < g_{F,m}(\epsilon), \quad t \in I.$$

Definition 2.8. The problem (1.1) is called Hyers–Ulam–Rassias stble on I with respect to (θ, μ)if there exists $M_{F,m,\theta} > 0$ such that, for every $\epsilon > 0$ and for every solution $y \in PC^n(I, \Re)$ of (2.3), there exists a solution $x_0 \in PC^n(I, \Re)$ of (1.1) with

$$|y(t) - x_0(t)| < M_{F,m,\theta}\epsilon(\theta(t) + \mu), \quad t \in I.$$

Definition 2.9. The problem (1.1) is said to be generalized Hyers–Ulam–Rassias stble on I with respect to (θ, μ) if there exists an $L_{F,m,\theta} > 0$ such that,for every $\epsilon > 0$ and for every solution $y \in PC^n(I, \Re)$ of (2.2), there exists a solution $x_0 \in PC^n(I, \Re)$ of (1.1) with

$$|y(t) - x_0(t)| < L_{F,m,\theta}(\theta(t) + \mu), \quad t \in I.$$

Remark 2.10. A function $w \in \mathbb{B}$ is a solution of the inequality (2.1), if there exists a function $\varpi \in \mathbb{B}$ and w dependent sequence ϖ_m, $m = 1, 2, \ldots, k$, such that

- $|\varpi(t)| \leq \varepsilon, \quad t \in I.$ $|\varpi_m| \leq \varepsilon, \quad m = 1, 2, \ldots, k.$
- ${}^C D^{\beta}_{0,t} w(t) = f(t, w(t), {}^C D^{\beta}_{0,t} w(t)) + \varpi(t), \quad t \in I, m = 0, 1, \ldots, k.$
- $w(t) = I_m(w(t_m)) + \varpi_m, \qquad t \in I, m = 1, 2, \ldots, k.$

Remark 2.11. A function $w \in \mathbb{B}$ is a solution of the inequality (2.2), if there exists a function $\varpi \in \mathbb{B}$ and a sequence ϖ_m, $m = 1, 2, \ldots, k$ which depends on w, such that

- $|\varpi(t)| \leq \phi(t), \quad t \in I.$ $|\varpi_m| \leq \psi, \quad m = 1, 2, \ldots, k.$
- ${}^C D^{\beta}_{0,t} w(t) = f(t, w(t), {}^C D^{\beta}_{0,t} w(t)) + \varpi(t), \quad t \in I, m = 0, 1, \ldots, k.$
- $\Delta w(t) = I_m(w(t_m)) + \varpi_m, \qquad t \in I, m = 1, 2, \ldots, k.$

Remark 2.12. A function $w \in \mathbb{B}$ is a solution of the inequality (2.3), if there exists a function $\varpi \in \mathbb{B}$ and a sequence ϖ_m, $m = 1, 2, \ldots, k$ which depends on w, such that

- $|\varpi(t)| \leq \epsilon\phi(t), \quad t \in I.$ $|\varpi_m| \leq \epsilon\psi, \quad m = 1, 2, \ldots, k.$
- ${}^C D^{\beta}_{0,t} w(t) = f(t, w(t), {}^C D^{\beta}_{0,t} w(t)) + \varpi(t), \quad t \in I, m = 0, 1, \ldots, k.$
- $\Delta w(t) = I_m(w(t_m)) + \varpi_m, \qquad t \in I, m = 1, 2, \ldots, k.$

Theorem 2.13. [13] ***(Banach fixed point theorem)*** *Let C be a non–empty closed subset of a Banach space $\mathbb{B}$. Then any contraction mapping $\daleth : \mathbb{B} \to \mathbb{B}$ has a unique fixed point.*

Theorem 2.14. [13] ***(Schaefer's fixed point theorem)*** *Let $\mathbb{B}$ be a Banach space and $\daleth : \mathbb{B} \to \mathbb{B}$ is a completely continuous operator and the set $D = \{u \in \mathbb{B} : u = \mu\daleth u, 0 < \mu < 1\}$ is bounded.Then $\daleth$ has a fixed point in $\mathbb{B}$.*

Theorem 2.15. [20] ***(Arzelà–Ascoli's theorem)*** *Let $H \in C(I, \Re)$, H is relatively compact if the following two conditions satisfy*

- *H is uniformly bounded that is there exists $\varepsilon > 0$, such that $|f(\upsilon)| < \varepsilon \qquad$ for each $g \in H$ and $\upsilon \in I$.*
- *H is equicontinuous, that is for every $\epsilon > 0$, there exists $\delta > 0$ such that for any $w, v \in I$, $|v - v'| \leq \delta$ implies $|g(v) - g(v')| \leq \epsilon$, for each $g \in H$.*

3. Main Results

In this section, we investigate the existence and uniqueness of solution to the proposed class of impulsive integral boundary value problem of implicit fractional differential equations.

Lemma 3.1. *Let $0 < \beta \leq 1$ and assume that $y : I \to \Re$ be a continuous function. Then a function $u \in \mathbb{B}$ is a solution of the following fractional integral boundary value* *problem of impulsive differential equations.*

$$\begin{cases} {}^cD^{\beta}_{0,t}u(t) = y(t), \quad t \neq t_m \in I, \\ \Delta u(t_m) = I_m(u(t_m)), \quad m = 1, 2, \dots, q-1, \\ \eta_1 u(0) + \xi_1 I^{\beta} u(t) \mid_{t=0} = \nu_1, \\ \eta_2 u(T) + \xi_2 I^{\beta} u(t) \mid_{t=T} = \nu_2, \end{cases} \tag{3.1}$$

if and only if $u \in \mathbb{B}$ is the solution of the fractional differential equation given by

$$u(t) = \begin{cases} \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} y(s)ds + \frac{\nu_1}{\eta_1}, \quad t \in [0, t_1] \\ \frac{1}{\Gamma(\beta)} \int_{t_m}^t (t-s)^{\beta-1} y(s)ds + \sum_{i=1}^m \left[\frac{1}{\Gamma(\beta)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\beta-1} y(s)ds + I_i(u(t_i)) \right] \\ +\frac{\nu_1}{\eta_1}, \quad t \in (t_m, t_{m+1}] \\ \frac{1}{\Gamma(\beta)} \int_{t_q}^t (t-s)^{\beta-1} y(s)ds + \frac{1}{\Gamma(\beta)} \int_{t_m}^{t_q} (t_q - s)^{\beta-1} y(s)ds \\ + \sum_{i=1}^m \left(\frac{1}{\Gamma(\beta)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\beta-1} y(s)ds + I_i(u(t_i)) \right) + I_q(u(t_q)) \\ +\frac{\nu_1}{\eta_1} + t \Bigg[\frac{\nu_2}{\eta_2 T} - \frac{\xi_2}{\eta_2 T} \int_0^T \frac{(T-s)^{\beta-1}}{\Gamma(\beta)} u(s)ds \\ +\frac{1}{T} \Bigg\{ \frac{1}{\Gamma(\beta)} \int_{t_q}^T (T-s)^{\beta-1} y(s)ds + \frac{1}{\Gamma(\beta)} \int_{t_m}^{t_q} (t_q - s)^{\beta-1} y(s)ds \\ + \sum_{i=1}^m \left(\frac{1}{\Gamma(\beta)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\beta-1} y(s)ds + I_i(u(t_i)) \right) + I_q(u(t_q)) + \frac{\nu_1}{\eta_1} \Bigg\} \Bigg], \\ \quad t \in (t_q, T]. \end{cases} \tag{3.2}$$

Proof. For $t \in [0, t_1]$, we consider

$${}^cD^{\beta}_{0,t}u(t) = y(t),$$

with $\eta_1 u(0) + \xi_1 I^{\beta} u(t) \mid_{t=0} = \nu_1$. By using Lemma 2.5, we get

$$u(t) = I^{\beta} y(t) + c_0.$$

$$u(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} y(s)ds + c_0. \tag{3.3}$$

Using the initial condition we have

$$c_0 = \frac{\nu_1}{\eta_1}.$$

Thus Eq. (3.3) becomes

$$u(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} y(s)ds + \frac{\nu_1}{\eta_1}. \tag{3.4}$$

For $t \in (t_1, t_2]$, we consider

$${}^cD^{\beta}_{0,t}u(t) = y(t),$$

with $\Delta u(t_1) = I_1(u(t_1))$. By Lemma 2.5 we obtain

$$u(t) = I^{\beta} y(t) + d_1.$$

$$u(t) = \frac{1}{\Gamma(\beta)} \int_{t_1}^t (t-s)^{\beta-1} y(s)ds + d_1. \tag{3.5}$$

Since we know that

$$\Delta u(t_1) = u(t_1^+) - u(t_1^-) = I_1(u(t_1)).$$
$$u(t_1^-) = \frac{1}{\Gamma(\beta)} \int_0^{t_1} (t_1 - s)^{\beta-1} y(s)ds + \frac{\nu_1}{\eta_1}.$$
$$d_1 = u(t_1^+) = u(t_1^-) + I_1(u(t_1)).$$
$$d_1 = \frac{1}{\Gamma(\beta)} \int_0^{t_1} (t_1 - s)^{\beta-1} y(s)ds + \frac{\nu_1}{\eta_1} + I_1(u(t_1)),$$

now putting in Eq. (3.5), we get

$$u(t) = \frac{1}{\Gamma(\beta)} \int_{t_1}^{t} (t-s)^{\beta-1} y(s)ds + \frac{1}{\Gamma(\beta)} \int_0^{t_1} (t_1 - s)^{\beta-1} y(s)ds + I_1(u(t_1)) + \frac{\nu_1}{\eta_1}. \quad (3.6)$$

On the similar procedure for $t \in (t_m, t_{m+1}]$, where $m = 1, 2, \ldots, q-1$, we get

$$u(t) = \frac{1}{\Gamma(\beta)} \int_{t_m}^{t} (t-s)^{\beta-1} y(s)ds + \sum_{i=1}^{m} \left[\frac{1}{\Gamma(\beta)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\beta-1} y(s)ds + I_i(u(t_i)) \right] + \frac{\nu_1}{\eta_1}. \quad (3.7)$$

Finally for $t \in (t_q, T]$, we consider

$${}^cD_{0,t}^{\beta} u(t) = y(t),$$

with impulse $\Delta u(t_q) = I_q(u(t_q))$, and boundary condition $\eta_2 u(T) + \xi_2 I^{\beta} u(t) \mid_{t=T} = \nu_2$. Now using Lemma 2.5, we have

$$u(t) = I^{\beta} y(t) + d_q + ct.$$

$$u(t) = \frac{1}{\Gamma(\beta)} \int_{t_q}^{t} (t-s)^{\beta-1} y(s)ds + d_q + ct. \quad (3.8)$$

Since

$$\Delta u(t_q) = u(t_q^+) - u(t_q^-).$$

Thus $u(t_q^-)$ can be calculated from Eq. (3.7) by putting $t = t_q$ i. e.

$$u(t_q^-) = \frac{1}{\Gamma(\beta)} \int_{t_m}^{t_q} (t_q - s)^{\beta-1} y(s)ds + \sum_{i=1}^{m} \left[\frac{1}{\Gamma(\beta)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\beta-1} y(s)ds + I_i(u(t_i)) \right] + \frac{\nu_1}{\eta_1}. \quad (3.9)$$

$$\begin{aligned} d_q = u(u(t_q^+)) = & \frac{1}{\Gamma(\beta)} \int_{t_m}^{t_q} (t_q - s)^{\beta-1} y(s)ds \\ & + \sum_{i=1}^{m} \left[\frac{1}{\Gamma(\beta)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\beta-1} y(s)ds + I_i(u(t_i)) \right] + \frac{\nu_1}{\eta_1} + I_q(u(t_q)). \quad (3.10) \end{aligned}$$

Eq. (3.8) implies

$$\begin{aligned} u(t) = & \frac{1}{\Gamma(\beta)} \int_{t_q}^{t} (t-s)^{\beta-1} y(s)ds + \frac{1}{\Gamma(\beta)} \int_{t_m}^{t_q} (t_q - s)^{\beta-1} y(s)ds \\ & + \sum_{i=1}^{m} \left[\frac{1}{\Gamma(\beta)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\beta-1} y(s)ds + I_i(u(t_i)) \right] + \frac{\nu_1}{\eta_1} + I_q(u(t_q)) + ct. \quad (3.11) \end{aligned}$$

Using condition $\eta_2 u(T) + \xi_2 I^\beta u(t) \mid_{t=T} = \nu_2$, for finding the value of c, we obtained

$$\begin{aligned} u(T) &= \frac{1}{\Gamma(\beta)} \int_{t_q}^{T} (T-s)^{\beta-1} y(s) ds + \frac{1}{\Gamma(\beta)} \int_{t_m}^{t_q} (t_q - s)^{\beta-1} y(s) ds \\ &+ \sum_{i=1}^{m} \left[\frac{1}{\Gamma(\beta)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\beta-1} y(s) ds + I_i(u(t_i)) \right] + \frac{\nu_1}{\eta_1} + I_q(u(t_q)) + cT. \end{aligned}$$

Multiplying η_2 and adding $\xi_2 I^\beta \mid_{t=T}$ to both sides, we get

$$\begin{aligned} c &= \frac{\nu_2}{\eta_2 T} - \frac{\xi_2}{\eta_2 T} \int_0^T \frac{(T-s)^{\beta-1}}{\Gamma(\beta)} u(s) ds + \frac{1}{T} \left[\frac{1}{\Gamma(\beta)} \int_{t_q}^{T} (T-s)^{\beta-1} y(s) ds \right. \\ &+ \frac{1}{\Gamma(\beta)} \int_{t_m}^{t_q} (t_q - s)^{\beta-1} y(s) ds \\ &\left. + \sum_{i=1}^{m} \left(\frac{1}{\Gamma(\beta)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\beta-1} y(s) ds + I_i(u(t_i)) \right) + I_q(u(t_q)) + \frac{\nu_1}{\eta_1} \right]. \end{aligned}$$

Now putting the value of c in Eq. (3.11), we have

$$\begin{aligned} u(t) &= \frac{1}{\Gamma(\beta)} \int_{t_q}^{t} (t-s)^{\beta-1} y(s) ds + \frac{1}{\Gamma(\beta)} \int_{t_m}^{t_q} (t_q - s)^{\beta-1} y(s) ds \\ &+ \sum_{i=1}^{m} \left(\frac{1}{\Gamma(\beta)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\beta-1} y(s) ds + I_i(u(t_i)) \right) + I_q(u(t_q)) + \frac{\nu_1}{\eta_1} \\ &+ t \left[\frac{\nu_2}{\eta_2 T} - \frac{\xi_2}{\eta_2 T} \int_0^T \frac{(T-s)^{\beta-1}}{\Gamma(\beta)} u(s) ds \right. \\ &+ \frac{1}{T} \left(\frac{1}{\Gamma(\beta)} \int_{t_q}^{T} (T-s)^{\beta-1} y(s) ds + \frac{1}{\Gamma(\beta)} \int_{t_m}^{t_q} (t_q - s)^{\beta-1} y(s) ds \right. \\ &\left. \left. + \sum_{i=1}^{m} \left(\frac{1}{\Gamma(\beta)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\beta-1} y(s) ds + I_i(u(t_i)) \right) + I_q(u(t_q)) + \frac{\nu_1}{\eta_1} \right) \right]. \end{aligned}$$

Conversely, assume that u is a solution of the integral Eq. (3.1), then we can easily verify that the solution $u(t)$ given by Eq. (3.2) satisfies problem (3.1) along with its impulsive and integral boundary conditions. □

For obtaining our results, we consider the following assumptions

(H_1) The function $g : I \times \Re \times \Re \to \Re$ is continuous;

(H_2) There exist constants $K_g > 0$ and $0 < L_g < 1$ such that

$$|g(t, u_1, w_1) - g(t, u_2, w_2)| \le K_g |u_1 - u_2| + L_g |w_1 - w_2|,$$

for $t \in I$, and $u_1, u_2, w_1, w_2 \in \Re$;

(H_3) There exists a constant $L_I > 0$ such that

$$|I_m(u_1) - I_m(u_2)| \le L_I |u_1 - u_2|,$$

for each $u_1, u_2 \in \Re$, $\quad t \in I_m$ and $m = 1, 2, \dots, q-1$;

(H_4) There exist constants $\alpha, \beta, \gamma \in C(I, \Re_+)$, such that

$$|f(t, u(t), w(t))| \le \alpha(t) + \beta(t)|u| + \gamma(t)|w|,$$

for $t \in I, u, w \in \Re$, with $\alpha^* = \sup_{t\in I} \alpha(t)$, $\beta^* = \sup_{t\in I} \beta(t)$, $\gamma^* = \sup_{t\in I} \gamma(t) < 1$;
(H_5) The functions $I_m : \Re \to \Re$ are continuous and there exist constants $N, N^* > 0$, such that

$$|I_m(u)| \leq N|u(t)| + N^*,$$

for each $u \in \Re, \quad m = 1, 2, \ldots, q-1$.
(H_6) Suppose a function $\varphi \in PC(I, \Re_+)$, which is increasing. Then there exists $\lambda_\varphi > 0$, such that the following inequality holds

$$I^p\varphi(t) \leq \lambda_\varphi \varphi(t),$$

for each $t \in I$.

Theorem 3.2. *Let the assumptions $(H_1) - (H_3)$ are satisfied and if*

$$\begin{aligned}||\daleth w - \daleth u||_{PC} \leq \Bigg[&\frac{2K_gT^\beta}{(1-L_g)\Gamma(\beta+1)} + m(\frac{K_gT^\beta}{(1-L_g)\Gamma(\beta+1)} + L_I) + L_I \\ &+ t\Bigg[\frac{1}{T}\Bigg\{\frac{2K_gT^\beta}{(1-L_g)\Gamma(\beta+1)} + m(\frac{K_gT^\beta}{(1-L_g)\Gamma(\beta+1)} + L_I) + L_I\Bigg\}\Bigg]\Bigg] < 1,\end{aligned} \tag{3.12}$$

then the problem (1.2) has a unique solution in $\mathbb{B}$.

Proof. For this we define a mapping $\daleth : \mathbb{B} \to \mathbb{B}$ by

$$(\daleth(u))(t) = \begin{cases} \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} f(s, u(s), g(s))ds + \frac{\nu_1}{\eta_1}, \quad t \in [0, t_1]; \\ \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} f(s, u(s), g(s))ds \\ +\sum_{i=1}^m \left(\frac{1}{\Gamma(\beta)}\int_{t_{i-1}}^{t_i} (t_i-s)^{\beta-1} f(s, u(s), g(s))ds + I_i(u(t_i))\right) \\ +\frac{\nu_1}{\eta_1}, \quad t \in (t_m, t_{m+1}]; \\ \frac{1}{\Gamma(\beta)}\int_{t_q}^t (t-s)^{\beta-1} f(s, u(s), g(s))ds + \frac{1}{\Gamma(\beta)}\int_{t_m}^{t_q} (t_q-s)^{\beta-1} f(s, u(s), g(s))ds \\ +\sum_{i=1}^m \left(\frac{1}{\Gamma(\beta)}\int_{t_{i-1}}^{t_i} (t_i-s)^{\beta-1} f(s, u(s), g(s))ds + I_i(u(t_i))\right) \\ +I_q(u(t_q)) + \frac{\nu_1}{\eta_1} + t\Bigg[\frac{\nu_2}{\eta_2 T} - \frac{\xi_2}{\eta_2 T} \\ \int_0^T \frac{(T-s)^{\beta-1}}{\Gamma(\beta)} u(s)ds + \frac{1}{T}\Bigg\{\frac{1}{\Gamma(\beta)}\int_{t_q}^T (T-s)^{\beta-1} f(s, u(s), g(s))ds \\ +\frac{1}{\Gamma(\beta)}\int_{t_m}^{t_q} (t_q-s)^{\beta-1} f(s, u(s), g(s))ds \\ +\sum_{i=1}^m \left(\frac{1}{\Gamma(\beta)}\int_{t_{i-1}}^{t_i} (t_i-s)^{\beta-1} f(s, u(s), g(s))ds + I_i(u(t_i))\right) \\ +I_q(u(t_q)) + \frac{\nu_1}{\eta_1}\Bigg\}, \quad t \in (t_q, T]. \end{cases}$$

For $t \in I_q = (t_q, T]$, we consider

$$\begin{aligned}\Big|(\daleth w)(t) - (\daleth u)(t)\Big| \leq &\frac{1}{\Gamma(\beta)}\int_{t_q}^t (t-s)^{\beta-1}\Big|x(s) - g(s)\Big|ds \\ &+ \frac{1}{\Gamma(\beta)}\int_{t_m}^{t_q} (t_q-s)^{\beta-1}\Big|x(s) - g(s)\Big|ds + \sum_{i=1}^m \frac{1}{\Gamma(\beta)}\int_{t_{i-1}}^{t_i} (t_i-s)^{\beta-1}\Big|x(s) - g(s)\Big|\end{aligned}$$

$$+\sum_{i=1}^{m}|I_i(w(t_i))-I_i(u(t_i))|+\Big|I_q(w(t_q))-I_q(u(t_q))\Big|$$
$$+t\Big[\frac{1}{T}\Big\{\frac{1}{\Gamma(\beta)}\int_{t_q}^{T}(T-s)^{\beta-1}\Big|x(s)-g(s)\Big|ds+\frac{1}{\Gamma(\beta)}\int_{t_m}^{t_q}(t_q-s)^{\beta-1}\Big|x(s)-g(s)\Big|ds$$
$$+\sum_{i=1}^{m}\frac{1}{\Gamma(\beta)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\beta-1}\Big|x(s)-g(s)\Big|+\sum_{i=1}^{m}\Big|I_i(w(t_i))-I_i(u(t_i))\Big|$$
$$+\Big|I_q(w(t_q))-I_q(u(t_q))\Big|\Big\}\Big],$$

where

$$x(s)=f(s,w(s),x(s)) \text{ and } g(s)=f(s,u(s),g(s)).$$

Then by (H_2), we have

$$\begin{aligned}\Big|x(s)-g(s)\Big| &= \Big|f(s,w(s),x(s))-f(s,u(s),g(s)\Big|\\ &\le K_g\Big|w(s)-u(s)\Big|+L_g\Big|x(s)-g(s)\Big|.\end{aligned}$$

Thus

$$\Big|x(s)-g(s)\Big|\le\frac{K_g}{1-L_g}\Big|w(s)-u(s)\Big|. \tag{3.13}$$

Now using the inequality (3.13) and assumption (H_3), we have

$$\begin{aligned}\Big|(\daleth w)(t)-(\daleth u)(t)\Big| &\le \frac{K_g}{1-L_g}\int_{t_q}^{t}\frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\Big|w(s)-u(s)\Big|ds\\
&+\frac{K_g}{1-L_g}\int_{t_m}^{t_q}\frac{(t_q-s)^{\beta-1}}{\Gamma(\beta)}\Big|w(s)-u(s)\Big|ds\\
&+\frac{K_g}{1-L_g}\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\beta-1}}{\Gamma(\beta)}\Big|w(s)-u(s)\Big|ds+L_I\sum_{i=1}^{m}\Big|w(s)-u(s)\Big|\\
&+L_I\Big|w(s)-u(s)\Big|+t\Big[\frac{1}{T}\Big(\frac{K_g}{1-L_g}\int_{t_q}^{T}\frac{(T-s)^{\beta-1}}{\Gamma(\beta)}\Big|w(s)-u(s)\Big|ds\\
&+\frac{K_g}{1-L_g}\int_{t_m}^{t_q}\frac{(t_q-s)^{\beta-1}}{\Gamma(\beta)}\Big|w(s)-u(s)\Big|ds+L_I\sum_{i=1}^{m}\Big|w(s)-u(s)\Big|\\
&+\frac{K_g}{1-L_g}\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\beta-1}}{\Gamma(\beta)}\Big|w(s)-u(s)\Big|ds+L_I\Big|(w(s)-u(s)\Big|\Big)\Big].\end{aligned} \tag{3.14}$$

Which implies that

$$\Big\|\daleth w-\daleth u\Big\|_{PC}\leq\Big[\frac{K_gT^\beta}{(1-L_g)\Gamma(\beta+1)}+\frac{K_gT^\beta}{(1-L_g)\Gamma(\beta+1)}+\frac{mK_gT^\beta}{(1-L_g)\Gamma(\beta+1)}+mL_I$$
$$+L_I+t\Big[\frac{1}{T}\Big(\frac{K_gT^\beta}{(1-L_g)\Gamma(\beta+1)}+\frac{K_gT^\beta}{(1-L_g)\Gamma(\beta+1)}$$
$$+\frac{mK_gT^\beta}{(1-L_g)\Gamma(\beta+1)}+mL_I+L_I\Big)\Big]\Big]\|w-u\|_{PC}. \tag{3.15}$$

Similarly for $t\in I_m$, we obtain

$$\Big\|\daleth w-\daleth u\Big\|_{PC}\leq\Big[\frac{K_gT^\beta}{(1-L_g)\Gamma(\beta+1)}+\frac{mK_gT^\beta}{(1-L_g)\Gamma(\beta+1)}+mL_I\Big]\Big\|w-u\Big\|_{PC}. \tag{3.16}$$

By the same procedure for $t\in I_0$, we have the following result

$$\Big\|\daleth w-\daleth u\Big\|_{PC}\leq\frac{K_gT^\beta}{(1-L_g)\Gamma(\beta+1)}\Big\|w-u\Big\|_{PC}. \tag{3.17}$$

As $I=I_0\cup I_m\cup I_q$, Thus combining Eq. (3.14), Eq. (3.15) and Eq. (3.16), we have

$$\Big\|\daleth w-\daleth u\Big\|_{PC}\leq\Big[\frac{K_gT^\beta}{(1-L_g)\Gamma(\beta+1)}+\frac{K_gT^\beta}{(1-L_g)\Gamma(\beta+1)}+\frac{mK_gT^\beta}{(1-L_g)\Gamma(\beta+1)}$$
$$+mL_I+L_I+t\Big[\frac{1}{T}\Big\{\frac{K_gT^\beta}{(1-L_g)\Gamma(\beta+1)}+\frac{K_gT^\beta}{(1-L_g)\Gamma(\beta+1)}$$
$$+\frac{mK_gT^\beta}{(1-L_g)\Gamma(\beta+1)}+mL_I+L_I\Big\}\Big]\Big]\Big\|w-u\Big\|_{PC}.$$

Now since

$$\Big[\quad\frac{2K_gT^\beta}{(1-L_g)\Gamma(\beta+1)}+m(\frac{K_gT^\beta}{(1-L_g)\Gamma(\beta+1)}+L_I)+L_I$$
$$+t\Big\{\frac{1}{T}\Big(\frac{2K_gT^\beta}{(1-L_g)\Gamma(\beta+1)}+m(\frac{K_gT^\beta}{(1-L_g)\Gamma(\beta+1)}+L_I)+L_I\Big)\Big\}\Big]<1,$$

hence by Banach contraction theorem $\daleth$ is a contraction operator and thus it has a unique fixed point, which is the corresponding unique solution of problem (1.2). This completes the proof. □

Theorem 3.3. *If the assumptions* (H_1)*–*(H_5) *are satisfied and if*

$$mN+\frac{\beta^*T^\beta(m+1)}{(1-\gamma^*)\Gamma(\beta+1)}<1,$$

then the problem has at least one solution.

Proof. Consider the operator $\daleth$ defined in Theorem 3.2. We use Schaefers fixed point theorem to prove our required result.

Step 1: First we prove that $\daleth$ is continuous. For this take a sequence $\{u_n\}\in\mathbb{B}$, such

that $u_n \to u \in \mathbb{B}$.
For $t \in I_q$, we have

$$\begin{aligned}\Big|(\urcorner u_n)(t) - (\urcorner u)(t)\Big| &\le \frac{1}{\Gamma(\beta)}\int_{t_q}^{t}(t-s)^{\beta-1}\Big|y_n(s)-y(n)\Big|ds\\ &+ \frac{1}{\Gamma(\beta)}\int_{t_m}^{t_q}(t_q-s)^{\beta-1}\Big|y_n(s)-y(s)\Big|ds\\ &+ \sum_{i=1}^{m}\frac{1}{\Gamma(\beta)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\beta-1}\Big|y_n(s)-y(s)\Big| + \sum_{i=1}^{m}\Big|I_i(u_n(t_i))-I_i(u(t_i))\Big|\\ &+ \Big|I_q(u_n(t_q))-I_q(u(t_q))\Big| + t\Bigg[\frac{1}{T}\Bigg(\frac{1}{\Gamma(\beta)}\int_{t_q}^{T}(T-s)^{\beta-1}\Big|y_n(s)-y(s)\Big|ds\\ &+ \frac{1}{\Gamma(\beta)}\int_{t_m}^{t_q}(t_q-s)^{\beta-1}\Big|y_n(s)-y(s)\Big|ds + \sum_{i=1}^{m}\frac{1}{\Gamma(\beta)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\beta-1}\Big|y_n(s)-y(s)\Big|\\ &+ \sum_{i=1}^{m}\Big|I_i(u_n(t_i))-I_i(u(t_i))\Big| + \Big|I_q(u_n(t_q))-I_q(u(t_q))\Big|\Bigg)\Bigg],\end{aligned} \tag{3.18}$$

where $y_n, y \in C(I, \Re)$ and given as

$$y_n(s) = f(s, u_n(s), y_n(s)), \quad y(s) = f(s, u(s), y(s)).$$

So by (H_2), we have

$$\begin{aligned}\Big|y_n(s)-y(s)\Big| &= \Big|f(s,u_n(s),y_n(s)) - f(s,u(s),y(s)\Big|\\ &\le K_g\Big|u_n(s)-u(s)\Big| + L_g\Big|y_n(s)-y(s)\Big|.\end{aligned}$$

Thus

$$\Big|y_n(s)-y(s)\Big| \le \frac{K_g}{1-L_g}\Big\|u_n(s)-u(s)\Big\|_{PC}. \tag{3.19}$$

Now since $u_n \to u$ as $n \to \infty$, which implies that $y_n(s) \to y(s)$ as $n \to \infty$ for each $s \in I_q$. As a consequence of Lebesgue dominated convergence theorem, the right hand side of inequality (3.18) tends to zero as $n \to \infty$, hence

$$\Big|(\urcorner u_n)(t) - (\urcorner u)(t)\Big| \to 0, \quad as \quad n \to \infty.$$

Which implies that

$$\Big\|(\urcorner u_n) - (\urcorner u)\Big\|_{PC} \to 0, \quad as \quad n \to \infty.$$

For $t \in I_m$, we have

$$\Big\|(\urcorner u_n) - (\urcorner u)\Big\|_{PC} \to 0, \quad as \quad n \to \infty.$$

And similarly for $t \in I_0$, we obtain

$$\Big\|(\urcorner u_n) - (\urcorner u)\Big\|_{PC} \to 0, \quad as \quad n \to \infty.$$

Thus $\urcorner$ is continuous.
Step 2: Now to prove that $\urcorner$ maps bounded sets into bounded sets in $\mathbb{B}$. In fact we just need to show that for any positive constant μ, there exists a constant $\upsilon > 0$, such that for each $u \in B_\mu = \{u \in \mathbb{B} : ||u||_{PC} \leq \mu\}$, we have $||\urcorner(u)||_{PC} \leq \upsilon$. For $t \in I_q$, we get

$$\begin{aligned}(\urcorner u)(t) &= \frac{1}{\Gamma(\beta)}\int_{t_q}^{t}(t-s)^{\beta-1}y(s)ds + \frac{1}{\Gamma(\beta)}\int_{t_m}^{t_q}(t_q-s)^{\beta-1}y(s)ds \\ &+ \sum_{i=1}^{m}\left(\frac{1}{\Gamma(\beta)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\beta-1}y(s)ds + I_i(u(t_i))\right) + I_q(u(t_q)) + \frac{\nu_1}{\eta_1} \\ &+ t\Bigg[\frac{\nu_2}{\eta_2 T} - \frac{\xi_2}{\eta_2 T}\int_0^T \frac{(T-s)^{\beta-1}}{\Gamma(\beta)}u(s)ds \\ &+ \frac{1}{T}\Bigg\{\frac{1}{\Gamma(\beta)}\int_{t_q}^{T}(T-s)^{\beta-1}y(s)ds + \frac{1}{\Gamma(\beta)}\int_{t_m}^{t_q}(t_q-s)^{\beta-1}y(s)ds \\ &+ \sum_{i=1}^{m}\left(\frac{1}{\Gamma(\beta)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\beta-1}y(s)ds + I_i(u(t_i))\right) + I_q(u(t_q)) + \frac{\nu_1}{\eta_1}\Bigg\}\Bigg],\end{aligned} \tag{3.20}$$

where $y \in C(I, \Re)$, is given by

$$y(s) = f(s, u(s), y(s)).$$

By (H_4) for $t \in I_q$, we can write

$$\begin{aligned}|y(s)| &= |f(s, u(s), y(s))| \\ &\leq \alpha(s) + \beta(s)|u| + \gamma(s)|y| \\ &\leq \alpha(s) + \beta(s)\mu + \gamma(s)|y(s)|\end{aligned}$$

So

$$|y(s)| \leq \alpha^* + \beta^*\mu + \gamma^*|y(s)|, \tag{3.21}$$

where $\alpha^* = \sup_{t\in I}\alpha(t)$, $\beta^* = \sup_{t\in I}\beta(t)$, $\gamma^* = \sup_{t\in I}\gamma(t) < 1$ and from Eq (3.20), we get

$$|y(s)| \leq \frac{\alpha^* + \beta^*\mu}{1-\gamma^*} = M.$$

Thus by (H_4) and (H_5), Eq. (3.20) becomes

$$\begin{aligned}\Big|(\urcorner u)(t)\Big| \leq\; & \frac{MT^\beta}{\Gamma(\beta+1)} + \frac{MT^\beta}{\Gamma(\beta+1)} + \frac{qMT^\beta}{\Gamma(\beta+1)} + q(\mu N + N^*) + (\mu N + N^*) + \frac{\nu_1}{\eta_1} \\ & + t\Bigg[\frac{\nu_2}{\eta_t} - \frac{\xi_2 T^\beta}{\eta_2 T\Gamma(\beta+1)} + \frac{1}{T}\Bigg\{\frac{MT^\beta}{\Gamma(\beta+1)} \\ & + \frac{MT^\beta}{\Gamma(\beta+1)} + \frac{qMT^\beta}{\Gamma(\beta+1)} + q(\mu N + N^*) + (\mu N + N^*) + \frac{\nu_1}{\eta_1}\Bigg\}\Bigg] = Q.\end{aligned}$$

For $t \in I_m$, we have

$$\Big|(\urcorner u)(t)\Big| \leq \frac{MT^\beta}{\Gamma(\beta+1)} + \frac{qMT^\beta}{\Gamma(\beta+1)} + q(\mu N + N^*) + \frac{\nu_1}{\eta_1} = Q^*.$$

And similarly for $t \in I_0$, we have

$$\Big|(\daleth u)(t)\Big| \leq \frac{MT^{\beta}}{\Gamma(\beta+1)} + \frac{\nu_1}{\eta_1} = Q^{**}.$$

Thus the function is bounded.
Step 3:Now we need to show that $\daleth$ maps bounded set into equicontinuous set of $\mathbb{B}$. Let $t_1, t_2 \in I_q$ with $t_1 < t_2$ and let B_μ be a bounded set in the second step. Then for $u \in B_\mu$, we have

$$\begin{aligned}
&\Big|(\daleth u)(t_2) - (\daleth u)(t_1)\Big| \\
&\leq \int_{t_q}^{t_1} \Big|\frac{(t_2-s)^{\beta-1} - (t_1-s)^{\beta-1}}{\Gamma(\beta)} y(s)\Big| ds + \int_{t_1}^{t_2} \Big|\frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)} y(s)\Big| ds \\
&\quad + \int_{t_1}^{t_2} \Big|\frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)} y(s)\Big| ds + \sum_{0<t_m<t_2-t_1} \Big|I_m u(t_m)\Big| + t\Big\{\frac{1}{T}\Big(\int_{t_1}^{T} \Big|\frac{(T-s)^{\beta-1}}{\Gamma(\beta)} y(s)\Big| ds \\
&\quad + \int_{t_1}^{t_2} \Big|\frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)} y(s)\Big| ds + \sum_{0<t_m<t_2-t_1} \Big|I_m u(t_m)\Big| + \int_{t_1}^{t_2} \Big|\frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)} y(s)\Big| ds\Big)\Big\} \\
&\leq \frac{M}{\Gamma(\beta+1)}\Big[(t_2-t_q)^{\beta} - 2(t_2-t_1)^{\beta} - 2(t_2-t)^{\beta}\Big] + (t_2-t_1)(N||u||_{PC} + N^*) \\
&\quad + t\Big\{\frac{1}{T}\frac{M}{\Gamma(\beta+1)}\Big((t_2-t_q)^{\beta} - 2(t_2-t_1)^{\beta} - 2(t_2-t)^{\beta}\Big) + (t_2-t_1)(N||u||_{PC} + N^*)\Big\}.
\end{aligned}$$

We see that the right hand side of the above equation tends to zero as $t_1 \to t_2$. Thus by Arzelà–Ascoli theorem, we can say that $\daleth : \mathbb{B} \to \mathbb{B}$ is completely continuous.
Step 4: Now in the final step, we show that the set defined by

$$S = \{u \in \mathbb{B} : u = \delta(\daleth u) \text{ for some } 0 < \delta < 1\}$$

is bounded. Let $u \in S$, then for some $0 < \delta < 1$, $u = \delta(\daleth u)$. Therefore for $t \in I_q$, we have

$$\begin{aligned}
u(t) \;=\; & \delta\frac{1}{\Gamma(\beta)}\int_{t_q}^{t}(t-s)^{\beta-1}y(s)ds + \delta\frac{1}{\Gamma(\beta)}\int_{t_m}^{t_q}(t_q-s)^{\beta-1}y(s)ds \\
& + \sum_{i=1}^{m}\frac{1}{\Gamma(\beta)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\beta-1}y(s)ds + \delta\sum_{i=1}^{m} I_i(u(t_i)) \\
& + \delta I_q(u(t_q)) + \frac{\nu_1}{\eta_1} + t\Big[\frac{\nu_2}{\eta_2 T} - \frac{\xi_2}{\eta_2 T}\delta\int_0^T \frac{(T-s)^{\beta-1}}{\Gamma(\beta)}u(s)ds \\
& + \frac{1}{T}\Big\{\delta\frac{1}{\Gamma(\beta)}\int_{t_q}^{T}(T-s)^{\beta-1}y(s)ds + \delta\frac{1}{\Gamma(\beta)}\int_{t_m}^{t_q}(t_q-s)^{\beta-1}y(s)ds \\
& + \sum_{i=1}^{m}\delta\frac{1}{\Gamma(\beta)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\beta-1}y(s)ds + \delta\sum_{i=1}^{m} I_i(u(t_i)) + \delta I_q(u(t_q)) + \frac{\nu_1}{\eta_1}\Big\}\Big],
\end{aligned}$$

i.e.

$$\begin{aligned}u(t) \leq & \frac{1}{\Gamma(\beta)}\int_{t_q}^{t}(t-s)^{\beta-1}y(s)ds + \frac{1}{\Gamma(\beta)}\int_{t_m}^{t_q}(t_q-s)^{\beta-1}y(s)ds \\ & + \sum_{i=1}^{m}\frac{1}{\Gamma(\beta)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\beta-1}y(s)ds + \sum_{i=1}^{m} I_i(u(t_i)) + I_q(u(t_q)) + \frac{\nu_1}{\eta_1} \\ & + t\Bigg[\frac{\nu_2}{\eta_2 T} - \frac{\xi_2}{\eta_2 T}\int_0^T \frac{(T-s)^{\beta-1}}{\Gamma(\beta)}u(s)ds \\ & + \frac{1}{T}\Bigg\{\frac{1}{\Gamma(\beta)}\int_{t_q}^{T}(T-s)^{\beta-1}y(s)ds + \frac{1}{\Gamma(\beta)}\int_{t_m}^{t_q}(t_q-s)^{\beta-1}y(s)ds \\ & + \sum_{i=1}^{m}\frac{1}{\Gamma(\beta)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\beta-1}y(s)ds + \sum_{i=1}^{m} I_i(u(t_i)) + I_q(u(t_q)) + \frac{\nu_1}{\eta_1}\Bigg\}\Bigg]. \quad (3.22)\end{aligned}$$

Also we have $|y(s)| \leq \frac{\alpha^*+\beta^*||u||_{PC}}{1-\gamma^*} = M$. Thus inequality (3.22) becomes

$$\begin{aligned}\Big|u(t)\Big| \leq & \frac{\alpha^*+\beta^*||u||_{PC}}{(1-\gamma^*)\Gamma(\beta)}\int_{t_q}^{t}(t-s)^{\beta-1}y(s)ds + \frac{\alpha^*+\beta^*||u||_{PC}}{(1-\gamma^*)\Gamma(\beta)}\int_{t_m}^{t_q}(t_q-s)^{\beta-1}y(s)ds \\ & + \frac{\alpha^*+\beta^*||u||_{PC}}{(1-\gamma^*)\Gamma(\beta)}\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}(t_i-s)^{\beta-1}y(s)ds + q(N||u||_{PC}+N^*) + (N||u||_{PC}+N^*) \\ & + \frac{\nu_1}{\eta_1} + t\Bigg[\frac{\nu_2}{\eta_2 T} - \frac{\xi_2}{\eta_2 T}\int_0^T \frac{(T-s)^{\beta-1}}{\Gamma(\beta)}u(s)ds + \frac{1}{T}\Bigg\{\frac{\alpha^*+\beta^*||u||_{PC}}{(1-\gamma^*)\Gamma(\beta)}\frac{1}{\int_{t_q}}^{T}(T-s)^{\beta-1}y(s)ds \\ & + \frac{\alpha^*+\beta^*||u||_{PC}}{(1-\gamma^*)\Gamma(\beta)}\int_{t_m}^{t_q}(t_q-s)^{\beta-1}y(s)ds + \frac{\alpha^*+\beta^*||u||_{PC}}{(1-\gamma^*)\Gamma(\beta)}\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}(t_i-s)^{\beta-1}y(s)ds \\ & + q(N||u||_{PC}+N^*) + (N||u||_{PC}+N^*) + \frac{\nu_1}{\eta_1}\Bigg\}\Bigg]\end{aligned}$$

implies

$$\begin{aligned}\Big\|u\Big\|_{PC} & \leq \frac{\frac{(q+2)\alpha^* T^\beta}{(1-\gamma^*)\Gamma(\beta+1)} + (1+q)N^* + \frac{\nu_1}{\eta_1} + t\mathcal{L}}{1 - \frac{(q+2)\beta^* T^\beta}{(1-\gamma^*)\gamma(\beta+1)} + (q+1)N + t\Big[\frac{1}{T}\Big\{\frac{(q+2)\beta^* T^\beta}{(1-\gamma^*)\gamma(\beta+1)} + (q+1)N\Big\}\Big]} \\ & = M\end{aligned}$$

where

$$\mathcal{L} = \Bigg[\frac{\nu_2}{\eta_2 T} - \frac{\xi_2 T^\beta}{\eta_2 T\Gamma(\beta+1)} + \frac{1}{T}\Bigg\{\frac{(q+2)\alpha^* T^\beta}{(1-\gamma^*)\Gamma(\beta+1)} + (1+q)N^* + \frac{\nu_1}{\eta_1}\Bigg\}\Bigg].$$

It means that the set S is bounded. Thus by Schaefer's fixed point theorem, we prove that S has a fixed point which is the solution of the problem (1.2). □

4. Ulam–Hyers Stability Analysis

In this section, we discuss various types of Ulam–Hyers stability.

Theorem 4.1. *If the assumptions (H_1) – (H_3) and the inequality (3.12) are satisfied, then the problem (1.2) is Ulam–Hyers stable and consequently generalized Ulam–Hyers stable.*

Proof. Let $w \in \mathbb{B}$ is the solution of inequality (2.1) and u be the unique solution of the following problem.

$$\begin{cases} {}^cD^{\beta}_{0,t}u(t) = g(t), \quad t \neq t_m \in I = [0,T], \\ \Delta u(t) = I_m(u(t_m)), \quad m = 1,2,\dots,q-1, \\ \eta_1 u(0) + \xi_1 I^{\beta} u(t) \mid_{t=0} = \nu_1, \\ \eta_2 x(T) + \xi_2 I^{\beta} u(t) \mid_{t=T} = \nu_2. \end{cases} \tag{4.1}$$

so by Lemma 3.1, for each $t \in I_m$ we have

$$\begin{aligned} u(t) \;=\; & \frac{1}{\Gamma(\beta)}\int_{t_q}^{t}(t-s)^{\beta-1}g(s)ds + \frac{1}{\Gamma(\beta)}\int_{t_m}^{t_q}(t_q-s)^{\beta-1}g(s)ds \\ & + \sum_{i=1}^{m}\left[\frac{1}{\Gamma(\beta)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\beta-1}g(s)ds + I_i(u(t_i))\right] + I_q(u(t_q)) + \frac{\nu_1}{\eta_1}, \end{aligned}$$

where $g \in C(I,\Re)$, and is given by

$$g(s) = f(s,u(s),g(s)).$$

Since w is the solution of inequality (2.1) hence by Remark 2.10, we have

$$\begin{cases} {}^cD^{\beta}_{0,t}u(t) = g(t) + \varpi(t), \quad t \neq t_m \in I, \\ \Delta u(t) = I_m(u(t_m)) + \varpi_m, m = 1,2,\dots,q-1, \\ \eta_1 x(0) + \xi_1 I^{\alpha} x(t) \mid_{t=0} = \nu_1, \\ \eta_2 x(T) + \xi_2 I^{\alpha} x(t) \mid_{t=T} = \nu_2. \end{cases} \tag{4.2}$$

Obviously the solution of the Eq. (4.2) will be

$$x(t) = \begin{cases} \frac{1}{\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}y(s)ds + \frac{1}{\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}\varpi(s)ds + \frac{\nu_1}{\eta_1}, \quad t \in I_0 \\ \frac{1}{\Gamma(\beta)}\int_{t_m}^t(t-s)^{\beta-1}y(s)ds + \frac{1}{\Gamma(\beta)}\int_{t_1}^t(t-s)^{\beta-1}\varpi(s)ds \\ \;+\sum_{i=1}^m\frac{1}{\Gamma(\beta)}\int_{t_{i-1}}^t(t_i-s)^{\beta-1}y(s)ds + \sum_{i=1}^m\frac{1}{\Gamma(\beta)}\int_{t_{i-1}}^t(t_i-s)^{\beta-1}\varpi(s)ds \\ \;+\sum_{i=1}^m I_i(u(t_i)) + \sum_{i=1}^m\varpi_i + \frac{\nu_1}{\eta_1}, \quad t \in I_m \\ \frac{1}{\Gamma(\beta)}\int_{t_q}^t(t-s)^{\beta-1}y(s)ds + \frac{1}{\Gamma(\beta)}\int_{t_q}^t(t-s)^{\beta-1}\varpi(s)ds \\ \;+\frac{1}{\Gamma(\beta)}\int_{t_m}^{t_q}(t_q-s)^{\beta-1}y(s)ds + \frac{1}{\Gamma(\beta)}\int_{t_m}^{t_q}(t_q-s)^{\beta-1}\varpi(s)ds \\ \;+\sum_{i=1}^m\frac{1}{\Gamma(\beta)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\beta-1}y(s)ds + \sum_{i=1}^m\frac{1}{\Gamma(\beta)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\beta-1}\varpi(s)ds \\ \;+\sum_{i=1}^m I_i(w(t_i)) + \sum_{i=1}^m\varpi_i + I_q(w(t_q)) + \frac{\nu_1}{\eta_1} \\ \;+t\Bigg[\frac{\nu_2}{\eta_2 T} - \frac{\xi_2}{\eta_2 T}\int_0^T\frac{(T-s)^{\beta-1}}{\Gamma(\beta)}u(s)ds + \frac{1}{T}\Bigg\{\frac{1}{\Gamma(\beta)}\int_{t_q}^T(T-s)^{\beta-1}y(s)ds \\ \;+\frac{1}{\Gamma(\beta)}\int_{t_q}^T(T-s)^{\beta-1}\varpi(s)ds + \frac{1}{\Gamma(\beta)}\int_{t_m}^{t_q}(t_q-s)^{\beta-1}y(s)ds \\ \;+\frac{1}{\Gamma(\beta)}\int_{t_m}^{t_q}(t_q-s)^{\beta-1}\varpi(s)ds + \sum_{i=1}^m\frac{1}{\Gamma(\beta)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\beta-1}y(s)ds \\ \;+\sum_{i=1}^m\frac{1}{\Gamma(\beta)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\beta-1}\varpi(s)ds + \sum_{i=1}^m I_i(w(t_i)) + \sum_{i=1}^m\varpi_i\Bigg\} \\ \;+I_q(w(t_q)) + \frac{\nu_1}{\eta_1}\Bigg], \quad t \in I_q, \end{cases}$$

where $x \in C(I, \Re)$, and is given by

$$x = g(s, u(s), x(s)).$$

Therefore, for each $t \in I_q$, we have

$$\begin{aligned}
\Big|w(t) - u(t)\Big| &\le \frac{1}{\Gamma(\beta)} \int_{t_q}^{t} (t-s)^{\beta-1} \Big|g(s) - y(s)\Big| + \frac{1}{\Gamma(\beta)} \int_{t_q}^{t} (t-s)^{\beta-1} |\varpi(s)| \\
&+ \frac{1}{\Gamma(\beta)} \int_{t_m}^{t_q} (t_q-s)^{\beta-1} \Big|g(s) - y(s)\Big| + \frac{1}{\Gamma(\beta)} \int_{t_m}^{t_q} (t_q-s)^{\beta-1} |\varpi(s)| \\
&+ \sum_{i=1}^{m} \frac{1}{\Gamma(\beta)} \int_{t_{i-1}}^{t_i} (t_i-s)^{\beta-1} \Big|g(s) - y(s)\Big| ds + \sum_{i=1}^{m} \frac{1}{\Gamma(\beta)} \int_{t_{i-1}}^{t_i} (t_i-s)^{\beta-1} |\varpi(s)| ds \\
&+ \sum_{i=1}^{m} \Big|I_i(w(t_i)) - I_i(u(t_i))\Big| + \sum_{i=1}^{m} |\varpi_i| + \Big|I_q(w(t_q)) - I_q(u(t_q))\Big| \\
&+ t\Bigg[\frac{1}{T}\Bigg(\frac{1}{\Gamma(\beta)} \int_{t_q}^{t} (T-s)^{\beta-1} \Big|g(s) - y(s)\Big| + \frac{1}{\Gamma(\beta)} \int_{t_q}^{t} (T-s)^{\beta-1} |\varpi(s)| \\
&+ \frac{1}{\Gamma(\beta)} \int_{t_m}^{t_q} (t_q-s)^{\beta-1} \Big|g(s) - y(s)\Big| ds + \frac{1}{\Gamma(\beta)} \int_{t_m}^{t_q} (t_q-s)^{\beta-1} |\varpi(s)| ds \\
&+ \sum_{i=1}^{m} \frac{1}{\Gamma(\beta)} \int_{t_{i-1}}^{t_i} (t_i-s)^{\beta-1} \Big|g(s) - y(s)\Big| ds + \sum_{i=1}^{m} \frac{1}{\Gamma(\beta)} \int_{t_{i-1}}^{t_i} (t_i-s)^{\beta-1} |\varpi(s)| ds \\
&+ \sum_{i=1}^{m} |\varpi_i| + \sum_{i=1}^{m} \Big|I_i(w(t_i)) - I_i(u(t_i))\Big| + \Big|I_q(w(t_q)) - I_q(u(t_q))\Big|\Bigg)\Bigg].
\end{aligned}$$

By (H_2), we get

$$||x - y|| \le \frac{K_g}{1 - L_g} ||w - u||_{PC}.$$

Hence by $(H_2), (H_3), (H_4)$ and (i) of Remark 2.10, we get

$$\begin{aligned}
\Big\|w(t) - u(t)\Big\| &\le \frac{K_g ||w-u||_{PC}}{1 - L_g} \int_{t_q}^{t} \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} + \epsilon \int_{t_q}^{t} \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \\
&+ \frac{K_g ||w-u||_{PC}}{1 - L_g} \int_{t_m}^{t_q} \frac{(t_q-s)^{\beta-1}}{\Gamma(\beta)} ds + \epsilon \int_{t_m}^{t_q} \frac{(t_q-s)^{\beta-1}}{\Gamma(\beta)} \\
&+ \frac{K_g ||w-u||_{PC}}{1 - L_g} \sum_{i=1}^{m} \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\beta-1}}{\Gamma(\beta)} ds + \epsilon \sum_{i=1}^{m} \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\beta-1}}{\Gamma(\beta)} \\
&+ ||w-u||_{PC} \sum_{i=1}^{m} L_I + \sum_{i=1}^{m} \epsilon + L_I ||w-u||_{PC} \\
&+ t\Bigg[\frac{1}{T}\Bigg(\frac{K_g ||w-u||_{PC}}{1 - L_g} \int_{t_q}^{T} \frac{(T-s)^{\beta-1}}{\Gamma(\beta)} + \epsilon \int_{t_q}^{T} \frac{(T-s)^{\beta-1}}{\Gamma(\beta)} \\
&+ \frac{K_g ||w-u||_{PC}}{1 - L_g} \int_{t_m}^{t_q} \frac{(t_q-s)^{\beta-1}}{\Gamma(\beta)} + \epsilon \int_{t_m}^{t_q} \frac{(t_q-s)^{\beta-1}}{\Gamma(\beta)}
\end{aligned}$$

$$+\frac{K_g||w-u||_{PC}}{1-L_g}\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\beta-1}}{\Gamma(\beta)}ds+\epsilon\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\beta-1}}{\Gamma(\beta)}$$
$$+||w-u||_{PC}\sum_{i=1}^{m}L_I+\sum_{i=1}^{m}\epsilon+L_I||w-u||_{PC}\Big)\Big]$$
$$\leq\epsilon\Big[\frac{T^\beta}{\Gamma(\beta+1)}+\frac{T^\beta}{\Gamma(\beta+1)}+m\frac{T^\beta}{\Gamma(\beta+1)}+m+t\Big(\frac{1}{T}\frac{T^\beta}{\Gamma(\beta+1)}$$
$$+\frac{T^\beta}{\Gamma(\beta+1)}+m\frac{T^\beta}{\Gamma(\beta+1)}+m\Big)\Big]$$
$$+\Big[\frac{K_gT^p}{1-L_g\Gamma(\beta+1)}+\frac{K_gT^p}{1-L_g\Gamma(\beta+1)}+m\frac{K_gT^p}{1-L_g\Gamma(\beta+1)}+mL_I+L_I$$
$$+t\Big(\frac{1}{T}\frac{K_gT^p}{1-L_g\Gamma(\beta+1)}+\frac{K_gT^p}{1-L_g\Gamma(\beta+1)}+m\frac{K_gT^p}{1-L_g\Gamma(\beta+1)}+mL_I+L_I\Big)\Big].$$

$$||w-u||_{PC}\leq\frac{\epsilon\Big[2+2m+t\Big(\frac{1}{T}(2+m)\frac{T^\beta}{\Gamma(\beta+1)}+m\Big)\Big]}{1-\Big[\frac{K_gT^\beta}{(1-L_g)\Gamma(\beta+1)}\Big(2+m(t+1)+t(\frac{1}{T}+1)\Big)+(t+1)(m+1)L_I\Big]}.\tag{4.3}$$

Similarly for $t\in I_0$, we have

$$\Big\|w-u\Big\|_{PC}\leq\frac{\epsilon\Big[\frac{T^\beta}{\Gamma(\beta+1)}\Big]}{\Big[1-\frac{K_gT^\beta}{(1-L_g)\Gamma(\beta+1)}\Big]}.\tag{4.4}$$

Now repeat the same procedure for $t\in I_m$, we obtain

$$\Big\|w-u\Big\|_{PC}\leq\frac{\epsilon\Big(\frac{T^\beta}{\Gamma(\beta+1)}+m\frac{T^\beta}{\Gamma(\beta+1)}+m\Big)}{1-\Big[\frac{K_gT^\beta}{(1-L_g)\Gamma(\beta+1)}+m\frac{K_gT^\beta}{(1-L_g)\Gamma(\beta+1)}+mL_I\Big]}.\tag{4.5}$$

Combining Eq. (4.3), Eq. (4.4) and Eq. (4.5), we get

$$\Big\|w-u\Big\|_{PC}\leq\epsilon\Bigg\{\frac{2+2m+t\Big(\frac{1}{T}(2+m)\frac{T^\beta}{\Gamma(\beta+1)}+m\Big)}{1-\Big[\frac{K_gT^\beta}{(1-L_g)\Gamma(\beta+1)}\Big(2+m(t+1)+t(\frac{1}{T}+1)\Big)+(t+1)(m+1)L_I\Big]}$$
$$+\frac{\frac{T^\beta}{\Gamma(\beta+1)}}{1-\frac{K_gT^\beta}{(1-L_g)\Gamma(\beta+1)}}+\frac{\frac{T^\beta}{\Gamma(\beta+1)}+m\frac{T^\beta}{\Gamma(\beta+1)}+m}{1-\frac{K_gT^\beta}{(1-L_g)\Gamma(\beta+1)}+m\frac{K_gT^\beta}{(1-L_g)\Gamma(\beta+1)}+mL_I}\Bigg\}.$$

Thus

$$||w-u||_{PC}\leq m_{g,p,q,\sigma}\epsilon,$$

where

$$m_{g,p,q,\sigma} = \frac{2+2m+t\Big(\frac{1}{T}(2+m)\frac{T^\beta}{\Gamma(\beta+1)}+m\Big)}{1-\Big[\frac{K_gT^\beta}{(1-L_g)\Gamma(\beta+1)}\Big(2+m(t+1)+t(\frac{1}{T}+1)\Big)+(t+1)(m+1)L_I\Big]}$$

$$+\frac{\frac{T^\beta}{\Gamma(\beta+1)}}{1-\frac{K_gT^\beta}{(1-L_g)\Gamma(\beta+1)}}+\frac{\frac{T^\beta}{\Gamma(\beta+1)}+m\frac{T^\beta}{\Gamma(\beta+1)}+m}{1-\frac{K_gT^\beta}{(1-L_g)\Gamma(\beta+1)}+m\frac{K_gT^\beta}{(1-L_g)\Gamma(\beta+1)}+mL_I}.$$

Hence the problem (1.2) is Ulam–Hyers stable. Moreover if we set $\theta(\epsilon) = m_{g,p,q,\sigma}$; $\theta(0) = 0$, then the problem (1.2) is generalized Ulam-Hyers stable. □

Theorem 4.2. *Let us suppose that the inequalities* $(H_1 - H_3)$, H_6 *and (3.12) are satisfied then the problem (1.2) is Ulam–Hyers–Rassias stable with respect to* (ϕ, ψ), *consequently generalized Ulam–Hyers–Rassias stable.*

Proof. Let $w \in I$ be a solution of the inequality (2.3) and let u be a unique solution of the following problem

$$\begin{cases} {}^cD^\beta_{0,t}u(t) = g(t,u(t),{}^cD^\beta_{0,t}u(t)), \quad t \neq t_m \in I \\ \Delta u(t) = I_m(u(t_m)), m = 1,2,\dots,q-1, \\ \eta_1 u(0) + \xi_1 I^\beta u(t)\mid_{t=0} = \nu_1, \\ \eta_2 u(T) + \xi_2 I^\beta u(t)\mid_{t=T} = \nu_2. \end{cases}$$

From the proof of Theorem 4.1, for each $t \in I_q$, we obtain

$$\begin{aligned} \Big|w(t)-u(t)\Big| &\le \frac{1}{\Gamma(\beta)}\int_{t_q}^t (t-s)^{\beta-1}\Big|g(s)-y(s)\Big| + \frac{1}{\Gamma(\beta)}\int_{t_q}^t (t-s)^{\beta-1}|\varpi(s)| \\ &+ \frac{1}{\Gamma(\beta)}\int_{t_m}^{t_q}(t_q-s)^{\beta-1}\Big|g(s)-y(s)\Big| + \frac{1}{\Gamma(\beta)}\int_{t_m}^{t_q}(t_q-s)^{\beta-1}|\varpi(s)| \\ &+ \sum_{i=1}^m \frac{1}{\Gamma(\beta)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\beta-1}\Big|g(s)-y(s)\Big|ds + \sum_{i=1}^m\frac{1}{\Gamma(\beta)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\beta-1}|\varpi(s)|ds \\ &+ \sum_{i=1}^m\Big|I_i(w(t_i))-I_i(u(t_i))\Big| + \sum_{i=1}^m|\varpi_i| + \Big|I_q(w(t_q))-I_q(u(t_q))\Big| \\ &+ t\Bigg[\frac{1}{T}\Bigg(\frac{1}{\Gamma(\beta)}\int_{t_q}^t(T-s)^{\beta-1}\Big|g(s)-y(s)\Big| + \frac{1}{\Gamma(\beta)}\int_{t_q}^t (T-s)^{\beta-1}|\varpi(s)| \\ &+ \frac{1}{\Gamma(\beta)}\int_{t_m}^{t_q}(t_q-s)^{\beta-1}\Big|g(s)-y(s)\Big|ds + \frac{1}{\Gamma(\beta)}\int_{t_m}^{t_q}(t_q-s)^{\beta-1}|\varpi(s)|ds \\ &+ \sum_{i=1}^m\frac{1}{\Gamma(\beta)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\beta-1}\Big|g(s)-y(s)\Big|ds + \sum_{i=1}^m\frac{1}{\Gamma(\beta)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\beta-1}|\varpi(s)|ds \\ &+ \sum_{i=1}^m|\varpi_i| + \sum_{i=1}^m\Big|I_i(w(t_i))-I_i(u(t_i))\Big| + \Big|I_q(w(t_q))-I_q(u(t_q))\Big|\Bigg)\Bigg]. \end{aligned}$$

By (H_2) we get that

$$||x-y|| \leq \frac{K_g}{1-L_g}||w-u||_{PC}.$$

Hence by $(H_2),(H_3),(H_4)$ and (i) of Remark 2.11, we get

$$\begin{aligned}
\Big|w(t)-u(t)\Big| \leq & \frac{K_g||w-u||_{PC}}{1-L_g}\int_{t_q}^{t}\frac{(t-s)^{\beta-1}}{\Gamma(\beta)} + \epsilon\int_{t_q}^{t}\frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\varphi(t) \\
& + \frac{K_g||w-u||_{PC}}{1-L_g}\int_{t_m}^{t_q}\frac{(t_q-s)^{\beta-1}}{\Gamma(\beta)}ds + \epsilon\int_{t_m}^{t_q}\frac{(t_q-s)^{\beta-1}}{\Gamma(\beta)}\varphi(t) \\
& + \frac{K_g||w-u||_{PC}}{1-L_g}\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\beta-1}}{\Gamma(\beta)}ds + \epsilon\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\beta-1}}{\Gamma(\beta)}\varphi(t) \\
& + ||w-u||_{PC}\sum_{i=1}^{m}L_I + \epsilon\sum_{i=1}^{m}\psi + L_I||w-u||_{PC} \\
& + t\Bigg[\frac{1}{T}\Bigg(\frac{K_g||w-u||_{PC}}{1-L_g}\int_{t_q}^{T}\frac{(T-s)^{\beta-1}}{\Gamma(\beta)} + \epsilon\int_{t_q}^{T}\frac{(T-s)^{\beta-1}}{\Gamma(\beta)}\varphi(t) \\
& + \frac{K_g||w-u||_{PC}}{1-L_g}\int_{t_m}^{t_q}\frac{(t_q-s)^{\beta-1}}{\Gamma(\beta)} + \epsilon\int_{t_m}^{t_q}\frac{(t_q-s)^{\beta-1}}{\Gamma(\beta)}\varphi(t) \\
& + \frac{K_g||w-u||_{PC}}{1-L_g}\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\beta-1}}{\Gamma(\beta)}ds \\
& + \epsilon\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\beta-1}}{\Gamma(\beta)}\varphi(t) + ||w-u||_{PC}\sum_{i=1}^{m}L_I + \epsilon\sum_{i=1}^{m}\psi + L_I||w-u||_{PC}\Bigg)\Bigg].
\end{aligned}$$

Using (H_6) we have

$$\begin{aligned}
\Big|w(t)-u(t)\Big| \leq & \epsilon\Bigg[\lambda_\phi\phi(t) + \lambda_\phi\phi(t) + m\lambda_\phi\phi(t) + m\psi + t\Bigg\{\frac{1}{T}\Bigg(\lambda_\phi\phi(t) \\
& + \lambda_\phi\phi(t) + m\lambda_\phi\phi(t) + m\psi\Bigg)\Bigg\}\Bigg] + \Bigg[\frac{K_gT^p}{(1-L_g)\Gamma(\beta+1)} + \frac{K_gT^p}{(1-L_g)\Gamma(\beta+1)} \\
& + m\frac{K_gT^p}{(1-L_g)\Gamma(\beta+1)} + mL_I + L_I + t\Bigg\{\frac{1}{T}\Bigg(\frac{K_gT^p}{(1-L_g)\Gamma(\beta+1)} + \frac{K_gT^p}{(1-L_g)\Gamma(\beta+1)} \\
& + m\frac{K_gT^p}{(1-L_g)\Gamma(\beta+1)} + mL_I + L_I\Bigg)\Bigg\}\Bigg]||w-u||_{PC} \\
\leq & \epsilon(\phi(t)+\psi(t))\Bigg[\lambda_\phi + \lambda_\phi + m\lambda_\phi + m\psi + t\Bigg\{\frac{1}{T}\Bigg(\lambda_\phi + \lambda_\phi + m\lambda_\phi + m\psi\Bigg)\Bigg\}\Bigg] \\
& + \Bigg[\frac{K_gT^p}{(1-L_g)\Gamma(\beta+1)} + \frac{K_gT^p}{(1-L_g)\Gamma(\beta+1)} + m\frac{K_gT^p}{(1-L_g)\Gamma(\beta+1)} + mL_I + L_I \\
& + t\Bigg\{\frac{1}{T}\Bigg(\frac{K_gT^p}{(1-L_g)\Gamma(\beta+1)} + \frac{K_gT^p}{(1-L_g)\Gamma(\beta+1)} \\
& + m\frac{K_gT^p}{(1-L_g)\Gamma(\beta+1)} + mL_I + L_I\Bigg)\Bigg\}\Bigg]||w-u||_{PC}.
\end{aligned}$$

From which we get

$$\Big\|w-u\Big\|_{PC} \leq \frac{\epsilon(\phi(t)+\psi(t))\Big[\lambda_\phi(2+m)+m\psi+t\Big\{\frac{1}{T}\Big((2+m)\lambda_\phi+m\psi\Big)\Big\}\Big]}{1-\Big[\frac{K_gT^\beta}{(1-L_g)\Gamma(\beta+1)}\Big(2+m(t+1)+t(\frac{1}{T}+1)\Big)+(t+1)(m+1)L_I\Big]}. \quad (4.6)$$

Similarly for $t \in I_0$, we obtain

$$\Big\|w(t)-u(t)\Big\| \leq \frac{\epsilon\lambda_\phi\phi(t)}{\Big[1-\frac{K_gT^\beta}{(1-L_g)\Gamma(\beta+1)}\Big]}. \quad (4.7)$$

Now similarly for $t \in I_m$, we get

$$\Big\|w(t)-u(t)\Big\| \leq \frac{\epsilon\Big(\lambda_\phi\phi(t)+m\lambda_\phi\phi(t)+m\psi\Big)}{1-\Big[\frac{K_gT^\beta}{(1-L_g)\Gamma(\beta+1)}+m\frac{K_gT^\beta}{(1-L_g)\Gamma(\beta+1)}+mL_I\Big]}. \quad (4.8)$$

Combining (4.4), (4.5) and (4.6), we get

$$\Big\|w-u\Big\|_{PC} \leq \frac{\epsilon(\phi(t)+\psi(t))\Big[\lambda_\phi(2+m)+m\psi+t\Big\{\frac{1}{T}\Big((2+m)\lambda_\phi+m\psi\Big)\Big\}\Big]}{1-\Big[\frac{K_gT^\beta}{(1-L_g)\Gamma(\beta+1)}\Big(2+m(t+1)+t(\frac{1}{T}+1)\Big)+(t+1)(m+1)L_I\Big]}$$
$$+\frac{\epsilon(\phi(t)+\psi(t))\lambda_\phi}{1-\frac{K_gT^\beta}{(1-L_g)\Gamma(\beta+1)}}+\frac{\epsilon(\phi(t)+\psi(t))\Big(\lambda_\phi+m\lambda_\phi+m\Big)}{1-\frac{K_gT^\beta}{(1-L_g)\Gamma(\beta+1)}+m\frac{K_gT^\beta}{(1-L_g)\Gamma(\beta+1)}+mL_I}.$$

Thus

$$||w-u||_{PC} \leq m_{g,p,q,\sigma,\phi}\epsilon(\phi(t)+\psi(t)),$$

where

$$m_{g,p,q,\sigma,\phi} = \frac{\Big[\lambda_\phi(2+m)+m\psi+t\Big((\frac{1}{T}+1+m)\lambda_\phi+m\psi\Big)\Big]}{1-\Big[\frac{K_gT^\beta}{(1-L_g)\Gamma(\beta+1)}\Big(2+m(t+1)+t(\frac{1}{T}+1)\Big)+(t+1)(m+1)L_I\Big]}$$
$$+\frac{\lambda_\phi}{1-\frac{K_gT^\beta}{(1-L_g)\Gamma(\beta+1)}}+\frac{\Big(\lambda_\phi+m\lambda_\phi+m\Big)}{1-\frac{K_gT^\beta}{(1-L_g)\Gamma(\beta+1)}+m\frac{K_gT^\beta}{(1-L_g)\Gamma(\beta+1)}+mL_I}.$$

Thus the problem (1.2) is Ulam–Hyers–Rassias stable. Hence it is also obvious that the proposed problem (3.1) is generalized Ulam–Hyers–Rassias stable. □

Finally, we give an example.

Example 4.1. Consider the instantaneous impulsive boundary value problem

$$\begin{cases} {}^C D^{\frac{1}{2}}_{0,t} u(t) = \frac{|u(t)| + {}^C D^{\frac{1}{2}}_{0,t} u(t)}{e^{t+2} + |u(t)| + {}^C D^{\frac{1}{2}}_{0,t} u(t)}, \quad t \in [0,3] \quad t \neq \frac{3}{2}, \\ I_1(u(\frac{3}{2})) = \frac{|u(\frac{3}{2})|}{20 + |u(\frac{3}{2})|}, \\ u(0) + I^{\frac{1}{2}} u(t) \mid_{t=0} = \frac{1}{2}, \\ u(3) + I^{\frac{1}{2}} u(t)|_{t=3} = \frac{1}{2}, \quad \nu_i = \frac{1}{2}, (i = 1, 2). \end{cases}$$

Here, $\beta = \frac{1}{2}$, $m = 1$, $I_0 = [0, \frac{3}{2}]$, $I_1 = (\frac{3}{2}, 3]$, $t_0 = 0$, $t_1 = \frac{3}{2}$,and we set the function $g : I \times \Re \times \Re \to \Re$ as

$$g(t, w, u) = \frac{|u(t)| + w(t)}{e^{t+2} + |u(t)| + w(t)} \quad t \in \left[0, \frac{3}{2}\right].$$

Also, for any $u, w, \bar{u}, \bar{w} \in \Re$, we have

$$|g(t, w, u) - g(t, \bar{w}, \bar{u})| \leq \frac{1}{40e}[|u - \bar{u}| + |w - \bar{w}|],$$

so we have $K_g = L_g = \frac{1}{40e}$, Thus (H_2) holds, and also we see

$$|g(t, w, u)| \leq \frac{1}{40e^{t+2}}(|u(t)| + |w(t)|) \quad t \in I.$$

Also we have $\alpha(t) = \frac{1}{2e^{2+t}}$, $\beta(t) = \gamma(t) = \frac{1}{4e^{2+t}}$ so for this we find $\alpha^* = \frac{1}{2e^2}$, $\beta^* = \frac{1}{4e^2}$ $\gamma^* = \frac{1}{4e^2}$. Further, we see that

$$\left|I_1 u(\frac{3}{2})\right| \leq \frac{1}{20}\left|u(\frac{1}{2})\right| + 1.$$

For this we can see that $N = \frac{1}{20}, N^* = 1$. so

$$\left|I_1 u(\frac{3}{2}) - I_1 x(\frac{3}{2})\right| \leq \frac{1}{20}|u - x|$$

$$mN + \frac{\beta^* T^\beta (m+1)}{(1 - \gamma^*)\Gamma(\beta + 1)} = \frac{1}{20} + \frac{4\frac{1}{40e^2}}{1 - \frac{1}{40e^2}\Gamma(\frac{3}{2})} \approx 0.2993 < 1.$$

Thus by Theorem 3.2 we can say that the problem has unique solution. Now

$$\left[\frac{2K_g T^\beta}{(1 - L_g)\Gamma(\beta + 1)} + m(\frac{K_g T^\beta}{(1 - L_g)\Gamma(\beta + 1)} + L_I) + L_I \right.$$
$$\left. + t\left[\frac{1}{T}\left\{\frac{2K_g T^\beta}{(1 - L_g)\Gamma(\beta + 1)} + m(\frac{K_g T^\beta}{(1 - L_g)\Gamma(\beta + 1)} + L_I) + L_I\right\}\right]\right] < 1.$$

putting all values in the above equation

$$\frac{2\frac{2}{40e}}{(1 - 40e)\sqrt(\Pi)} + 1(\frac{240e}{(1 - 40e)\sqrt(\Pi)} + 0.012) + 0.012 + \frac{1}{2}$$
$$\left\{\frac{1}{4}\left(2\frac{2}{40e}(1 - 40e)\sqrt(\Pi) + 1(\frac{240e}{(1 - 40e)\sqrt(\Pi)} + 0.012) + 0.012\right)\right\} \approx 0.0317 < 1.$$

Thus by Theorem 3.1 the problem has at least one solution on the similar way we check the condition of Theorem 4.1 and 4.2.

Conclusion

We proved the existence and uniqueness conditions for a class of nonlinear implicit type impulsive boundary value problem by using Schaefer's fixed point theorem, Banach contraction theorem and Arzelà–Ascoli theorem. Further, we proved different types of Ulam's type stability.

Competing interest

The authors claim no competing interest for this research paper.

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