Annals of University of Craiova, Math. Comp. Sci. Ser.
Volume 30, 2003, Pages 45–52
ISSN: 1223-6934

On Mixed Methods for Signorini Problems

Faker Ben Belgacem, Yves Renard, and Leila Slimane

Abstract. We present here some mixed strategies to solve numerically the Signorini problems. A general Framework for the discretization of such problems and convergence results in the nearly incompressible case are given. Two and three-dimensional numerical experiments are presented.

2000 Mathematics Subject Classification. 35J85, 74S05, 74M15.
Key words and phrases. Signorini problem, unilateral contact, mixed methods.

1. General framework

A so-called Signorini problem is a boundary value problem of a scalar or vectorial partial differential equation with a Signorini condition on a part of its boundary. A Signorini condition is a complementary condition which modeled threshold phenomena like contact problem, thermostatic device or semi-permeable membranes.

1.1. Scalar Signorini problem. Let Ω be a Lipschitz bounded domain in $\mathbb{R}^2$. The boundary $\partial\Omega$ is a union of three non-overlapping portions Γ_D, Γ_N and Γ_C. The part Γ_D of nonzero measure is subjected to Dirichlet condition while on Γ_N a Neumann condition is prescribed, and Γ_C is the part of the boundary submitted to a Signorini condition.

For a given data $f \in L^2(\Omega)$ and $g \in H^{-1/2}(\Gamma_N)$, the scalar Signorini problem consists in finding u such that

$$\begin{cases} -\triangle u = f, & \text{in } \Omega, \\ u = 0, & \text{on } \Gamma_D, \\ \dfrac{\partial u}{\partial \mathbf{n}} = g, & \text{on } \Gamma_N, \\ u \geq 0,\ \dfrac{\partial u}{\partial \mathbf{n}} \geq 0,\ u\dfrac{\partial u}{\partial \mathbf{n}} = 0, & \text{on } \Gamma_C, \end{cases} \tag{1}$$

where $\mathbf{n}$ is the outward unit normal on $\partial\Omega$. It is assumed here an homogeneous Dirichlet condition and no initial gap for the Signorini condition. The analysis can be extended straightforwardly to the case of non-vanishing Dirichlet condition and initial gap.

Received: 11 December 2002.

1.2. Vectorial Signorini problem. In the framework of deformable solid mechanics, the displacement of a linearly elastic body Ω supported by a frictionless rigid foundation Γ_C, fixed along a part Γ_D of the boundary and subjected to external forces $f_{|\Omega}$ and $g_{|\Gamma_N}$ (see figure 1) is solution to the following problem

$$\begin{cases} -\text{div } \sigma(u) = f, & \text{in } \Omega, \\ u = 0, & \text{on } \Gamma_D, \\ \sigma(u)\mathbf{n} = g, & \text{on } \Gamma_N, \\ u_N \leq 0, \quad \sigma_N \leq 0, \quad u_N \sigma_N = 0, & \\ \sigma_T = 0, & \text{on } \Gamma_C, \end{cases} \tag{2}$$

where the stress tensor is obtained from the displacement through the constitutive law $\sigma(u) = \mathcal{A}\varepsilon(u)$, $\mathcal{A} \in L^\infty(\Omega; \mathbb{R}^{16})$ is the fourth order Hook tensor, symmetric and elliptic, $\varepsilon(u) = \frac{1}{2}(\nabla u + \nabla u^T)$ is the small strain tensor, and $\sigma\mathbf{n} = \sigma_N \mathbf{n} + \sigma_T$ and $u = u_N \mathbf{n} + u_T$ are the decompositions into normal and tangential components on Γ_C.

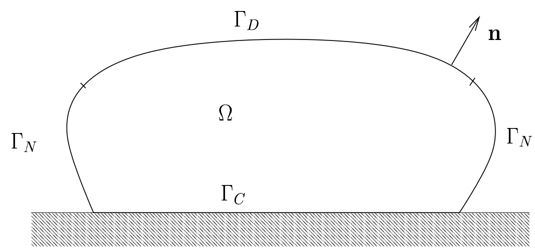

FIGURE 1. Deformable solid Ω on contact with a rigid foundation

1.3. weak formulation. Both the two models have the same weak formulation, introducing

scalar model	vectorial model
$V = \{v \in H^1(\Omega) : v \leq 0 \text{ on } \Gamma_D\}$	$V = \{v \in H^1(\Omega, \mathbb{R}^2) : v \leq 0 \text{ on } \Gamma_D\}$
$K_0 = \{v \in V : v \geq 0 \text{ on } \Gamma_C\}$	$K_0 = \{v \in V : v_N \geq 0 \text{ on } \Gamma_C\}$
$X_N = \{v_{\|\Gamma_C} : v \in V\}$	$X_N = \{v_{N\|\Gamma_C} : v \in V\}$
$a(u,v) = \int_\Omega \nabla u . \nabla v dx$	$a(u,v) = \int_\Omega \sigma(u) : \varepsilon(v) dx$
$l(v) = \int_\Omega f v dx + \int_{\Gamma_N} g v d\Gamma$	$l(v) = \int_\Omega f.v dx + \int_{\Gamma_N} g.v d\Gamma$

which is

$$\begin{cases} \text{Find } u \in K_0 \text{ satisfying} \\ a(u, v-u) \geq l(v-u), \quad \forall\, v \in K_0. \end{cases} \tag{3}$$

Another weak formulation is obtained introducing $F_{N} \in X'_{N}$ a multiplier which represents the contact force on Γ_C and

$$
\begin{aligned}
&A : V \longrightarrow V', < Au, v >_{V',V} = a(u, v),\ \forall u, v \in V,\\
&\mathcal{F} \in V', < \mathcal{F}, v >_{V',V} = l(v),\ \forall v \in V,\\
&B_{N} : X'_{N} \longrightarrow V', F_{N} \longmapsto < F_{N}, . >_{X'_{N},X_{N}}
\end{aligned}
$$

$$
\left\{
\begin{array}{l}
\text{Find } u \in V, F_{N} \in X'_{N} \text{ satisfying}\\
Au = \mathcal{F} + B_{N}F_{N}, \quad \text{in } V'\\
F_{N} + N_{K_{N}}(u_{N}) \ni 0, \quad \text{in } X'_{N},
\end{array}
\right. \tag{4}
$$

where $K_{N} = \{v_{N} \in X_{N} : v_{N} \leq 0\}$, $N_{K_{N}}(u_{N}) = \partial I_{K_{N}}(u_{N}) = \{F_{N} \in X'_{N} :< F_{N}, w_{N} - u_{N} > \leq 0,\ \forall w_{N} \in K_{N}\}$. The solution to Problem 4 satisfies

$$
u_{N} \leq 0, \quad < F_{N}, v_{N} >_{X'_{N},X_{N}} \geq 0, \quad < F_{N}, u_{N} >_{X'_{N},X_{N}} = 0.
$$

The inclusion of Problem 4 can also be inverted:

$$
\left\{
\begin{array}{l}
\text{Find } u \in V, F_{N} \in X'_{N} \text{ satisfying}\\
Au = \mathcal{F} + B_{N}F_{N}, \quad \text{in } V'\\
u_{N} \in N_{K^*_{N}}(-F_{N}), \quad \text{in } X_{N},
\end{array}
\right. \tag{5}
$$

where K^*_{N} is the polar cone to K_{N}

$$
K^*_{N} = \{F_{N} \in X'_{N} :< F_{N}, v_{N} > \leq 0,\ \forall\ v_{N} \in K_{N}\}.
$$

1.4. Discretization. A discretization of Problem 5 will be the choice of three components. The first choice consists in a finite dimensional discretization space $V^h \subset V$. This finite element discretization is defined on a regular triangulation $\mathcal{T}^h$ of Ω. Then

$$
X^h_{N} = \{v^h_{N|\Gamma_C} : v^h \in V^h\},
$$

is given. The second choice is a discretization X'^h_{N} of X'_{N}. Most of the time $X'^h_{N} \subset L^2(\Gamma_C)$ and we will consider it the case in the following. For instance, a direct discretization leads to X'^h_{N} isomorphic to X^h_{N}. The third choice is a discretization K^{*h}_{N} and K^h_{N} of the cones K^*_{N} and K_{N}. They are linked by the relation

$$
K^h_{N} = \{v^h_{N} \in X^h_{N} : \int_{\Gamma_C} v^h_{N} F^h_{N} d\Gamma \leq 0,\ \forall F^h_{N} \in K^{*h}_{N}\}.
$$

Since the mass matrix on the boundary Γ_C is generally not diagonal, it is not possible to have both $K^h_{N} \subset K_{N}$ and $K^{*h}_{N} \subset K^*_{N}$. In the following, we will see that for polynomial of order greater or equal to two, both the two discretizations are non-conformal in that sense.

The discrete problem is obtained with the Galerkin procedure

$$
\left\{
\begin{array}{l}
\text{Find } u^h \in V^h, F^h_{N} \in X'^h_{N} \text{ satisfying}\\
Au^h = \mathcal{F} + B_{N}F^h_{N}, \quad \text{in } V'\\
u^h_{N} \in N_{K^{*h}_{N}}(-F^h_{N}), \quad \text{in } X^h_{N},
\end{array}
\right. \tag{6}
$$

where $N_{K^{*h}_{N}}(-F^h_{N}) = \{v^h_{N} \in L^2(\Gamma_C) : \displaystyle\int_{\Gamma_C} v^h_{N}(w^h_{N} + F^h_{N}) d\Gamma \leq 0\}$.

A condition for the discretized problem to be well posed is the so-called discrete inf-sup condition:

$$
\inf_{F^h_{N} \in X'^h_{N}} \frac{\sup_{v^h \in V^h} < B_{N}F^h_{N}, v^h >_{V',V}}{\|v^h\|_V \|F^h_{N}\|_{X'_{N}}} \geq C, \tag{7}
$$

where $C > 0$ is a constant preferably independent of h.

1.5. Standard discretizations.

1.5.1. *direct discretization.* The implicit choice of a direct discretization (i.e. a direct Galerkin procedure applied to Problem 1) is the choice of $X_N'^h$ isomorphic to X_N^h (a problem could happen when $\bar{\Gamma}_C \cap \bar{\Gamma}_D \neq \phi$, see [4]). this discretization satisfies the Inf-Sup condition (7) since the operator B_N is represented by the finite element mass matrix on Γ_C.

1.5.2. P_1-P_0 *discretization.* When the finite element space V^h is a classical P_1 continuous element (i.e. continuous and piecewise polynomial of degree ≤ 1 on simplexes) and since the derivative of a P_1 function is a P_0 function, representing the contact force with a P_0 element seems to be a good choice. Unfortunately, the inf-sup condition (7) is often not satisfied, due to the fact that P_0 element may have more degrees of freedom that the continuous P_1 element. This is alway the case for three dimensional problems, and also the case for two-dimensional problems when $\overline{\Gamma_C} \cap \overline{\Gamma_D} \neq \phi$. One solution is to stabilized the finite element with additional bubble functions. A complete study of this problem is presented in [4].

1.5.3. P_2 *Element.* Quadratic finite element often gives a largely better approximation than P_1 element, especially in linear elasticity framework. If one tries to have a conformal discretization of the Signorini problem, one has to consider the following convex of admissible displacements on the contact boundary:

$$K_N^h = \{v_N \in X_N^h : v_N \leq 0\},$$

where

$$X_N^h = \{v_N \in \mathcal{C}^0(\Gamma_C) : v_N|_{T\cap\Gamma_C} \in P_2(T \cap \Gamma_C),\ \forall T \in \mathcal{T}^h\}.$$

Unfortunately, this convex is hard to describe. There is no simple basis of X_N^h such that the set of components which represent K_N^h is itself a convex.

A way to remedy for this is to approximate K_N^h. A standard choice is to prescribe the non-positiveness of the displacement on each finite element node. This gives the convex set:

$$K_N^{h(2)} = \{v_N \in X_N^h : v_N(a_i) \leq 0, \quad \text{for all finite element node } a_i \text{ on } \Gamma_C\},$$

Another possible choice (see [3]) is to prescribe the non-positiveness of the displacement only on the vertices of the elements and to prescribe non-positiveness of the mean value on each edge:

$$\begin{aligned} K_N^{h(3)} \quad = \quad & \{v_N \in X_N^h : v_N(a_i) \leq 0, \quad \text{for all vertex of the mesh } a_i \text{ on } \Gamma_C, \\ & \int_e u^h d\Gamma \leq 0, \quad \text{for all edge } e \text{ of the mesh on } \Gamma_C\}, \end{aligned}$$

This gives two non-conformal discretizations in the sense that $K_N^{h(i)} \not\subset K_N$. The set $K_N^{h(3)}$ has good properties for numerical analysis (see [3] and [6]).

2. Taylor-Hood element for nearly incompressible linearized elasticity

Standard low degree finite element methods applied to linear elasticity problems are known to produce unsatisfactory results when the material becomes almost incompressible (locking phenomena). A way is to consider a mixed formulation which take into account the pressure p as an independent unknown. Let λ and μ be the Lamé coefficients of the material. The incompressibility of the material means that λ is very large, or equivalently that the Poisson coefficient ν is very closed to $\frac{1}{2}$. with

$$a_\lambda(u,v) = \int_\Omega 2\mu\varepsilon(u) : \varepsilon(v)dx + \frac{\lambda}{|\Omega|}\int_\Omega div(u)dx \int_\Omega div(v)dx,$$

$$b(v,p) = \int_\Omega q div(u)dx,$$

the Signorini problem can be written in a mixed formulation as

$$\begin{cases} u \in K_0, p \in L^2_0(\Omega), \\ a_\lambda(u, v-u) + b(v-u, p) \geq l(v-u), \quad \forall v \in K_0, \\ b(u,q) = \dfrac{1}{\lambda}\int_\Omega pq dx, \quad \forall q \in L^2_0(\Omega), \end{cases} \tag{8}$$

where $L^2_0(\Omega) = \{f \in L^2(\Omega) : \int_\Omega f dx = 0\}$.

Taylor-Hood element consists in taking a continuous P_2 element to discretize u and a continuous P_1 element to discretize the pressure p. In [6] the following result is proved:

Theorem 2.1. *When $u \in H^2(\Omega; \mathbb{R}^2)$ and $p \in H^1(\Omega)$ one has*

- *If $K_N^{h(2)}$ is used then*

$$\|u - u^h\|_{1,\Omega} + \|p - p_h\|_{0,\Omega} \leq (1 + \sqrt{\lambda})Ch^{3/4}(\|u\|_{2,\Omega} + \|p\|_{1,\Omega}),$$

- *If $K_N^{h(3)}$ is used then*

$$\|u - u^h\|_{1,\Omega} + \|p - p_h\|_{0,\Omega} \leq Ch^{3/4}(\|u\|_{2,\Omega} + \|p\|_{1,\Omega}),$$

where $C > 0$ does not depend on λ and u^h, p^h are the Taylor-Hood approximated solutions.

This result indicates that, at least for the choice $K_N^{h(3)}$, the quality of the solution does not depend on λ, which means that λ can be arbitrary large. This is not the case for the choice $K_N^{h(2)}$ but we think that this represents only a technical difficulty of the numerical analysis since numerical experiments does not present a degradation when λ is very large.

2.1. Numerical experiments. A rectangular nearly incompressible elastic body is originally at rest on a rigid foundation. It is slightly lifted from its above edge (Dirichlet condition $u = (0, \alpha)$ on the top). Under the effect of it own weight the solid undergoes an elastic deformation and a part of its bottom edge $\Gamma_C = \{0\} \times [0,1]$ may leave the ground. On the vertical edges the body is free of constraints (homogeneous Neumann condition).

Figure 2 shows the reference configuration of the solid (dotted lines) and the configuration after deformation (solid lines, with an exaggerated scale).

The algorithm used to compute the solution of the discrete problem is an Usawa algorithm along with a Polak-Ribière conjugate gradient for the displacement u^h. The finite element code used, GETFEM++, is freely available [7].

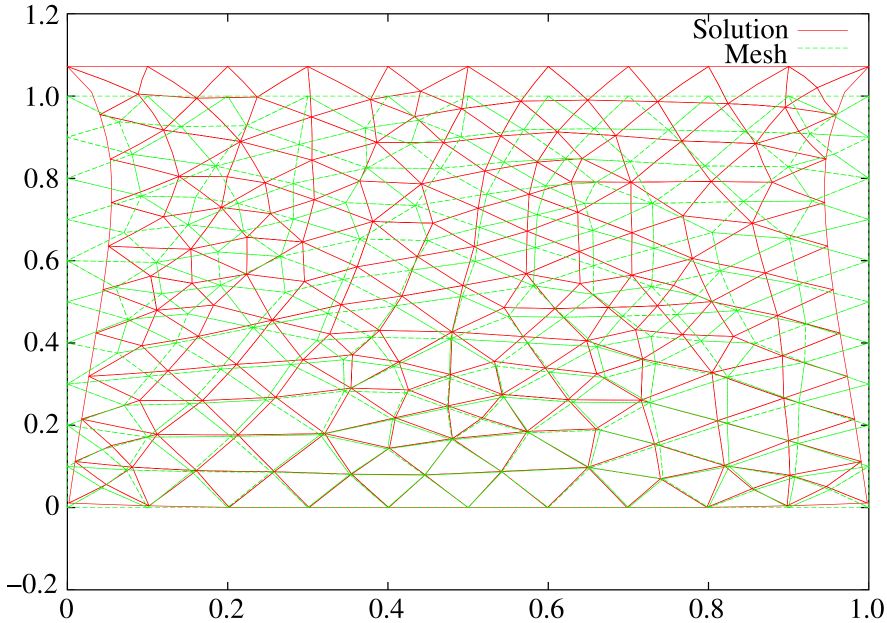

FIGURE 2. reference and deformed configuration for a particular computation

Figure 3 shows the L^∞ norm between a very refined solution and the approximated solution computed with three methods : mixed method with Taylor-Hood element, direct discretization with a P_1 element and with a P_2 element.

The Locking phenomenon is clearly visible when the Poisson coefficient ν goes to $\frac{1}{2}$. The P_2 element has a better behavior but a significant deterioration appears when the number of degrees of freedom increases. The condition number of the stiffness matrix becomes very large and the C.G. is very slow and fails to compute a satisfactory solution. The mixed Taylor-Hood element solution is not affected by the nearly incompressible characteristic and we did not observe any slow down of the C.G. procedure.

Figure 4 and 5 show experiments in dimension 3. The results are quite similar as in dimension two.

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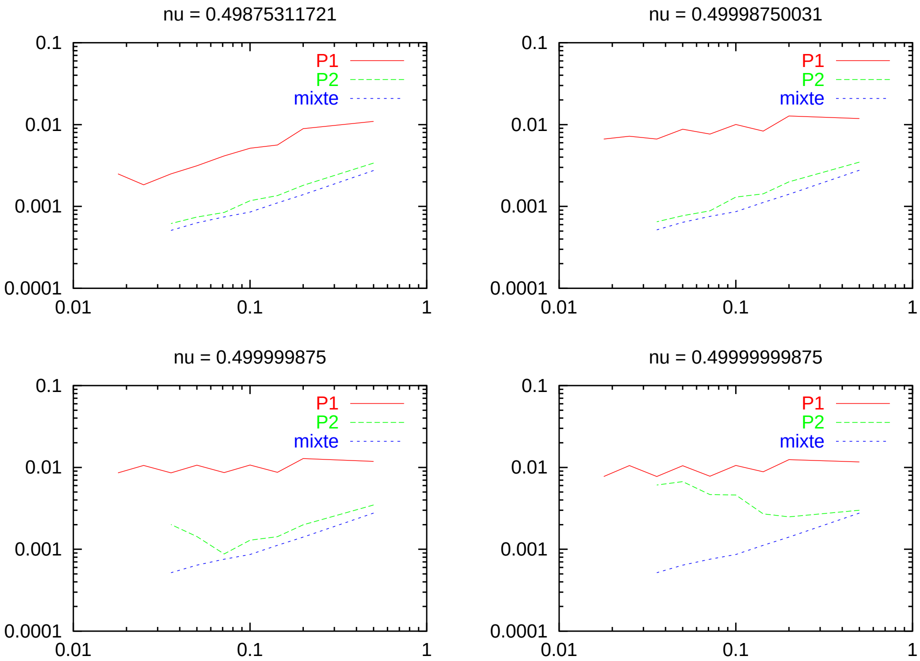

Figure 3. Result for a 2D domain, convergence curves for P1 element, P2 element and mixed Taylor-Hood element for different values of ν. max error vs elements size (h)

[7] Y. Renard, J. Pommier, Getfem++, a C++ generic toolbox for finite element methods, freely distributed under LGPL License, http://www.gmm.insa-tlse.fr/getfem.

(Faker Ben Belgacem) Mathématiques pour l'Industrie et la Physique, UMR 5640, Université Paul Sabatier, 118 route de Narbonne, 31062 Toulouse cedex 04, France
E-mail address: belgacem@mip.ups-tlse.fr

(Yves Renard) Mathématiques pour l'Industrie et la Physique, UMR 5640, Institut National des Sciences Appliquées, Département de Mathématiques, 135 avenue de Rangueil, 31077 Toulouse cedex 04, France
E-mail address: Yves.Renard@gmm.insa-tlse.fr

(Leila Slimane) Université de Moncton, campus de Shippagan, 218, boulevard J.-D.-Gauthier, Shippagan (Nouveau-Brunswick) E8S 1P6, Canada
E-mail address: LSLIMANE@umcs.ca

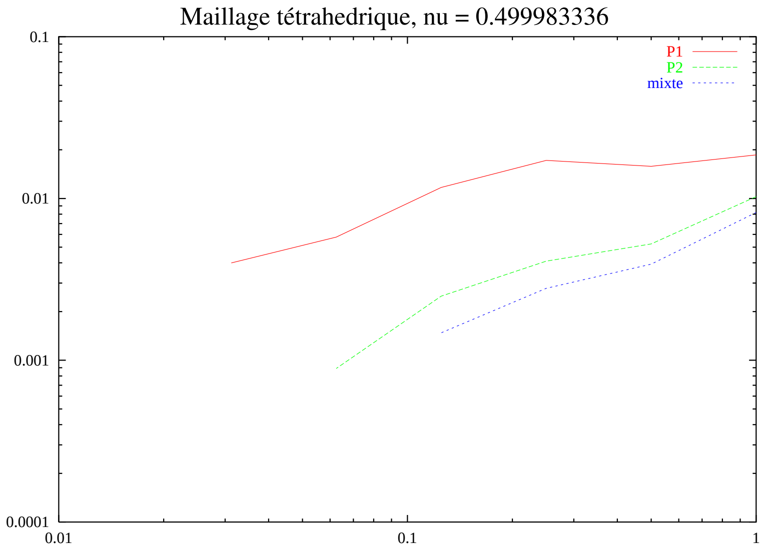

Figure 4. Result for a 3D domain, convergence curves for P1 element, P2 element and mixed Taylor-Hood element. max error vs elements size (h). $\nu = 0.4999833$

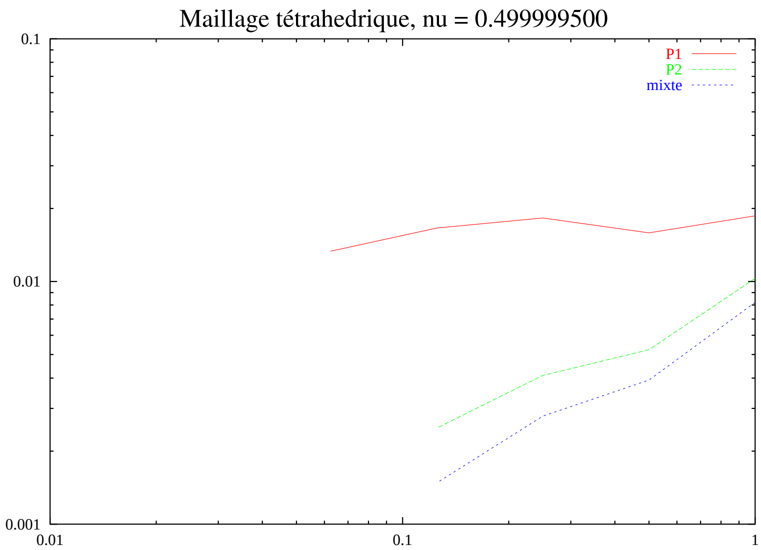

Figure 5. Result for a 3D domain, convergence curves for P1 element, P2 element and mixed Taylor-Hood element. max error vs elements size (h). $\nu = 0.4999995$