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Nonlinear parabolic capacity and renormalized solutions for PDEs with diffuse measure data and variable exponent

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Abstract. We extend the theory of capacity to generalized Sobolev spaces for the study of nonlinear parabolic equations. We introduce the definition and some properties of renormalized solutions and we show that diffuse measure can be decomposed in space and time. As consequence, we show the existence and uniqueness of renormalized solutions. The main technical tools used include estimates and compactness convergence.

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1. Introduction

The concept of capacity is of fundamental importance in the study of solutions of partial differential equations and classical potential theory. For example, a characterization of the relationship between sets and zero parabolic $p-$capacity sets is fundamental. In the stationary case, capacity is related to the underlying Sobolev space, but the situation is more delicate for parabolic partial differential equations. Indeed the theory of capacity seems to be related more closely to the existence and uniqueness of the solution of some elliptic and parabolic problems. The principal prototype for evolution case is the $p-$parabolic equation

$$\begin{cases} u_t - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = \mu & \text{in } (0,T)\times\Omega, \\ u(0,x) = u_0(x) & \text{in } \Omega, \\ u(t,x) = 0 & \text{on } (0,T)\times\partial\Omega, \end{cases} \tag{1}$$

with $1 < p < \infty$, $u \mapsto -\operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the $p-$Laplace operator, and μ is a non-negative Radon measure. When $p = 2$, the thermal capacity related to the heat equation and its generalization have been studied, for example, by Lanconelli [33] and Watson [45]. Capacities defined in terms of functions spaces are introduced in [3, 24, 29, 36, 47]. For non-quadratic case, Droniou, Porretta and Prignet [22], as well as Saraiva [43], introduced and studied the notion of parabolic capacity to get a representation theorem for measures that are zero on subsets of Q of null capacity. One of essential results (Theorem 2.7 below), gives a generalization of the decomposition result using the $p(\cdot)-$parabolic capacity developed in [35] (this extends [22, Theorem 2.28]). In this paper we prove the existence of renormalized solutions to the parabolic problems for arbitrary $\mathcal{M}_0(Q)-$data using the compactness results. The paper is

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organized as follows: in Section 2, we recall some basic properties on Sobolev spaces with variable exponents and $p(\cdot)-$parabolic capacity. In section 3, we state the precise hypotheses on the data and the main results of this paper. We then quickly prove some a priori estimates on the solutions of (1). Finally, in Section 4, we show how these estimates allow to obtain existence of solutions. Our argument will be based on a special type of distributional solutions, the so-called "renormalized solutions" and also on the strong convergence of truncates.

2. Mathematical preliminaries

2.1. Sobolev spaces with variable exponents. We recall some definitions and basic properties of the generalized Lebesgue-Sobolev spaces $L^{p(\cdot)}(\Omega)$, $W^{1,p(\cdot)}(\Omega)$ and $W_0^{1,p(\cdot)}(\Omega)$, where Ω is an open subset of $\mathbb{R}^N$. We refer to Fan and Zhao [26, 27] for further properties on variable exponent Lebesgue-Sobolev spaces. Let $p(\cdot):\overline{\Omega}\to[1,+\infty)$ be a continuous, real-valued function (the variable exponent) and let $p_-=\min_{x\in\overline{\Omega}} p(x)$ and $p_+=\max_{x\in\overline{\Omega}} p(x)$. We define the variable exponent Lebesgue spaces

$$L^{p(\cdot)}(\Omega)=\{u:\Omega\to\mathbb{R}, u \text{ is measurable with } \int_\Omega |u(x)|^{p(x)}dx<\infty\}.$$

We define a norm, the so-called Luxembourg norm, on this space by the formula

$$\|u\|_{L^{p(\cdot)}(\Omega)}=\inf\{\lambda>0\,;\int_\Omega \big|\frac{u(x)}{\lambda}\big|^{p(x)}dx\le 1\}.$$

The following inequality will be used later

$$\min\{\|u\|^{p-}_{L^{p(\cdot)}(\Omega)},\|u\|^{p+}_{L^{p(\cdot)}(\Omega)}\}\le\int_\Omega |u(x)|^{p(x)}dx\le\max\{\|u\|^{p-}_{L^{p(\cdot)}(\Omega)},\|u\|^{p+}_{L^{p(\cdot)}(\Omega)}\}. \quad (2)$$

If $p^->1$, then $L^{p(\cdot)}(\Omega)$ is reflexive and its dual can be identified with $L^{p'(\cdot)}(\Omega)$, where $\frac{1}{p(\cdot)}+\frac{1}{p'(\cdot)}=1$. For any $u\in L^{p(\cdot)}(\Omega)$ and $v\in L^{p'(\cdot)}(\Omega)$, the Hölder type inequality

$$\int_\Omega |uv|dx\le\Big(\frac{1}{p_-}+\frac{1}{p'_-}\Big)\|u\|_{L^{p(\cdot)}(\Omega)}\|v\|_{L^{p'(\cdot)}(\Omega)} \quad (3)$$

holds true. Extending a variable exponent $p(\cdot):\overline{\Omega}\to[1,+\infty)$ to $\overline{Q}=[0,T]\times\overline{\Omega}$ by setting $p(t,x):=p(x)$ for all $(t,x)\in\overline{Q}$, we may also consider the generalized Lebesgue space (which, of course, shares the same type of properties as $L^{p(\cdot)}(\Omega)$)

$$L^{p(\cdot)}(Q)=\{u:Q\to\mathbb{R};\ u \text{ is measurable with } \int_Q |u(t,x)|^{p(x)}d(t,x)<\infty\},$$

endowed with the norm

$$\|u\|_{L^{p(\cdot)}(Q)}=\inf\{\lambda>0\,;\int_Q \big|\frac{u(t,x)}{\lambda}\big|^{p(x)}d(t,x)\le 1\}.$$

We define also the variable Sobolev space

$$W^{1,p(\cdot)}(\Omega)=\{u\in L^{p(\cdot)}(\Omega);|\nabla u|\in L^{p(\cdot)}(\Omega)\}.$$

On $W^{1,p(\cdot)}(\Omega)$ we may consider one of the following equivalent norms

$$\begin{cases}\|u\|_{W^{1,p(\cdot)}(\Omega)}=\|u\|_{L^{p(\cdot)}(\Omega)}+\|\nabla u\|_{L^{p(\cdot)}(\Omega)}\\ \|u\|_{W^{1,p(\cdot)}(\Omega)}=\inf\{\lambda>0\,;\int_\Omega(\big|\frac{\nabla u(x)}{\lambda}\big|^{p(x)}+\big|\frac{u(x)}{\lambda}\big|^{p(x)})dx\le 1\}.\end{cases}$$

We define also $W_0^{1,p(\cdot)}(\Omega) = \overline{C_c^\infty(\Omega)}^{W^{1,p(\cdot)}(\Omega)}$. Assuming $p^- > 1$, the spaces $W^{1,p(\cdot)}(\Omega)$ and $W_0^{1,p(\cdot)}(\Omega)$ are separable and reflexive Banach spaces and the space $W^{-1,p'(\cdot)}(\Omega)$ denotes the dual of $W_0^{1,p(\cdot)}(\Omega)$.

Remark 2.1. The variable exponent $p(\cdot) : \overline{\Omega} \to [1, +\infty)$ satisfies the Lög-continuity condition if

$$\forall x_1, x_2 \in \overline{\Omega}, \quad |x_1 - x_2| < 1, \quad |p(x_1) - p(x_2)| < \omega(|x_1 - x_2|), \tag{4}$$

where $\omega : (0, \infty) \to \mathbb{R}$ is a nondecreasing function with $\limsup_{\alpha \to 0^+} \omega(\alpha) \ln(\frac{1}{\alpha}) < +\infty$. The Log-Hölder continuity condition is used to obtain regularity results on Sobolev spaces with variable exponents; in particular, $C^\infty(\overline{\Omega})$ is dense in $W^{1,p(\cdot)}(\Omega)$ and $W_0^{1,p(\cdot)}(\Omega) = W^{1,p(\cdot)}(\Omega) \cap W_0^{1,1}(\Omega)$. Moreover, if $p(\cdot)$ satisfies the Lög-continuity condition and $1 < p_- \leq p_+ < N$, then the Sobolev embedding holds also for $r(\cdot) = p^*(\cdot)$, i.e. $W^{1,p(\cdot)}(\Omega) \subset L^{p^*(\cdot)}(\Omega)$ where $p^*(x) = \frac{Np(x)}{N - p(x)}$ (see [17] for more details). We do not need these regularity properties to prove our results and will most exclusively work with Lebesgue and Lebesgue-Sobolev spaces with only continuous variable exponents $p(\cdot) : \overline{\Omega} \to [1, +\infty)$ such that $p^- > 1$.

Remark 2.2. Note that the following inequality in general does not hold [26]

$$\int_\Omega |u|^{p(x)} dx \leq C \int_\Omega |\nabla u|^{p(x)} dx.$$

Remark 2.3. Note also that the generalized Lebesgue and Sobolev spaces can also be defined in the same way for only measurable real-valued variable exponents $p(\cdot)$ satisfying $1 \leq p_{\inf} \leq p_{\sup} < \infty$ where $p_{inf} = \text{ess-inf}_{x \in \Omega} p(x)$, $p_{sup} = \text{ess-sup}_{x \in \Omega} p(x)$. According to [26], such variable exponent Lebesgue and Sobolev spaces are Banach spaces, the Hölder type inequality holds, they are reflexive if and only if we have $1 < p_{\inf} \leq p_{\sup} < \infty$. The inclusion between Lebesgue spaces also generalizes naturally: if $0 < |\Omega| < \infty$ and r_1, r_2 are variable exponents so that $r_1(\cdot) \leq r_2(\cdot)$ almost everywhere in Ω, then there exists the continuous embedding $L^{r_2(\cdot)}(\Omega) \hookrightarrow L^{r_1(\cdot)}(\Omega)$ whose norm does not exceed $|\Omega| + 1$. For $u \in W_0^{1,p(\cdot)}(\Omega)$ with $p \in C(\overline{\Omega})$ and $p^- \geq 1$, the Poincaré inequality holds [28]

$$\|u\|_{L^{p(\cdot)}(\Omega)} \leq C \|\nabla u\|_{L^{p(\cdot)}(\Omega)}, \tag{5}$$

for some constant C which depends on Ω and the function $p(\cdot)$. For $p(\cdot) \in C(\overline{\Omega})$ with $1 < p^- \leq p^+ < N$, the Sobolev embedding hold [25]

$$W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{r(\cdot)}(\Omega), \tag{6}$$

for any measurable function $r(\cdot) : \Omega \to [1, +\infty)$ such that $\underset{x \in \Omega}{\text{ess-inf}} \left(\frac{Np(x)}{N - p(x)} - r(x) \right) > 0$.

We will also use the standard notations for Bochner spaces, i.e., if $q \geq 1$ and X is a Banach space, then $L^q(0, T; X)$ denotes the space of strongly measurable functions $u : [0, T] \to X$ for which $t \to \|u(t)\|_X \in L^q(0, T)$. Moreover, $C([0, T]; X)$ denotes the space of continuous functions u endowed with $\|u\|_{C([0,T];X)} := \max_{t \in [0,T]} \|u(t)\|_X$.

2.2. Parabolic Capacity. In this part, we shall mainly work with capacities of compact sets, since we are interested in local properties, we restrict our attention to $U \subset Q$, where U is an open set. Then, we begin with a general definition (in the same spirit of Pierre [36]) of the space $W_{p(\cdot)}(0,T)$ and the parabolic $p(\cdot)-$capacity.

Definition 2.1. Let us define $V = W_0^{1,p(\cdot)}(\Omega) \cap L^2(\Omega)$, endowed with its natural norm $\|.\|_{W_0^{1,p(\cdot)}(\Omega)} + \|\cdot\|_{L^2(\Omega)}$ and the space

$$W_{p(\cdot)}(0,T) = \{u \in L^{p_-}(0,T;V); \nabla u \in L^{p(\cdot)}(Q), u_t \in L^{(p_-)'}(0,T;V')\}$$

endowed with its natural norm

$$\|u\|_{W_{p(\cdot)}(0,T)} = \|u\|_{L^{p_-}(0,T;V)} + \|\nabla u\|_{L^{p(\cdot)}(Q)} + \|u_t\|_{L^{(p_-)'}(0,T;V')}.$$

Definition 2.2. The parabolic $p(\cdot)-$capacity of an arbitrary subset E of Q is

$$\mathrm{cap}_{p(\cdot)}(E) = \inf\{\|u\|_{W_{p(\cdot)}(0,T)}; u \in W_{p(\cdot)}(0,T), u > \chi_U \text{ a.e. in } Q\}. \tag{7}$$

If the set, over which the infimum is taken, is not bounded from above, we set $\mathrm{cap}_{p(\cdot)}(E) = 0$.

Remark 2.4. Notice also that

(i) The parabolic capacity can be expressed in terms of Borelian subsets as

$$\mathrm{cap}_{p(\cdot)}(B) = \inf\{\mathrm{cap}_{p(\cdot)}(U), U \text{ open subset of } Q, B \subset U\}. \tag{8}$$

(ii) It also follows immediately from the definition that if $E_1 \subset E_2$, then

$$\mathrm{cap}_{p(\cdot)}(E_1) \leq \mathrm{cap}_{p(\cdot)}(E_2). \tag{9}$$

Thus, the parabolic capacity is a monotonic set function.

(iii) For E_i, $i \in \mathbb{N}$, be arbitrary subsets of Q and $E = \cup_{i=1}^{\infty} E_i$. Then,

$$\mathrm{cap}_{p(\cdot)}(E) \leq \sum_{i=1}^{\infty} \mathrm{cap}_{p(\cdot)}(E_i). \tag{10}$$

The parabolic capacity is also countably sub-additive.

The next result shows that the capacity is inner regular.

Lemma 2.1. *Let Ω be a bounded subset of $\mathbb{R}^N$ and $1 < p_- < p_+ < \infty$. Then $C_c^\infty([0,T] \times \Omega)$ is dense in $W_{p(\cdot)}(0,T)$.*

Proof. See [35, Proposition 3.3]. □

Definition 2.3. Let K be a compact subset of Q. the capacity of K is defined as

$$\mathrm{cap}_{p(\cdot)}(K) = \inf\{\|u\|_{W_{p(\cdot)}(0,T)} : u \in C_c^\infty([0,T] \times \Omega); u > \chi_K\}.$$

The capacity of any open subset U of Q is then defined by

$$\mathrm{cap}_{p(\cdot)}(U) = \sup\{\mathrm{cap}_{p(\cdot)}(K), K \text{ compact }, K \subset U\}$$

and the capacity of any Borelian set $B \subset Q$ by

$$\mathrm{cap}_{p(\cdot)}(B) = \inf\{\mathrm{cap}_{p(\cdot)}(U), U \text{ open subset of } Q, B \subset U\}.$$

Definition 2.4. A claim is said to hold $\mathrm{cap}_{p(\cdot)}-$quasi everywhere if it holds everywhere except on a set of zero $p(\cdot)-$capacity. A function $u : Q \to \mathbb{R}$ is said to be $\mathrm{cap}_{p(\cdot)}-$quasi continuous if for $\epsilon > 0$, there exists an open set U_ϵ with $\mathrm{cap}_{p(\cdot)}(U_\epsilon) < \epsilon$ such that u restricted to $Q \backslash U_\epsilon$ is continuous.

In fact, the natural space that appears in the study of nonlinear parabolic operators is not $W_{p(\cdot)}(0,T)$ but $\overline{W}_{p(\cdot)}(0,T) \subset W_{p(\cdot)}(0,T)$. Following the outlines of [35]

$$\overline{W}_{p(\cdot)}(0,T) = \{u \in L^{p_-}(0,T; W_0^{1,p(\cdot)}(\Omega)) \cap L^\infty(0,T;L^2(\Omega)); \nabla u \in (L^{p(\cdot)}(Q))^N, \\ u_t \in L^{(p_-)'}(0,T;W^{-1,p'(\cdot)}(\Omega))\}$$

and for all $z \in \overline{W}_{p(\cdot)}(0,T)$, let us denote

$$[z]_{W_{p(\cdot)}(0,T)} = \|z\|^{p_-}_{L^{p_-}(0,T;W_0^{1,p(\cdot)}(\Omega))} + \|z_t\|^{(p_-)'}_{L^{(p_-)'}(0,T;V')} + \|z\|^2_{L^\infty(0,T;L^2(\Omega))}.$$

In [35], the authors has shown the following result that we present in this paper as a Lemma. For the sake of simplicity, we use the notations

$$\begin{cases} [u]_* &= \rho_{p(\cdot)}(|\nabla u|) + \|u_t\|^2_{L^{(p')_-}(0,T;V')} + \|u\|^2_{L^\infty(0,T;L^2(\Omega))} + \|u_t\|^{(p'_-)}_{L^{(p'_-)}(0,T;V')} \\ &\quad + \|u_t\|_{L^{p'_-}(0,T;V')} + \|u_t\|_{L^{(p'_-)}(0,T;V')} \|u\|_{L^\infty(0,T;L^2(\Omega))} \\ [u]_{**} &= \rho_{p(\cdot)}(|\nabla u|) + \|u\|^2_{L^\infty(0,T;L^2(\Omega))} + \|u_t\|^{p'_-}_{L^{p'_-}(0,T;W^{-1,p'(\cdot)}(\Omega))+L^1(Q)} \\ &\quad + \|u_t\|_{L^{p'_-}(0,T;W^{-1,p'(\cdot)}(\Omega))+L^1(Q)} + \|u_t\|_{L^{p'_-}(0,T;W^{-1,p'(\cdot)}(\Omega))+L^1(Q)} \|u\|_{L^\infty(Q)}. \end{cases}$$

Lemma 2.2. *Let $u \in W_{p(\cdot)}(0,T)$, there exists $z \in \overline{W}_{p(\cdot)}(0,T)$ such that $|u| \leq z$ and*

$$[z]_{W_{p(\cdot)}} \leq C([u]_{**} + [u]_{**}^{\frac{1}{p_-}} + [u]_{**}^{\frac{1}{p_+}} + [u]_{**}^{\frac{1}{(p')_-}} + [u]^{\frac{1}{(p')_+}}),$$

where $u \in L^{p_-}(0,T;W_0^{1,p(\cdot)}(\Omega)) \cap L^\infty(Q), u_t \in L^{p_-}(0,T;W^{-1,p'(\cdot)}(\Omega)) + L^1(Q)$ and

$$\|z\|_{\overline{W}_{p(\cdot)}(0,T)} \leq C([u]_*^{\frac{1}{2}} + [u]_*^{\frac{1}{p_-}} + [u]_*^{\frac{1}{p_+}} + [u]_*^{\frac{1}{(p')_-}} + [u]_*^{\frac{1}{(p')_+}}).$$

Now our aim is to prove the following result.

Theorem 2.3. *Let $u \in W_{p(\cdot)}(0,T)$; then u admits a unique $cap_{p(\cdot)}-$quasi continuous representative defined $cap_{p(\cdot)}-$quasi everywhere.*

To prove Theorem 2.3, we need first a capacitary estimate, that is the goal of the following result.

Lemma 2.4. *Let $u \in W_{p(\cdot)}(0,T)$ be $cap_{p(\cdot)}-$quasi continuous, then for every $k > 0$,*

$$cap_{p(\cdot)}(\{|u| > k\}) \leq \frac{c}{k} \max(\|u\|^{\frac{p_-}{p'_-}}_{W_{p(\cdot)}(0,T)}, \|u\|^{\frac{p'_-}{p_-}}_{W_{p(\cdot)}(0,T)}). \tag{11}$$

Proof. See [35, Proposition 3.16]. □

Proof of Theorem 2.3. Let us first observe that we can approximate a function $u \in W_{p(\cdot)}(0,T)$ with smooth functions $u^m \in C_0^\infty([0,T]\times\Omega)$ in the norm of $W_{p(\cdot)}(0,T)$ using convolution arguments; so let u^m be a sequence such that

$$\sum_{m=1}^{\infty} 2^m \max\{\|u^{m+1} - u^m\|^{\frac{p_-}{p'_-}}_{W_{p(\cdot)}(0,T)}, \|u^{m+1} - u^m\|^{\frac{p'_-}{p_-}}_{W_{p(\cdot)}(0,T)}\} \text{ is finite.}$$

For every m and r, let us define

$$\omega^m = \{|u^{m+1} - u^m| > \frac{1}{2^m}\} \quad \text{and} \quad \Omega^r = \underset{m \geq r}{\cup} \omega^m.$$

Now we can apply Lemma 2.4 to obtain

$$\mathrm{cap}_{p(\cdot)}(\omega^m) \leq C\ 2^m \max\{\|u^{m+1} - u^m\|_{W_{p(\cdot)}(0,T)}^{\frac{p_-}{p'_-}}, \|u^{m+1} - u^m\|_{W_{p(\cdot)}(0,T)}^{\frac{p'_-}{p_-}}\}$$

and so, by sub-additivity,

$$\mathrm{cap}_{p(\cdot)}(\Omega^r) \leq C \sum_{m\geq r} 2^m \max\{\|u^{m+1} - u^m\|_{W_{p(\cdot)}(0,T)}^{\frac{p_-}{p'_-}}, \|u^{m+1} - u^m\|_{W_{p(\cdot)}(0,T)}^{\frac{p'_-}{p_-}}\}$$

which implies that

$$\lim_{r\to\infty} \mathrm{cap}_{p(\cdot)}(\Omega^r) = 0. \tag{12}$$

Moreover, for every $y \notin \Omega^r$ we have

$$|u^{m+1} - u^m|(y) \leq \frac{1}{2^m}.$$

For any $m \geq r$, u^m converges uniformly on the complement of Ω^r and pointwise on the complement of $\cap_{r=1}^{\infty}\Omega^r$. But, for any $l \in \mathbb{N}$, we have

$$\mathrm{cap}_{p(\cdot)}(\cap_{r=1}^{\infty}\Omega^r) \leq \mathrm{cap}_{p(\cdot)}(\Omega^l),$$

and so, by (12), we conclude that $\mathrm{cap}_{p(\cdot)}(\cap_{r=1}^{\infty}\Omega^r) = 0$; therefore the limit of u^m is $\mathrm{cap}_{p(\cdot)}-$quasi continuous and is defined $\mathrm{cap}_{p(\cdot)}-$quasi everywhere. Let us denote $\tilde{u}$ this $\mathrm{cap}_{p(\cdot)}-$quasi continuous representative of u, and let z be another $\mathrm{cap}_{p(\cdot)}-$quasi continuous representative of u; thanks to Lemma 2.4, for any $\epsilon > 0$, we have

$$\mathrm{cap}_{p(\cdot)}(\{|\tilde{u} - z| > \epsilon\}) \leq \frac{C}{\epsilon}(\|\tilde{u} - z\|_{W_{p(\cdot)}(0,T)}^{\frac{p_-}{p'_-}}, \|\tilde{u} - z\|_{W_{p(\cdot)}(0,T)}^{\frac{p'_-}{p_-}}) = 0,$$

since $\tilde{u} = z$ in $W_{p(\cdot)}(0,T)$ and this conclude the proof.

2.3. Diffuse measures. We denote by $\mathcal{M}_b(Q)$ the space of bounded measures on the $\sigma-$algebra of Borelian of Q, and $\mathcal{M}_b^+(Q)$ will denote the subsets of nonnegative measures of $\mathcal{M}_b(Q)$, with the symbol $\mathcal{M}_0(Q)$ we mean a measure with bounded variation over Q which does not charge the sets of zero $p(\cdot)-$capacity, this measure μ is called soft or diffuse measure. We refer the reader to [22] for further specifications about parabolic $p-$capacity and to [35] for $p(\cdot)-$capacity.

Definition 2.5. Let E be a subset of Q. the space $\mathcal{M}_0(Q)$ is defined as

$$\mathcal{M}_0(Q) = \{\mu \in \mathcal{M}_b(Q) : \mu(E) = 0, \quad \forall E \subset Q \text{ such that } \mathrm{cap}_{p(\cdot)}(E) = 0\}.$$

We denote by $\langle\langle \cdot, \cdot \rangle\rangle$ the duality pairing between $(W_{p(\cdot)}(0,T))'$ and $W_{p(\cdot)}(0,T)$, if $\gamma \in (W_{p(\cdot)}(0,T))'$ such that there exists $c > 0$ satisfying $\langle\langle \gamma, \varphi \rangle\rangle \leq C\|\varphi\|_{L^\infty(Q)}$ for every function $\varphi \in C_c^\infty(Q)$.Then, $\gamma \in (W_{p(\cdot)}(0,T))' \cap \mathcal{M}_b(Q)$ and is identified by unique linear application $\varphi \in C_c^\infty(Q) \to \int_Q \varphi\gamma^{\mathrm{meas}}$ when γ^{meas} belongs to $\mathcal{M}_b(Q)$. This shows that we need to detail the structure of the dual space $(W_{p(\cdot)}(0,T))'$.

Lemma 2.5. *Let $g \in (W_{p(\cdot)}(0,T))'$, then there exists $g_1 \in L^{p'_-}(0,T; W^{-1,p'(\cdot)}(\Omega))$, $g_2 \in L^{p_-}(0,T;V)$, $F \in (L^{p'(\cdot)}(Q))^N$ and $g_3 \in L^{p'_-}(0,T;L^2(\Omega))$ such that*

$$\langle\langle g, u \rangle\rangle = \int_0^T \langle g_1, u \rangle dt - \int_0^T \langle u_t, g_2 \rangle + \int_Q F \cdot \nabla u \, dxdt + \int_Q g_3 u \, dxdt, \ \forall u \in W_{p(\cdot)}(0,T)$$

and there exist a constant C (do not depend on g) such that

$$\|g_1\|_{L^{p'_-}(0,T;W^{-1,p'(\cdot)}(\Omega))} + \|g_2\|_{L^{p_-}(0,T;V)} + \|F\|_{L^{p'(\cdot)}(Q)} + \|g_3\|_{L^{p'_-}(0,T;L^2(\Omega))} \leq C\,\|g\|_{(W_{p(\cdot)}(0,T))'}.$$

Proof. See [35, Lemma 4.2]. □

The next Lemma will play an essential role in this context (see also [10, 48, 44]).

Lemma 2.6. *Let $\mu \in \mathcal{M}_0(Q)$, there exists a decomposition (g,h) of μ such that $g \in (W_{p(\cdot)}(0,T))'$, $h \in L^1(Q)$ and*

$$\int_Q \varphi d\mu = \langle\langle g, \varphi\rangle\rangle + \int_Q h\varphi dxdt \quad \textit{for all } \varphi \in C_c^\infty([0,T]\times\Omega). \tag{13}$$

Proof. See [35, Lemma 4.4]. □

Finally, the essential tool in our work is the following result.

Theorem 2.7. *Let $\mu \in \mathcal{M}_0(Q)$; there exists a decomposition (f, F, g_1, g_2) of μ with $f \in L^1(Q)$, $F \in (L^{p'(\cdot)}(Q))^N$, $g_1 \in L^{p'_-}(0,T;W^{-1,p'(\cdot)}(\Omega))$, $g_2 \in L^{p_-}(0,T;V)$ and*

$$\int_Q \varphi d\mu = \int_Q f\varphi dxdt + \int_Q F\cdot\nabla\varphi dxdt + \int_0^T \langle g_1,\varphi\rangle dt - \int_0^T \langle \varphi_t, g_2\rangle dt,\ \forall \varphi \in C_c^\infty(Q).$$

Proof. The proof is a combination of the proof of Lemmas 2.5 and 2.6 (see [35, 22]). □

Remark 2.5. In general, the decomposition in $\mathcal{M}_0(Q)$ is not unique.

Indeed, we have the following result.

Lemma 2.8. *Let $\mu \in \mathcal{M}_0(Q)$ and let (f, F, g_1, g_2), $(\tilde f, \tilde F, \tilde g_1, \tilde g_2)$ be two different decompositions of μ according to Theorem 2.7. Then, we have*

$$\int_0^T \langle (g_2-\tilde g_2)_t, \varphi\rangle dt = \int_Q (\tilde f - f)\varphi dxdt + \int_Q (\tilde F - F)\cdot\nabla\varphi dxdt + \int_0^T \langle \tilde g_1 - g_1, \varphi\rangle dt, \tag{14}$$

where $\varphi \in C_c^\infty([0,T]\times\Omega)$ and $g_2 - \tilde g_2 \in C([0,T]\times L^1(Q))$ with $(g_2-\tilde g_2)(0) = 0$.

Proof. See [35, Lemma 4.6]. □

3. Weak and renormalized solutions for problems with $\mathcal{M}_0(Q)-$data and approximation results

3.1. Variational case and weak solution. A large number of papers was devoted to the study of solutions for parabolic problems under various assumptions (for elliptic problems, the reader should consult [20] for more details): for a review on classical parabolic results (see [6, 9, 18, 30] and references therein). In [4, 5, 46] some anisotropic problems with variable exponents are studied and in [30, 23, 1] for weight Sobolev spaces and Orlicz spaces. Moreover, in the case when μ belongs to the dual of the parabolic Sobolev spaces, we refer to [32], see also [2, 11, 42] for L^1-data. General results for a finite Radon measure can be found in [10, 22, 36], another approaches can be found in [40, 41] for diffuse measures and in [37, 39] for singular measures (i.e. general measures). Throughout the paper, we assume that Ω is a bounded open set on $\mathbb{R}^N$, $N \geq 2$, $Q = \Omega\times(0,T)$, $T > 0$, and $a : Q\times\mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function (i.e. $a(\cdot,\cdot,\zeta)$ is measurable on Ω, for all $\zeta \in \mathbb{R}^N$, and $a(t,x,\cdot)$ is continuous on $\mathbb{R}^N$ for a.e. $(t,x) \in Q$) such that the following holds.

$$a(t,x,\zeta)\cdot\zeta \geq \alpha|\zeta|^{p(x)}, \tag{15}$$

$$|a(t,x,\xi)| \leq \beta[b(t,x) + |\zeta|^{p(x)-1}], \tag{16}$$

$$(a(t,x,\zeta) - a(t,x,\eta))\cdot(\zeta-\eta) > 0, \tag{17}$$

for a.e. $(t,x) \in Q$, for all $\zeta,\eta \in \mathbb{R}^N$ with $\zeta \neq \eta$, where $p_- > 1$, α,β are positive constants and b is a nonnegative function in $L^{p'(x)}(\Omega)$. For every $u \in L^{p_-}(0,T;W_0^{1,p(\cdot)}(\Omega))$ with $|\nabla u| \in (L^{p(\cdot)}(Q))^N$, let us define the differential operator $A(u) = -\mathrm{div}(a(t,x,\nabla u))$, which, thanks to the assumptions on a, turns out to be a coercive monotone operator acting from the space $L^{p_-}(0,T;W_0^{1,p}(\Omega))$ into its dual $L^{p'_-}(0,T;W^{-1,p'(\cdot)}(\Omega))$. We shall deal with the solutions of initial boundary-value problem

$$\begin{cases} u_t + A(u) = \mu & \text{in } (0,T)\times\Omega, \\ u(0,x) = u_0(x) & \text{in } \Omega, \\ u(t,x) = 0 & \text{on } (0,T)\times\partial\Omega, \end{cases} \tag{18}$$

where μ is a measure with bounded variation over $Q = (0,T)\times\Omega$, and $u_0 \in L^1(\Omega)$. Let us fix $T > 0$, if $\mu \in L^{p'_-}(0,T;W^{-1,p'(\cdot)}(\Omega))$, it is well known that problem (18) has a unique variational solution in $Q = (0,T)\times\Omega$ such that $u \in W_{p(\cdot)}(0,T) \cap C([0,T];L^2(\Omega))$, that is

$$\int_0^T \langle u_t,\varphi\rangle_{W^{-1,p'(\cdot)}(\Omega),W_0^{1,p(\cdot)}(\Omega)}\,dt + \int_Q a(t,x,\nabla u)\cdot\nabla\varphi\,dx\,dt = \int_0^T \langle \mu,\varphi\rangle_{W^{-1,p'(\cdot)},W_0^{1,p(\cdot)}}dt. \tag{19}$$

We mean that u is a weak solution of (18) if $u \in L^{p_-}(0,T;V)$, $|\nabla u| \in L^{p(\cdot)}(Q)$ and if

$$-\int_Q \langle\varphi_t,u\rangle dt - \int_\Omega u_0\varphi(0)dx + \int_Q a(t,x,\nabla u)\cdot\nabla\varphi dxdt = \langle\langle g,\varphi\rangle\rangle,$$

for any $\varphi \in C_c^\infty([0,T]\times\Omega)$. Since we are going to deal with measures, the solution we will find will not belong in general to Sobolev spaces. For this reason, we are going to justify the interest of $(W_{p(\cdot)}(0,T))'$ by the following existence and uniqueness theorem.

Theorem 3.1. *Let $g \in (W_{p(\cdot)}(0,T))'$, and let $u_0 \in L^2(\Omega)$. Then there exists a unique solution $u \in L^{p_-}(0,T;V)$ of (18) such that for every $\varphi \in W_{p(\cdot)}(0,T)$ with $\varphi(T) = 0$*

$$-\int_Q \langle\varphi_t,u\rangle dt - \int_\Omega u_0\varphi(0) + \int_Q a(t,x,\nabla u)\cdot\nabla\varphi dxdt = \langle\langle g,\varphi\rangle\rangle. \tag{20}$$

Remark 3.1. Since $g \in (W_{p(\cdot)}(0,T))'$, by Lemma 2.5 and (20), we deduce that

$$(u-g_2)_t = -Au + g_1 - div(F) + g_3 \in \; L^{p'_-}(0,T;W^{-1,p'(\cdot)}(\Omega)) + L^{p'_-}(0,T;L^2(\Omega))$$

and then to $L^{p'_-}(0,T;V'))$. Therefore, $u - g_2 \in W_{p(\cdot)}(0,T) \subset C([0,T];L^2(\Omega))$. Then by (20), $(u-g_2)(0) = u_0$. Moreover, for any two solutions u and v of (20), we have $u - v = u - g_2 - (v - g_2) \in W_{p(\cdot)}(0,T)$ and $(u-v)(0) = 0$.

Remark 3.2. Theorem 3.1 could also be stated with right-hand side in $(\overline{W}_{p(\cdot)}(0,T))'$ and test functions in $\overline{W}_{p(\cdot)}(0,T)$. Moreover, according to [19], one has

$$\begin{aligned}\overline{W}_{p(\cdot)}(0,T) = \{&u \in L^{p_-}(0,T;W_0^{1,p(\cdot)}(\Omega)) \cap L^2(0,T;L^2(\Omega)), |\nabla u| \in (L^{p(\cdot)}(Q))^N; \\ &u_t \in L^{(p_-)'}(0,T;W^{-1,p'(\cdot)}(\Omega))\},\end{aligned}$$

hence the right hand side $g_2 \in \overline{W}'_{p(\cdot)}(0,T)$ with $g_2 \in L^{p_-}(0,T;W_0^{1,p(\cdot)}(\Omega)) \supset L^{p_-}(0,T;V)$, the term $\int_0^T \langle \varphi_t, g_2\rangle$ makes sense also when $\varphi \in \overline{W}_{p(\cdot)}(0,T)$.

3.2. Approximating measures. We will argue by density for proving the existence of solutions, so that we need the following preliminary result that applies for equations to obtain additional regularity on the renormalized solutions.

Proposition 3.2. *Let $\mu \in \mathcal{M}_0(Q)$. Then there exists (f,F,g_1,g_2) of μ in the sense of Theorem 2.7 and $\mu^\epsilon \in C_c^\infty(Q)$ such that $\|\mu^\epsilon\|_{L^1(Q)} \le C$ and*

$$\int_Q \mu^\epsilon \varphi dxdt = \int_Q \varphi f^\epsilon dxdt + \int_Q F^\epsilon \cdot \nabla\varphi dxdt + \int_0^t \langle div\, G_1^\epsilon, \varphi\rangle dt - \int_0^t (\varphi, g_2^\epsilon)dt,$$

for every $\varphi \in C_c^\infty([0,T]\times\Omega)$ with (C not depending on ϵ)

$$\begin{cases} f^\epsilon \in C_c^\infty(Q) \text{ such that } \|f^\epsilon - f\|_{L^1(Q)} \le C\epsilon, \\ F^\epsilon \in (C_c^\infty(Q))^N \text{ such that } \|F^\epsilon - F\|_{(L^{p'(\cdot)}(Q))^N} \le C\epsilon, \\ G_1^\epsilon \in (C_c^\infty(Q))^N \text{ such that } \|G_1^\epsilon - G_1\|_{(L^{p'(\cdot)}(Q))^N} \le C\epsilon, \\ g_2^\epsilon \in C_c^\infty(Q) \text{ such that } \|g_2^\epsilon - g_2\|_{L^{p_-}(0,T;V)} \le C\epsilon. \end{cases}$$

Proof. From Definition 2.5, there exists $\gamma \in (W_{p(\cdot)}(0,T))' \cap \mathcal{M}_b^+(\Omega)$ and a non-negative Borel function $f \in C^1(Q, d\gamma^{\text{meas}})$ such that $\mu(B) = \int_B f d\gamma^{\text{meas}}$ for Borel set B in Q. From the fact that $C_c^\infty(Q)$ is dense in $L^1(Q, d\gamma^{\text{meas}})$, since γ^{meas} is a regular measure; there exists a sequence $f_n \in C_c^\infty(Q)$ such that f_n strongly converges to f in $L^1(Q, d\gamma^{\text{meas}})$. Then we can assume $\sum_{n=0}^\infty \|f_n - f_{n-1}\|_{L^1(Q,d\gamma^{\text{meas}})} < \infty$, and we define $\nu_n = (f_n - f_{n-1})\gamma \in (W_{p(\cdot)}(0,T))'$, we have $\nu_n \in (W_{p(\cdot)}(0,T))' \cap \mathcal{M}_b(Q)$ and $\sum_{n=0}^\infty \nu_n^{\text{meas}} = \sum_{n=0}^\infty (f_n - f_{n-1})\gamma^{\text{meas}} = \mu$ converges in the strong topology of measures, $\rho_l * \nu_n^{\text{meas}}$ strongly converges to ν_n in $(W_{p(\cdot)}(0,T))'$ as l tends to infinity, we can then extract a subsequence l_n such that $\|\rho_{l_n} * \nu_n^{\text{meas}} - \nu_n\|_{(W_{p(\cdot)}(0,T))'} \le \frac{1}{2^n}$. We have then $\sum_{k=0}^n \nu_k^{\text{meas}} = \sum_{k=0}^n \rho_{l_k} * \nu_k^{\text{meas}} + \sum_{k=0}^n (\nu_k^{\text{meas}} - \rho_{l_k} * \nu_k^{\text{meas}})$. Let us denote $m_n = \sum_{k=0}^n \nu_k^{\text{meas}}$, $h_n = \sum_{k=0}^n \rho_{l_k} * \nu_k^{\text{meas}}$, $g_n = \sum_{k=0}^n (\nu_k - \rho_{l_k} * \nu_k^{\text{meas}})$ and $g_n^{\text{meas}} = \sum_{k=0}^n (\nu_k^{\text{meas}} - \rho_{l_k} * \nu_k^{\text{meas}})$. We have that h_n strongly converges in $L^1(Q)$ (because $\|\rho_{l_k} * \nu_k^{\text{meas}}\|_{L^1(Q)} \le \|\nu_k^{\text{meas}}\|_{\mathcal{M}_b(Q)}$) and $\sum_{k=0}^\infty \nu_k^{\text{meas}}$ is totally convergent in $\mathcal{M}_b(Q)$; we denote by h its limit, we also have g_n is strongly convergent in $(W_{p(\cdot)}(0,T))'$ (because $\|\rho_{l_k} * \nu_k^{\text{meas}} - \nu_k\|_{(W_{p(\cdot)}(0,T))'} \le \frac{1}{2^k}$), denoting by g its limit. Now, we choose $\zeta_k \in C_c^\infty(Q)$ such that $\zeta_k \equiv 1$ on a neighborhood of $\operatorname{supp}(f_n - f_{n-1})$; then there exists $C(\zeta_k)$ only depending on ζ_k such that

$$\begin{cases} \|\zeta_k h\|_E \le C(\zeta_k)\|h\|_E \text{ if } E \subset \{(L^{p'(\cdot)}(Q))^N, L^{p'_-}(0,T;V), L^{p'_-}(0,T;L^2(\Omega))\} \text{ and } h \in E; \\ \|H\nabla\zeta_k\|_{L^{p'(\cdot)}(Q)} \le C(\zeta_k)\|H\|_{(L^{p'(\cdot)})^N} \text{ if } H \in (L^{p'(\cdot)}(Q))^N; \\ \|(\zeta_k)_t h\|_{L^{p_-}(0,T;L^2(\Omega))} \le C(\zeta_k)\|h\|_{L^{p_-}(0,T;L^2(\Omega))} \text{ if } h \in L^{p_-}(0,T;L^2(\Omega)). \end{cases}$$

We choose l_k such that $\|\rho_{l_k} * \nu_n^{meas} - \nu_k\|_{(W_{p(\cdot)}(0,T))'} \le \frac{1}{(2^k(C(\zeta_k)+1))}$ and $\zeta_k \equiv 1$ on a neighborhood of $\operatorname{supp}(\rho_{l_k} * \nu_k^{meas})$. Thanks to this choice and the decomposition $(b_0^k, div(B_1^k), b_2^k, b_3^k)$ of $\nu_k - \rho_{l_k} * \nu_k^{meas}$, there exists a constant C (not depending on k) such that

$$\begin{aligned} &\|b_0^k\|_{(L^{p'(\cdot)}(Q))^N} + \|B_1^k\|_{(L^{p'(\cdot)}(Q))^N} + \|b_2^k\|_{L^{p_-}(0,T;V)} + \|b_3^k\|_{L^{p'_-}(0,T;L^2(\Omega))} \\ &\le C\|\nu_k - \rho_{l_k} * \nu_k^{meas}\|_{(W_{p(\cdot)}(0,T))'}. \end{aligned}$$

So that we can write

$$\begin{cases} \sum_{k\geq 1}\zeta_k b_0^k \text{ converges in } (L^{p'(\cdot)}(Q))^N,\ \sum_{k\geq 1}\zeta_k B_1^k \text{ converges in } (L^{p'(\cdot)}(Q))^N,\\ \sum_{k\geq 1}\zeta_k b_2^k \text{ converges in } L^{p_-}(0,T;V),\ \sum_{k\geq 1}\zeta_k b_3^k \text{ converges in } L^{p'_-}(0,T;L^2(\Omega)),\\ \sum_{k\geq 1} b_0^k\nabla\zeta_k \text{ converges in } L^{p'(\cdot)}(Q),\ \sum_{k\geq 1} B_1^k\nabla\zeta_k \text{ converges in } L^{p'(\cdot)}(Q),\\ \sum_{k\geq 1}(\zeta_k)_t b_2^k \text{ converges in } L^{p_-}(0,T;L^2(\Omega)). \end{cases} \tag{21}$$

We denote by $F_0, G, -g_2, f_0, f_1, f_2$ and f_3 the respective limits of the terms above; (21) imply the convergence in $L^1(Q)$. Since $\nu_k - \rho_{l_k} * \nu_k^{meas} = \zeta_k(\nu_k - \rho_{l_k} * \nu_k^{meas})$ in $(W_{p(\cdot)}(0,T))'$ and thanks to the choice of ζ_k and ρ_k and the decomposition (b_0^k, $div(B_1^k)$, b_2^k, b_3^k) of $\nu_k - \rho_{l_k} * \nu_\epsilon^{meas}$, the last term admits a pseudo-decomposition $(\zeta_k b_0^k, \zeta_k B_1^k, \zeta_k b_2^k, \zeta_k b_3^k, -b_0^k\nabla\zeta_k, -B_1^k, (\zeta_k)_t b_2^k)$. Thus, as $\int_Q \varphi dm_n = \int_Q h_n\varphi dxdt + \langle g_n, \varphi\rangle$, we can write for all $\varphi \in C_c^\infty([0,T]\times\Omega)$,

$$\begin{aligned}\int_Q \varphi dm_n = \int_Q \varphi h_n + \int_0^t \langle div(\sum_{k=0}^n \zeta_k b_0^k), \varphi\rangle + \int_0^t \langle div(\sum_{k=0}^n \zeta_k B_1^k), \varphi\rangle + \int_0^t \langle \varphi_t, \sum_{k=0}^n \zeta_k b_2^k\rangle\\ + \int_0^t \sum_{k=0}^n \zeta_k b_3^k\varphi + \int_Q \sum_{k=0}^n (-F_0^k\nabla\zeta_k)\varphi + \int_Q \sum_{k=0}^n (-B_1^k\nabla\zeta_k)\varphi + \int_Q \sum_{k=0}^n (\zeta_k)_t b_2^k\varphi.\end{aligned}$$

From the convergences of m_n to μ, of h_n to h and using (21), we have

$$\int_Q \varphi d\mu = \int_Q (h + f_0 + f_1 - f_2 + f_3)\varphi + \int_0^t F\cdot\nabla\varphi + \int_0^t \langle div(G), \varphi\rangle - \int_0^T (\varphi_t, g_2).$$

That is $(f = h + f_0 + f_1 - f_2 + f_3, F, \mathrm{div}(G), g_2)$ is a decomposition of μ in the sense of Theorem 2.7. Taking n large enough and $\epsilon > 0$ fixed, we obtain

$$\begin{cases} \|\sum_{k=0}^n \zeta_k b_0^k - F\|_{(L^{p'(\cdot)}(Q))^N} \leq \epsilon,\\ \|\sum_{k=0}^n \zeta_k B_1^k - G_1\|_{(L^{p'(\cdot)}(Q))^N} \leq \epsilon,\\ \|\sum_{k=0}^n \zeta_k b_2^k + g_2\|_{L^{p_-}(0,T;V)} \leq \epsilon,\\ \|h_n + \sum_{k=0}^n \zeta_k b_3^k - \sum_{k=0}^n (b_0^k\nabla\zeta_k) - \sum_{k=0}^n (b_1^k\nabla\zeta_k) + \sum_{k=0}^n (\zeta)_t b_2^k - f\|_{L^1(Q)} \leq \epsilon. \end{cases} \tag{22}$$

Note that $\nu_k - \rho_{l_k} * \nu_k^{meas} = \zeta_k(\nu_k - \rho_{l_k} * \nu_k^{meas})$ and $(b_0^k, div(B_1^k), b_2^k, b_3^k)$ is a decomposition of $\nu_k - \rho_{l_k} * \nu_k^{meas}$, note also that, for j large enough, $((\zeta_k b_0^k) * \rho_j, (\zeta_k B_1^k) * \rho_j, (\zeta_k b_2^k) * \rho_j, (\zeta_k b_3^k) * \rho_j, (-f_0^k\nabla\zeta_k) * \rho_j, ((\zeta_k)_t b_2^k) * \rho_j)$ is a pseudo decomposition of $(\nu_k^{meas} - \rho_{l_k} * \nu_k^{meas}) * \rho_j \in C_c^\infty(Q)$. We take j_n such that, for all $k \in [0,n]$,

$$\begin{cases} \|(\zeta_k b_0^k) * \rho_{j_n} - \zeta_k b_0^k\|_{(L^{p'(\cdot)}(Q))^N} \leq \frac{\epsilon}{n+1},\\ \|(\zeta_k B_1^k) * \rho_{j_n}\| - \zeta_k B_1^k\|_{(L^{p'(\cdot)}(Q))^N} \leq \frac{\epsilon}{n+1},\\ \|(\zeta_k b_2^k) * \rho_{j_n} - \zeta_k b_2^k\|_{L^{p_-}(0,T;V)} \leq \frac{\epsilon}{n+1},\\ \|(\zeta_k b_3^k) * \rho_{j_n} - \zeta_k b_3^k\|_{L^1(Q)} + \|(b_0^k\nabla\zeta_k) * \rho_{j_n} - b_0^k\nabla\zeta_k\|_{L^1(Q)}\\ +\|(B_1^k\nabla\zeta_k) * \rho_{j_n} - B_1^k\nabla\zeta_k\|_{L^1(Q)} + \|(\zeta_k)_t b_2^k) * \rho_{j_n} - (\zeta_k)_t b_2^k\|_{L^1(Q)} \leq \frac{\epsilon}{n+1}. \end{cases} \tag{23}$$

Defining

$$\begin{cases} F^\epsilon = & \sum_{k=0}^n (\zeta_k b_0^k) * \rho_{j_n} \in (C_c^\infty(Q))^N,\\ G_1^\epsilon = & \sum_{k=0}^n (\zeta_k B_1^k) * \rho_{j_n} \in (C_c^\infty(Q))^N,\\ g_2^\epsilon = & -\sum_{k=0}^n (\zeta_k b_2^k) * \rho_{j_n} \in C_c^\infty(Q),\\ f^\epsilon = & h_n + \sum_{k=0}^n (\zeta_k b_3^k) * \rho_{j_n} - \sum_{k=0}^n (f_0^k\nabla\zeta_k) * \rho_{j_n}\\ & + \sum_{k=0}^n (B_1^k\nabla\zeta_k) * \rho_{j_n} + \sum_{k=0}^n ((\zeta_k)_t b_2^k) * \rho_{j_n} \in C_c^\infty(Q). \end{cases}$$

Then by (22)–(23), we get

$$\begin{cases} \|F^\epsilon - F\|_{(L^{p'(\cdot)}(Q))^N} \leq 2\epsilon, & \|G_1^\epsilon - G_1\|_{(L^{p'(\cdot)}(Q))^N} \leq 2\epsilon, \\ \|g_2^\epsilon - g_2\|_{L^{p_-}(0,T;V)} \leq 2\epsilon, & \|f^\epsilon - f\|_{L^1(Q)} \leq 2\epsilon. \end{cases}$$

Let us write μ^ϵ as follows $\mu^\epsilon = f^\epsilon + F^\epsilon + \mathrm{div}(G_1^\epsilon) + (g_2^\epsilon)_t \in C_c^\infty(Q)$; it remains to prove that $\|\mu^\epsilon\|_{L^1(Q)} \leq C$ with C not depending on ϵ. To see this, we recall that $((\zeta_k b_0^k) * \rho_{j_n}, (\zeta_k B_1^k) * \rho_{j_n}, (\zeta_k b_2^k) * \rho_{j_n}, (\zeta_k b_3^k) * \rho_{j_n}, (-f_0^k \nabla\zeta_k) * \rho_{j_n}, (-B_1^k \nabla\zeta_k) * \rho_{j_n}, ((\zeta_k)_t b_2^k) * \rho_{j_n})$ is a pseudo-decomposition of $(\nu_k^{meas} - \rho_{l_k} * \nu_k^{meas}) * \rho_{j_n}$, we have

$$\begin{aligned} \mu^\epsilon =& h_n + \sum_{k=0}^{n} (\nu_k^{meas} - \rho_{l_k} * \nu_k^{meas}) * \rho_{j_n} \\ =& h_n + (\sum_{k=0}^{n} (\nu_k^{meas} - \rho_{l_k} * \nu_k^{meas})) * \rho_{j_n} = h_n + g_n^{meas} * \rho_{j_n}. \end{aligned}$$

According to [22], $g_n^{meas} = m_n - h_n$. Then, it follows that $\|\mu^\epsilon\|_{L^1(Q)} \leq 2\|h_n\|_{L^1(Q)} + \|m_n\|_{\mathcal{M}_b(Q)}$. Since h_n converges in $L^1(Q)$ and m_n converges in $\mathcal{M}_b(Q)$, $\|h_n\|_{L^1(Q)}$ and $\|m_n\|_{\mathcal{M}_b(Q)}$ are bounded and we have the desired majoration on $\|\mu^\epsilon\|_{L^1(Q)}$.

□

3.3. Definition of renormalized solutions and main result. The notion of renormalized solutions was first introduced by DiPerna and Lions in [18] for the study of Boltzmann equation. It was then adapted to the study of some nonlinear elliptic and parabolic problems in fluid mechanics. We refer to [20] (see also [11, 13] for details). At the same time the notion of entropy solutions has been proposed by Bénilan and al. in [8] for nonlinear elliptic problems. This framework was extended to related problems with measures as data in [10, 42]. Recently, for elliptic problems and in [44] Sanchón and Urbano studied a Dirichlet problem of $p(x)-$Laplace equation and obtained the existence and uniqueness of entropy solutions for L^1- data, as well as integrability results for the solution and its gradient. The proofs rely crucially on a priori estimates in Marcinkiewicz spaces with variable exponents. Besides, Bendahmane and Wittbold in [16] proved the existence and uniqueness of renormalized solutions to nonlinear elliptic equations with variable exponents and L^1-data. Let $\mu \in \mathcal{M}_0(\Omega)$ be a measure with bounded variation over Q which does not charge the sets of zero elliptic $p(x)-$capacity; we call u a renormalized solution for the $p(x)-$Laplacian problem

$$-\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u) = \mu \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \tag{24}$$

if u is a function in $L^1(\Omega)$, satisfying the following conditions:

$$T_k(u) \in W_0^{1,p(\cdot)}(\Omega) \text{ for any } k > 0, \quad \lim_{n\to\infty} \int_{\{n\leq|u|\leq n+1\}} |\nabla u|^{p(x)} dx = 0.$$

For any renormalization $S \in C^\infty(\mathbb{R})$ such that supp $S' \subset [-M, M]$ for some $M > 0$,

$$-\mathrm{div}(S'(u)|\nabla u|^{p(x)-2}\nabla u) + S''(u)|\nabla u|^{p(x)} = fS'(u) + G \cdot \nabla u S''(u) - \mathrm{div}(S'(u)G)$$

holds in $D'(\Omega)$. For parabolic problems, we are naturally led to introduce the functional space

$$\begin{aligned} X = \{u : \overline{\Omega} \times (0,T) \to \mathbb{R} \text{ is measurable } |T_k(u) \in L^{p_-}(0,T; W_0^{1,p(\cdot)}(\Omega)), \\ \text{with } |\nabla T_k(u)| \in (L^{p(\cdot)}(Q))^N, \text{ for every } k > 0\}, \end{aligned} \tag{25}$$

which, endowed with the norm (or, the equivalence norm)

$$\|u\|_X := \|\nabla u\|_{L^{p(\cdot)}(Q)}, \text{ or } \|u\|_X := \|u\|_{L^{p_-}(0,T;W_0^{1,p(\cdot)}(\Omega))} + \|\nabla u\|_{L^{p(\cdot)}(Q)},$$

X is a separable and reflexive Banach space. The equivalence of the two norms is an easy consequence of the continuous embedding $L^{p(\cdot)}(Q) \hookrightarrow L^{p_-}(0,T;L^{p(\cdot)}(\Omega))$ and the Poincaré inequality. We state some further properties of X in the following lemma.

Lemma 3.3. *(i) We have the following continuous dense embeddings:*

$$L^{p_+}(0,T;W_0^{1,p(\cdot)}(\Omega)) \hookrightarrow X \hookrightarrow L^{p_-}(0,T;W_0^{1,p(\cdot)}(\Omega)). \tag{26}$$

In particular, since $\mathcal{D}(Q)$ is dense in $L^{p_+}(0,T;W_0^{1,p(\cdot)}(\Omega))$, it is dense in X and for the corresponding dual space we have

$$L^{(p_-)'}(0,T;(W_0^{1,p(\cdot)}(\Omega))^*) \hookrightarrow X^* \hookrightarrow L^{(p_+)'}(0,T;(W_0^{1,p(\cdot)}(\Omega))^*). \tag{27}$$

(ii) One can represent the elements of X^* as follows: If $T \in X^*$, then there exists $F = (f_1, \cdots, f_N) \in (L^{p'(\cdot)}(Q))^N$ such that $T = div_x F$ and $\langle T, \zeta\rangle_{X^*,X} = \int_0^T \int_\Omega F \cdot \nabla\zeta dx\, dt$, for every $\zeta \in V$. Moreover, we have $\|T\|_{X^*} = \max\{\|f_i\|_{L^{p(\cdot)}(Q)}, i = 1, \dots, n\}$.

Proof. See [16, Lemma 3.1]. □

For any non-negative real number k we denote by $T_k(s) = \max(-k, \min(k, s))$ the truncation function at level k and its primitive function $\Theta_k(z) = \int_0^z T_k(s)\,ds$. A function v such that $T_k(v) \in X$, for all $k > 0$, does not necessarily belongs to $L^1(0,T;W_0^{1,1}(\Omega))$. Thus ∇v in our equations is defined in a very weak sense.

Definition 3.1. For every measurable function $v : \overline{\Omega} \times (0,T) \to \mathbb{R}$ such that $T_k(v) \in X, \forall k > 0$, there exists a unique measurable function $w : Q \to \mathbb{R}^N$, called the very weak gradient of v and denoted by $w = \nabla v$, such that $\nabla T_k(v) = w\chi_{\{|v|<k\}}$ a.e. in Ω, where χ_E denotes the characteristic function of a measurable set E. Moreover, if v belongs to $L^1(0,T;W_0^{1,1}(\Omega))$, then w coincides with the weak gradient of v.

Now, let us define $\mu_0 = \mu - g_2 = f + F - \operatorname{div}(G)$ where g_2 is the time-derivative part of μ. In view of the definition given in [22] and the preceding remarks, we have the following definition.

Definition 3.2. Let $\mu \in \mathcal{M}_0(Q)$ and $u_0 \in L^1(\Omega)$. We say that a measurable function u is a renormalized solution of the problem (18) if, for all k, $T > 0$, we have

$$u - g_2 \in L^\infty(0,T;L^1(\Omega)), \quad T_k(u) \in X, \tag{28}$$

$$\lim_{n\to\infty} \int_{\{n\le|u-g_2|\le n+1\}} |\nabla u|^{p(x)} dxdt = 0. \tag{29}$$

Moreover, for all $S \in W^{2,\infty}(\mathbb{R})$ such that S' has compact support,

$$\begin{aligned} &-\int_Q S(u_0)\varphi(0)dx - \int_0^T \langle \varphi_t, S(u-g_2)\rangle dt + \int_Q S'(u-g_2)a(t,x,\nabla u)\cdot\nabla\varphi dxdt \\ &+ \int_Q S''(u-g_2)a(t,x,\nabla u)\cdot\nabla(u-g_2)\varphi dxdt = \int_Q S'(u-g_2)\varphi d\mu_0, \end{aligned} \tag{30}$$

$\forall \varphi \in L^{p_-}(0,T;W_0^{1,p(\cdot)}(\Omega)) \cap L^\infty(Q)$ with $\nabla\varphi \in (L^{p(\cdot)}(Q))^N$, $\varphi_t \in L^{p'_-}(0,T;W^{-1,p'(\cdot)}(\Omega))$ with $\varphi(T) = 0$ such that $S'(u-g_2)\varphi \in X$, and

$$S(u-g_2)(0) = S(u_0) \text{ in } L^1(\Omega). \tag{31}$$

Remark 3.3. Notice that, thanks to our regularity assumptions and the choice of S', all terms in (30) are well defined, also observe that (30) implies that equation

$$\begin{aligned}(S(u-g_2))_t - \operatorname{div}(a(t,x,\nabla u)S'(u-g_2)) + S''(u-g_2)a(t,x,\nabla u)\cdot\nabla(u-g_2)\\ = S'(u-g_2)f + S''(u-g_2)F\cdot\nabla(u-g_2) - \operatorname{div}(FS'(u-g_2))\\ +S''(u-g_2)G\cdot\nabla(u-g_2) - \operatorname{div}(GS'(u-g_2))\end{aligned} \tag{32}$$

is satisfied in the sense of distributions since $T_k(u-g_2)$ belongs to X for every $k>0$ and since S' has compact support. Indeed by taking M such that Supp $S'\subset(-M,M)$, since $S'(u-g_2)=S''(u-g_2)=0$ as soon as $|u-g_2|\geq M$, we can replace, everywhere in (30), $\nabla(u-g_2)$ by $\nabla T_M(u-g_2)\in(L^{p(\cdot)}(Q))^N$ and ∇u by $\nabla(T_M(u-g_2))+\nabla g_2\in(L^{p(\cdot)}(Q))^N$. Moreover, according to the assumption (16) and the definition of ∇u, $\nabla u=\nabla(u-g_2)+\nabla g_2$, we have $\nabla(u-g_2)$ is well defined and $|a(t,x,\nabla u)|\in L^{p'(x)}(Q)$.
We also have, for all S as above, $S(u-g_2)=S(T_M(u-g_2))\in X$ and $S'(u-g_2)f\in L^1(Q)$, $S'(u-g_2)F\in L^{p'(\cdot)}(Q)$, $S'(u-g_2)G_1\in L^{p'(\cdot)}(Q)$, $S'(u-g_2)a(t,x,\nabla u)\in(L^{p'(\cdot)}(Q))^N$, $S''(u-g_2)a(t,x,\nabla u)\cdot\nabla(u-g_2)\in L^1(Q)$, $S''(u-g_2)F\cdot\nabla(u-g_2)\in L^1(Q)$ and $S''(u-g_2)G_1\cdot\nabla(u-g_2)\in L^1(Q)$. Thus, by equation (32), $(S(u-g_2))_t$ belongs to the space $X'+L^1(Q)$, and therefore $S(u-g_2)$ belongs to $C([0,T];L^1(\Omega))$, one can say that the initial datum is achieved in a weak sense, that is $S(u-g_2)(0)=S(u_0)$ in $L^1(\Omega)$ for every renormalization S. Note also that, since $S(u-g_2)_t\in X^*+L^1(Q)$, we can use in (30) not only functions in $C_0^\infty(Q)$ but also in $X\cap L^\infty(Q)$.

Remark 3.4. Observe that (29) implies

$$\lim_{n\to\infty}\int_{\{n\leq|u-g_2|\leq n+c\}}|\nabla(u-g_2)|^{p(x)}dxdt=0,\ \text{for all } c>0. \tag{33}$$

Remark 3.5. Let us denote by $v=u-g_2$ the solution of (18), since $S(v)\in X\cap L^\infty(Q)$ and $(S_n(v))_t\in X^*+L^1(Q)$ and thanks to Theorem 2.3, $S_n(v)$ turns out to admit a $\mathrm{cap}_{p(\cdot)}$−quasi continuous representative finite $\mathrm{cap}_{p(\cdot)}$−quasi everywhere.

For classical Sobolev spaces, the definition of renormalized solution does not depend on the decomposition of the measures μ as shown in [22, Proposition 3.10]. Next result try to stress the fact that even for generalized Sobolev spaces this fact should be true.

Proposition 3.4. *Let u be a renormalized solution of (1). Then u satisfies Definition 3.2 for every $(\tilde f,\tilde F,-div(\tilde G_1),\tilde g_2)$ such that $g_2-\tilde g_2\in L^{p_-}(0,T;W_0^{1,p(\cdot)}(\Omega))\cap L^\infty(Q)$.*

Proof. Assume that u satisfies Definition 3.2 for $(f,F,-\mathrm{div}G,g_2)$ and let $(\tilde f,\tilde F,-\mathrm{div}\tilde G,\tilde g_2)$ be a different decomposition of μ_0 such that $g_2-\tilde g_2$ is bounded. Thanks to Lemma 2.8, we have $\tilde v=\tilde u-\tilde g_2\in L^\infty(0,T;L^1(\Omega))$; to prove that $T_k(u-\tilde g_2)\in L^{p_-}(0,T;W_0^{1,p(\cdot)}(\Omega))$ and $\nabla T_k(u-\tilde g_2)\in L^{p(\cdot)}(Q)$ with $k>0$ we can reason as in [22] by choosing $S=S_n$ and $T_k(S_n(u-g_2)+g_2-\tilde g_2)\in X\cap L^\infty(Q)$ as test function in

(30). Lemma 2.8 implies

$$
\begin{aligned}
&\int_0^T \langle (S_n(u-g_2)+g_2-\tilde{g}_2)_t, T_k(S_n(u-g_2)+g_2-\tilde{g}_2)\rangle dt && (A)\\
&+\int_Q S_n'(u-g_2)a(t,x,\nabla u)\nabla T_k(S_n(u-g_2)+g_2-\tilde{g}_2)dxdt && (B)\\
&=-\int_Q S_n''(u-g_2)a(t,x,\nabla u)\nabla(u-g_2)T_k(S_n(u-g_2)+g_2-\tilde{g}_2)dxdt && (C)\\
&+\int_Q ((S_n'(u-g_2)-1)f+\tilde{f})T_k(S_n(u-g_2)+g_2-\tilde{g}_2)dxdt && (D)\\
&+\int_Q (S_n'(u-g_2)-1)F+\tilde{F})\nabla T_k(S_n(u-g_2)+g_2-\tilde{g}_2)dxdt && (E)\\
&+\int_Q ((S_n'(u-g_2)-1)G_1+\tilde{G}_1)\nabla T_k(S_n(u-g_2)+g_2-\tilde{g}_2) && (F)\\
&+\int_Q S_n''(u-g_2)F\nabla(u-g_2)T_k(S_n(u-g_2)+g_2-\tilde{g}_2) && (G)\\
&+\int_Q S_n''(u-g_2)G\nabla(u-g_2)T_k(S_n(u-g_2)+g_2-\tilde{g}_2)dxdt && (H)
\end{aligned}
\tag{34}
$$

Let us analyze term by term the above identity. First of all, concerning the first term of (34) we integrate in time to get

$$
\begin{aligned}
(A) &= \int_0^T \langle (S_n(u-g_2)+g_2-\tilde{g}_2)_t, T_k(S_n(u-g_2)+g_2-\tilde{g}_2)\rangle dt\\
&= \Big[\int_\Omega \Theta_k(S_n(u-g_2)+g_2-\tilde{g}_2)dx\Big]_0^T\\
&= \int_\Omega \Theta_k(S_n(u-g_2)(T)+(g_2-\tilde{g}_2)(T)dx - \int_\Omega \Theta_k(S_n(u-g_2)(0)+(g_2-\tilde{g}_2)(0)dx.
\end{aligned}
$$

Since $S_n(u-g_2)(0)=S_n(u_0)$ and $(g_2-\tilde{g}_2)(0)=0$, we have $S_n(u-g_2)(0)+(g_2-\tilde{g}_2)(0)=S_n(u_0)$, so that using $0\le \Theta_k(s)\le k(s)$, the first term of (34), $(A)\le k\|u_0\|_{L^1(\Omega)}$. On the other hand, since $|S_n''(s)|\le 1$ and $S_n''(s)\ne 0$ if $|s|\in[n,n+1]$, using (16) and Young's inequality we obtain

$$
\begin{aligned}
|(C)+(G)+(H)| &\le \beta k\|S_n'(s)\|_{L^\infty(\mathbb{R})}\int_{\{n\le|u-g_2|\le n+1\}} |(b(t,x)+|\nabla u|^{p(x)-1})||\nabla(u-g_2)|\\
&\le Ck\Big[\int_{\{n\le|u-g_2|\le n+1\}} \frac{p^+-1}{p_-}(|b(t,x)|^{p'(x)}+|G_1|^{p'(x)}+|\nabla u|^{p'(x)(p(x)-1)})\\
&\quad+\int_{\{n\le|u-g_2|\le n+1\}}(|\nabla u|^{p(x)}+|\nabla g_2|^{p(x)})dxdt\Big]\\
&\le Ck\Big[\int_{\{n\le|u-g_2|\le n+1\}}(|b(t,x)|^{p'(x)}+|F|^{p'(x)}+|G_1|^{p'(x)}+|\nabla g_2|^{p'(x)})\\
&\quad+\int_{\{n\le|u-g_2|\le n+1\}}|\nabla u|^{p(x)}dxdt\Big].
\end{aligned}
$$

By the fact that meas$\{n\le|u-g_2|\le n+1\}\underset{n\to\infty}{\to}0$ and using (29), we get $|(C)+(G)+(H)|\le\omega(n)$, where $\omega(n)$ tends to zero as $n\to\infty$. Now, if $E_n=\{|S_n(u-g_2)+$

$g_2 - \tilde{g}_2| \le k\}$ we have (recalling that if $0 \le S_n'(s) \le 1$ then $|S_n'(s)|^{p'(x)} \le S_n'(s)$),

$$\begin{aligned}
|(D)+(E)+(F)| &\le \int_Q (|f|+|\tilde{f}|)|T_k(S_n(u-g_2)+g_2-\tilde{g}_2)|dxdt \\
&+ \int_{E_n} (|F|+|\tilde{F}|)(S_n'(u-g_2)|\nabla(u-g_2)|+|\nabla g_2|+|\nabla \tilde{g}_2|)dxdt \\
&+ \int_{E_n} (|G_1|+|\tilde{G}_1|)(S_n'(u-g_2)|\nabla(u-g_2)|+|\nabla g_2|+|\nabla \tilde{g}_2|)dxdt \\
&\le\ k(\|f\|_{L^1(Q)}+\|\tilde{f}\|_{L^1(Q)}) + \int_{E_n} (|F_1|+|\tilde{F}_1|)S_n'(u-g_2)|\nabla u|dxdt \\
&+ 2\int_Q (|F_1|+|\tilde{F}_1|)(|\nabla g_2|+|\nabla \tilde{g}_2|)dxdt + \int_{E_n} (|G_1|+|\tilde{G}_1|)S_n'(u-g_2)|\nabla u|dxdt \\
&+ 2\int_Q (|G_1|+|\tilde{G}_1|)(|\nabla g_2|+|\nabla \tilde{g}_2|)dxdt \\
&\le\ k(\|f\|_{L^1(Q)}+\|\tilde{f}\|_{L^1(Q)}) + 2\frac{p^+-1}{p_-}\int_Q |F|^{p'(x)}+|\tilde{F}|^{p'(x)}+|G_1|^{p'(x)}+|\tilde{G}_1|^{p'(x)}dxdt \\
&+ \frac{2}{p_-}\int_{\{n\le|u-g_2|\le n+1\}} |\nabla u|^{p(x)}dxdt + \frac{2}{p_-}\int_Q |\nabla g_2|^{p(x)}+|\nabla \tilde{g}_2|^{p(x)}dxdt \le\ C+\omega(n).
\end{aligned}$$

□

Our main result is the following Theorem.

Theorem 3.5. *Let $1 \le p_- \le p_+ < N$, $\mu \in \mathcal{M}_0(Q)$ and $u_0 \in L^1(\Omega)$, assume that $p_- > \frac{2N+1}{N+1}$. Assume that (15)–(17) hold true. Then there exists a renormalized solution u of problem (18).*

4. Proof of main result

We can now start the proof of the existence result (Theorem 3.5). Following a standard approach, we obtain the existence of a solution as limit of regular problems. For this purpose we consider the approximate problem

$$\begin{cases} u_t^\epsilon - \operatorname{div}(a(t,x,\nabla u^\epsilon)) = \mu^\epsilon & \text{in } (0,T)\times\Omega, \\ u^\epsilon(0,x) = u_0^\epsilon(x) & \text{in } \Omega, \\ u^\epsilon(t,x) = 0 & \text{on } (0,T)\times\partial\Omega, \end{cases} \tag{35}$$

where $\{\mu^\epsilon\}_{\epsilon>0}$, $\{u_0^\epsilon\}_{\epsilon>0}$ are smooth approximations of the data μ and u_0 with

$$\|u_0^\epsilon\|_{L^1(\Omega)} \le C\|u_0\|_{L^1(\Omega)}, \quad \|\mu^\epsilon\|_{L^1(Q)} \le C|\mu|.$$

Hence by the standard theory of monotone operators [32] or using [48, Lemma 2.5] with rather minor modifications, there exists a variational solution u^ϵ for each $\epsilon > 0$. Moreover, from Theorem 2.7, there exists a decomposition $(f^\epsilon, F^\epsilon, div(G_1^\epsilon), g_2^\epsilon)$ of μ^ϵ with $f^\epsilon \in C_c^\infty(Q)$ such that $\|f^\epsilon - f\|_{L^1(Q)} \le C\epsilon$, $F^\epsilon \in (C_c^\infty(Q))^N$ such that $\|F^\epsilon - F\|_{(L^{P(\cdot)}(Q))^N} \le C\epsilon$, $G_1^\epsilon \in (C_c^\infty(Q))^N$ such that $\|G_1^\epsilon - G_1\|_{(L^{p(\cdot)}(Q))^N} \le C\epsilon$ and $g_2^\epsilon \in C_c^\infty(Q)$ such that $\|g_2^\epsilon - g_2\|_{L^{p_-}(0,T;V)} \le C\epsilon$ (C not depending on ϵ) such that

$$\begin{aligned}
&\int_0^t \langle (u^\epsilon - g_2^\epsilon)_t, \varphi\rangle ds + \int_0^t\int_\Omega a(s,x,\nabla u^\epsilon)\nabla\varphi dxds \\
&= \int_0^t\int_\Omega f^\epsilon \varphi dxds + \int_0^t\int_\Omega F\cdot\nabla\varphi dxds + \int_0^t \langle div(G_1^\epsilon), \varphi\rangle ds,
\end{aligned} \tag{36}$$

$\forall\varphi \in L^{p_-}(0,T;V)$ with $\nabla\varphi \in (L^{p(\cdot)}(Q))^N$, $\forall t \in [0,T]$. Next, following the ideas of [7, 21], we can perform some estimates for the sequence $(u^\epsilon)_{\epsilon>0}$, to prove that u is actually the renormalized solution to the parabolic problem (18).

Proposition 4.1. *Let u^ϵ as defined before, then*

$$\begin{cases} \|u^\epsilon\|_{L^\infty(0,T;L^1(\Omega))} \leq C, \quad \int_Q |\nabla T_k(u^\epsilon)|^{p(x)}dxdt \leq Ck, \\ \|u^\epsilon - g_2^\epsilon\|_{L^\infty(0,T;L^1(\Omega))} \leq C, \quad \int_Q |\nabla T_k(u^\epsilon - g_2^\epsilon)|^{p(x)}dxdt \leq C(k+1). \end{cases} \tag{37}$$

Moreover, there exists a measurable functions u and $v = u - g_2$ such that $T_k(u)$ and $T_k(v)$ belongs to X, u and v belongs to $L^\infty(0,T;L^1(\Omega))$; and, up to a subsequence, for any $k > 0$, and for every $q(\cdot) < p(\cdot) - \frac{N}{N+1}$, we have

$$\begin{cases} u^\epsilon \to u \text{ a.e. in } Q \text{ weakly in } L^{q_-}(0,T;W_0^{1,q(\cdot)}(\Omega)) \text{ and strongly in } L^1(Q), \\ (u^\epsilon - g_2^\epsilon) \to (u-g_2) \text{ a.e. in } Q \text{ weakly in } L^{q_-}(0,T;W_0^{1,q(\cdot)}(\Omega)) \text{ and strongly in } L^1(Q), \\ (T_k(u^\epsilon) \rightharpoonup T_k(u) \text{ weakly in } L^{p_-}(0,T;W_0^{1,p(\cdot)}(\Omega)) \text{ and a.e. on } Q, \\ T_k(u^\epsilon - g_2^\epsilon) \rightharpoonup T_k(u-g_2) \text{ weakly in } L^{p_-}(0,T;W_0^{1,p(\cdot)}(\Omega)) \text{ and a.e. on } Q, \\ \nabla u^\epsilon \to \nabla u \text{ a.e. in } Q, \quad \nabla(u^\epsilon - g_2^\epsilon) \to \nabla(u-g_2) \text{ a.e. in } Q. \end{cases}$$

Proof. Here we give an idea on how (37) can be obtained following the outlines of [22]. Let $\epsilon > 0$, by taking $T_k(u^\epsilon)$ as test function in (35), we obtain

$$\int_0^t \langle \frac{\partial u^\epsilon}{\partial t}, T_k(u^\epsilon)\rangle dt + \int_Q a(t,x,\nabla u^\epsilon)\cdot \nabla T_k(u^\epsilon)dxdt = \int_Q \mu^\epsilon T_k(u^\epsilon)dxdt.$$

We have $\Theta_k(r) = \int_0^r T_k(s)ds$ and $|\Theta_k(r)| \leq k|r|$, then

$$\begin{aligned} \int_0^t \langle \frac{\partial u^\epsilon}{\partial t}, T_k(u^\epsilon)\rangle dt &= \int_\Omega \int_0^t \frac{\partial u^\epsilon}{\partial t} T_k(u^\epsilon)dxdt = \int_\Omega \int_0^t \frac{\partial \Theta_k(u^\epsilon)}{\partial t} dxdt \\ &= \int_\Omega \Theta_k(u^\epsilon(T))dx - \int_\Omega \Theta_k(u_0^\epsilon)dx \geq \int_\Omega \Theta_k(u^\epsilon(t))dx - k\|u_0^\epsilon\|_{L^1(\Omega)}. \end{aligned}$$

From (15) and using the fact that $\|u_0^\epsilon\|_{L^1(\Omega)}$ and $\|\mu^\epsilon\|_{L^1(Q)}$ are bounded, then

$$\int_\Omega \Theta_k(u^\epsilon(t))dx + \int_0^t \int_\Omega |\nabla T_k(u^\epsilon)|^{p(x)}dxdt \leq Ck, \quad \forall k \geq 0, \forall t \in [0,T].$$

Since $\Theta_k(s)$ is nonnegative and $|\Theta_1(s)| \geq |s| - 1$ for $k = 1$, we get

$$\int_\Omega |u^\epsilon(t)|dx + \int_0^t \int_\Omega |\nabla T_k(u^\epsilon)|^{p(x)}dxdt \leq C(k+1) \quad \forall t \in [0,T]. \tag{38}$$

Taking the supremum on $(0,T)$ we obtain the estimate

$$\|u^\epsilon\|_{L^\infty(0,T;L^1(\Omega))} \leq C.$$

To prove the estimate of $u^\epsilon - g_2^\epsilon$ in $L^\infty(0,T;L^1(\Omega))$, we will use the test function $T_k(u^\epsilon - g_2^\epsilon)$ in (36), this gives

$$\begin{aligned} &\int_0^t \langle \frac{\partial u^\epsilon}{\partial t}, T_k(u^\epsilon - g_2^\epsilon)\rangle dxdt - \int_0^t \langle (g_2^\epsilon)_t, T_k(u^\epsilon - g_2^\epsilon)\rangle dt + \int_Q a(t,x,\nabla u^\epsilon)\nabla T_k(u^\epsilon - g_2^\epsilon)dxdt \\ &= \int_Q f^\epsilon T_k(u^\epsilon - g_2^\epsilon)dxdt + \int_Q F\cdot \nabla T_k(u^\epsilon - g_2^\epsilon)dxdt - \int_0^t \langle div(G_1^\epsilon), T_k(u^\epsilon - g_2^\epsilon)\rangle. \end{aligned}$$

Now, since g_2^ϵ has compact support in Q, so that $(u^\epsilon - g_2^\epsilon)(0) = u^\epsilon(0) = u_0^\epsilon$. Using the integration by parts in time in the first term and using (15) we get

$$\int_\Omega \Theta_k(u^\epsilon - g_2^\epsilon)(t)dx - \int_\Omega \Theta_k(u_0^\epsilon)dx + \alpha \int_{\{|u^\epsilon - g_2^\epsilon| \le k\}} |\nabla u^\epsilon|^{p(x)} dxdt$$
$$- \int_{\{|u^\epsilon - g_2^\epsilon| \le k\}} a(t, x, \nabla u^\epsilon)\nabla g_2^\epsilon$$
$$\le \int_Q f^\epsilon T_k(u^\epsilon - g_2^\epsilon)dxdt + \int_{\{|u^\epsilon - g_2^\epsilon| \le k\}} F \cdot \nabla(u^\epsilon - g_2^\epsilon)dxdt$$
$$+ \int_{\{|u^\epsilon - g_2^\epsilon| \le k\}} G_1^\epsilon \nabla(u^\epsilon - g_2^\epsilon)dxdt.$$

Young's inequality then implies, using also (16),

$$\int_\Omega \Theta_k(u^\epsilon - g_2^\epsilon)(t)dx + \alpha \int_{\{|u^\epsilon - g_2^\epsilon| \le k\}} |\nabla u^\epsilon|^{p(x)} dxdt$$
$$\le C\beta\Big[\int_Q |b(t,x)|^{p'(x)}dxdt + \int_Q |\nabla u^\epsilon|^{p(x)}dxdt + \int_Q |\nabla g_2^\epsilon|^{p(x)}dxdt\Big]$$
$$+ k\Big[\|u_0^\epsilon\|_{L^1(\Omega)} + \|f^\epsilon\|_{L^1(Q)}\Big] + \frac{\alpha}{2}\Big[\int_Q |\nabla u|^{p(x)}dxdt + \int_Q |\nabla g_2^\epsilon|^{p(x)}dxdt$$
$$+ C_\alpha\Big[\int_Q |F|^{p'(x)}dxdt + \int_Q |G_1^\epsilon|^{p'(x)}dxdt\Big],$$

where C_α denote a positive constant which depends on p_+ and p_- but not depending on ϵ and k. In the same way we can deal with the right hand side of the last inequality, we used the fact that $f^\epsilon \in L^1(Q)$, $F^\epsilon \in (L^{p'(\cdot)}(Q))^N$, $g_1^\epsilon \in L^{p'_-}(0,T; W_0^{1,p(\cdot)}(\Omega))$, $g_2 \in L^{p_-}(0,T;V)$ and $u_0^\epsilon \in L^1(\Omega)$, (note that $\Theta_k(s)$ is nonnegative for any $k \ge 0$)

$$\Theta_1(u^\epsilon - g_2^\epsilon)(t) \le C \quad \forall t \in [0,T], \qquad \int_{\{|u^\epsilon - g_2^\epsilon| \le k\}} |\nabla u^\epsilon|^{p(x)}dxdt \le C(k+1).$$

Moreover, using the boundedness of g_2^ϵ in V, we have

$$\|u^\epsilon - g_2^\epsilon\|_{L^\infty(0,T;L^1(\Omega))} \le C, \qquad \int_Q |\nabla T_k(u^\epsilon - g_2^\epsilon|^{p(x)}dxdt \le C(k+1).$$

Now, we shall use the above estimates to prove some compactness results that will be useful to pass to the limit in the renormalized formulation for u^ϵ: If we multiply the first equation in (35) by $\gamma_k'(u^\epsilon - g_2^\epsilon)$ where γ is a $C^2(\mathbb{R})$ nondecreasing function with $\gamma(s) = s$ for $|s| \le \frac{k}{2}$ and $\gamma(s) = k$ for $|s| > k$ (γ_k', γ_k'' has compact support), we get

$$\begin{aligned}(\gamma_k(u^\epsilon - g_2^\epsilon))_t &- div(a(t,x,\nabla u^\epsilon)\gamma_k'(u^\epsilon - g_2^\epsilon)) + \gamma_k''(u^\epsilon - g_2^\epsilon)a(t,x,\nabla u^\epsilon)\nabla(u^\epsilon - g_2^\epsilon)\\ &= \gamma_k'(u^\epsilon - g_2^\epsilon)f^\epsilon - div(F^\epsilon\gamma_k'(u^\epsilon - g_2^\epsilon)) + \gamma_n''(u^\epsilon - g_2^\epsilon)F^\epsilon\nabla(u^\epsilon - g_2^\epsilon)\\ &+ \gamma_k''(u^\epsilon - g_2^\epsilon)G_1\nabla(u^\epsilon - g_2^\epsilon) - div(G_1^\epsilon\gamma_k'(u^\epsilon - g_2^\epsilon)).\end{aligned} \tag{39}$$

We also have $\gamma_k''(u^\epsilon - g_2^\epsilon)a(t,x,\nabla u^\epsilon)\cdot\nabla(u^\epsilon - g_2^\epsilon) \in L^1(Q)$, $\gamma_k''(u^\epsilon - g_2^\epsilon)F^\epsilon\cdot\nabla(u^\epsilon - g_2^\epsilon) \in L^1(Q)$, $\gamma_k''(u^\epsilon - g_2^\epsilon)G_1\cdot\nabla(u^\epsilon - g_2^\epsilon) \in L^1(Q)$, $\gamma_k'(u^\epsilon - g_2^\epsilon)a(t,x,\nabla u^\epsilon) \in (L^{p'(\cdot)}(Q))^N$, $\gamma_k'(u^\epsilon - g_2^\epsilon)G_1^\epsilon \in (L^{p'(\cdot)}(Q))^N$, $\gamma_k'(u^\epsilon - g_2^\epsilon)F^\epsilon \in (L^{p'(\cdot)}(Q))^N$. Thus, by equation (39), $(\gamma_k(u^\epsilon - g_2^\epsilon))_t$ belong to the space $X^* + L^1(Q)$. On the other hand, by the last equality

$T_k(u^\epsilon - g_2^\epsilon)$ is bounded in X for any $k > 0$, then we have

$$k \text{ meas}\{|u^\epsilon - g_2^\epsilon| > k\} = \int_{\{|u^\epsilon - g_2^\epsilon|>k\}} |T_k(u^\epsilon - g_2^\epsilon)|dxdt \leq \int_Q |T_k(u^\epsilon - g_2^\epsilon)|dxdt$$
$$\leq 2(\text{ meas}(Q)+1)^{\frac{1}{p'_-}} \|T_k(u^\epsilon - g_2^\epsilon)\|_X \leq Ck^{\frac{1}{p_-}},$$

which implies that

$$\text{meas}\{|u^\epsilon - g_2^\epsilon| > k\} \leq C\frac{1}{k^{1-\frac{1}{p_-}}} \to 0 \text{ as } k \to \infty. \tag{40}$$

Let $n, m \geq 0$, for all $\lambda > 0$, we have

$$\begin{aligned} \text{meas}\{|(u^n - g_2^n)| > \lambda\} &\leq \text{meas}\{|u_n - g_2^n| > k\} + \text{meas}\{|u_m - g_2^m)| > k\} \\ &\quad + \text{meas}\{|T_k(u_n - g_2^n) - T_k(u_m - g_2^m)| > \lambda\}. \end{aligned} \tag{41}$$

Using (40) we get that for all $\epsilon > 0$, there exists $k_0 > 0$ such that $\forall k \geq k_0(\epsilon)$,

$$\text{meas}\{|u_n - g_2^n| > k\} \leq \frac{\epsilon}{3}, \qquad \text{meas}\{|u_m - g_2^m| > k\} \leq \frac{\epsilon}{3}.$$

On the other hand, we have $(T_k(u_n - g_2^n))_{n\in\mathbb{N}}$ is bounded in X. Then, there exists a sequence still denoted $(T_k(u_n - g_2^n))_{n\in\mathbb{N}}$ such that

$$T_k(u_n - g_2^n) \rightharpoonup \eta_k \text{ in } X \text{ as } n \to \infty$$

and by the compact embedding $\{u : u \in X \text{ and } u_t \in X^*\}$ in $L^1(Q)$, we obtain

$$T_k(u_n - g_2^n) \to \eta_k \text{ in } L^1(Q) \text{ and a.e. in } Q.$$

Thus, we can assume that $(T_k(u_n - g_2^n))_{n\in\mathbb{N}}$ is a Cauchy sequence in Q, therefore for all $k > 0$ and $\lambda, \epsilon > 0$ there exists $n_0 = n_0(k, \lambda, \epsilon)$ such that

$$\text{meas}\{|T_k(u_n - g_2^n) - T_k(u_m - g_2^m)| > \lambda\} \leq \frac{\epsilon}{3} \quad \forall n, m \geq n_0. \tag{42}$$

By combining (40)–(42), we deduce that for $\epsilon, \lambda > 0$ there exits $n_0 = n_0(\lambda, \epsilon)$ such that

$$\text{meas}\{|(u_n - g_2^n) - (u_m - g_2^m)| > \lambda\} \leq \epsilon \quad \forall n, m \geq n_0.$$

It follows that $(u^\epsilon - g_2^\epsilon)_{\epsilon>0}$ is a Cauchy sequence in measure, then there exists a subsequence still denoted $(u^\epsilon - g_2^\epsilon)_{\epsilon>0}$ such that

$$u^\epsilon - g_2^\epsilon \to u - g_2 \text{ a.e. in } Q, \quad T_k(u^\epsilon - g_2^\epsilon > 0) \rightharpoonup T_k(u - g_2) \text{ weakly in } X.$$

In the view of the strong convergence of g_2^ϵ to g_2 in $L^{p_-}(0, T; W_0^{1,p(\cdot)}(\Omega))$, we have

$$u^\epsilon \to u \text{ a.e. in } Q, \quad T_k(u^\epsilon) \rightharpoonup T_k(u) \text{ weakly in } X.$$

Finally, the sequence $u^\epsilon - g_2^\epsilon$ satisfies the hypotheses of [7], and so we get

$$\nabla(u^\epsilon - g_2^\epsilon) \to \nabla(u - g_2) \text{ a.e. in } Q, \quad \nabla u^\epsilon \to \nabla(u) \text{ a.e. in } Q.$$

Next we shall prove the strong convergence of truncates of renormalized solutions of problem (18). To do that we will crossover the approach used in [38, 15]. With the symbol $T_k(v)_\mu$ we indicate the Landes time-regularization of the truncate function $T_k(v)$; this notion, introduced in [31], was fruitfully used in several papers afterwards (see in particular [7, 15, 21]). Let z_μ be a sequence of functions such that

$$\begin{cases} z_\mu \in W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega), \quad \|z_\mu\|_{L^\infty(\Omega)} \leq k, \\ z_\mu \to T_k(u_0) \text{ a.e. in } \Omega \text{ as } \mu \text{ tends to infinity}, \\ \frac{1}{\mu}\|z_\mu\|_{W_0^{1,p(\cdot)}(\Omega)} \to 0 \text{ as } \mu \text{ tends to infinity}. \end{cases}$$

Then, for fixed $k > 0$, and $\mu > 0$, we denote by $T_k(v)_\mu$ the unique solution of

$$\begin{cases} (T_k(v)_\mu)_t & = \mu(T_k(v) - T_k(v)_\mu) \text{ in the sense of distributions,} \\ T_k(v)_\mu(0) & = z_\mu \text{ in } \Omega. \end{cases}$$

Therefore $T_k(v)_\mu \in X \cap L^\infty(Q)$ and $\frac{d}{dt}T_k(v) \in V$, and it can be proved (see also [31]) that up to subsequences

$$T_k(v)_\mu \to T_k(v) \text{ strongly in } X \text{ and a.e. in } Q, \quad \|T_k(v)_\mu\|_{L^\infty(Q)} \le k, \quad \forall \mu > 0.$$

Choosing w^ϵ as a test function in the formulation (36), we obtain

$$\begin{aligned} &\int_0^T \int_\Omega (v^\epsilon)_t w^\epsilon dxdt + \int_0^T \int_\Omega a(t,x,\nabla u^\epsilon) \cdot \nabla w^\epsilon dxdt \\ &= \int_0^T \int_\Omega f^\epsilon w^\epsilon dxdt + \int_0^T \int_\Omega F^\epsilon \cdot \nabla w^\epsilon dxdt + \int_0^T \langle g_1^\epsilon, w^\epsilon \rangle dxdt. \end{aligned} \tag{43}$$

So, for the first term on the right-hand side of (43), we have

$$\begin{aligned} &|\int_0^T \int_\Omega f^\epsilon w^\epsilon dxdt| \le \int_0^T \int_\Omega |f^\epsilon - f||T_{2k}(v^\epsilon - T_h(v^\epsilon) + T_k(v^\epsilon) - (T_k(v))_\mu)|dxdt \\ &+ \int_0^T \int_\Omega |fT_{2k}(v^\epsilon - T_h(v^\epsilon) + T_k(v^\epsilon) - (T_k(v))_\mu)|dxdt \le 2k \int_0^T \int_\Omega |f^\epsilon - f|dxdt \\ &+ \int_0^T \int_\Omega |fT_{2k}(v^\epsilon - T_h(v^\epsilon) + T_k(v^\epsilon) - (T_k(v))_\mu)|dxdt. \end{aligned}$$

By using the fact that f^ϵ is strongly compact in $L^1(Q)$, the weak convergence of $T_k(v^\epsilon)$ to $T_k(v)$ in $L^{p_-}(0,T;W_0^{1,p(\cdot)}(\Omega))$ and a.e. in Q, the definition of $(T_k(v)_\mu)$ and the Lebesgue Dominated Convergence Theorem, we have

$$\lim_{h\to+\infty} \lim_{\mu\to+\infty} \lim_{\epsilon\to 0} |\int_0^T \int_\Omega f^\epsilon w^\epsilon dxdt| \le \lim_{h\to+\infty} \int_0^T \int_\Omega |fT_{2k}(v - T_h(v))|dxdt = 0.$$

Using the notations $\omega(\epsilon, \mu, h)$, we obtain

$$\int_0^T \int_\Omega f^\epsilon w^\epsilon dxdt = \omega(\epsilon,\mu,h), \quad \int_0^T \int_\Omega F^\epsilon \cdot \nabla w^\epsilon dxdt = \omega(\epsilon,\mu,h). \tag{44}$$

Let us analyze the second term in (43). By the fact that $\nabla w^\epsilon = 0$ if $|v^\epsilon| > M = h + 4k$

$$\int_0^T \int_\Omega a(t,x,\nabla u^\epsilon) \cdot \nabla w^\epsilon dxdt = \int_0^T \int_\Omega a(t,x,\nabla u^\epsilon \chi_{\{|v^\epsilon| \le M\}}) \cdot \nabla w^\epsilon dxdt.$$

Next we split the integral in the sets $\{|v^\epsilon| \le k\}$ and $\{|v^\epsilon| > k\}$, so that we have, recalling that for $h > 2k$,

$$\begin{aligned} &\int_0^T \int_\Omega a(t,x,\nabla u^\epsilon \chi_{\{|u^\epsilon|k\}}) \cdot \nabla T_{2k}(v^\epsilon - T_h(v^\epsilon) + T_k(v^\epsilon) - (T_k(v))_\mu)dxdt \\ &= \int\int_{\{|v^\epsilon| \le k\}} a(t,x,\nabla u^\epsilon) \nabla(v^\epsilon - T_k(v)_\mu)dxdt \\ &+ \int\int_{\{|v^\epsilon| > k\}} a(t,x,\nabla u^\epsilon \chi_{\{|v^\epsilon| \le M\}}) \nabla(v^\epsilon - T_h(v^\epsilon))dxdt \\ &- \int\int_{\{|v^\epsilon| > k\}} a(t,x,\nabla u^\epsilon \chi_{\{|v^\epsilon| \le M\}}) \nabla T_k(v)_\mu dxdt = I_1 + I_2 + I_3. \end{aligned} \tag{45}$$

Let us estimate I_2. Since $v^\epsilon = T_h(v^\epsilon) = 0$ if $|v^\epsilon| \leq h$, using (16) and young's inequality, we have

$$
\begin{aligned}
|I_2| &= |\int_{\{|v^\epsilon|>k\}} a(t,x,\nabla u^\epsilon \chi_{\{|v^\epsilon|\leq M\}})\nabla(v^\epsilon - T_h(v^\epsilon))dxdt| \leq \int_{\{h\leq|v^\epsilon|\leq M\}} |a(t,x,\nabla u^\epsilon)||\nabla v^\epsilon| \\
&\leq \int_{\{h\leq|v^\epsilon|\leq M\}} \beta(b(t,x) + |\nabla u^\epsilon|^{p(x)-1})|\nabla(u^\epsilon - g_2^\epsilon)|dxdt \\
&\leq \int_{\{h\leq|v^\epsilon|\leq M\}} \beta(b(t,x)|\nabla(u^\epsilon - g_2^\epsilon)|)dxdt + \int_{\{h\leq|v^\epsilon|\leq M\}} C|\nabla u|^{p(x)-1}|\nabla(u^\epsilon - g_2^\epsilon)|dxdt \\
&\leq \int_{\{h\leq|v^\epsilon|\leq M\}} \frac{C}{p'_-}|b(t,x)|^{p'(x)}dxdt + \int_{\{h\leq|v^\epsilon|\leq M\}} \frac{C}{p'_-}|\nabla u^\epsilon|^{p(x)}dxdt \\
&+ \int_{\{h\leq|v^\epsilon|\leq M\}} \frac{C}{p'_-}|\nabla g_2^\epsilon|^{p(x)}dxdt + \int_{\{h\leq|v^\epsilon|\leq M\}} |\nabla u^\epsilon|^{p(x)}dxdt \\
&+ \int_{\{h\leq|v^\epsilon|\leq M\}} \frac{C}{p'_-}|\nabla u^\epsilon|^{p(x)}dxdt + \int_{\{h\leq|v^\epsilon|\leq M\}} \frac{C}{p_-}|\nabla g_2^\epsilon|^{p(x)}dxdt \\
&\leq C\int_{\{h\leq|v^\epsilon|\leq M\}} |\nabla u^\epsilon|^{p(x)}dxdt + C\int_{\{h\leq|v^\epsilon|\leq M\}} |b(t,x)|^{p'(x)}dxdt + C\int_{\{h\leq|v^\epsilon|\leq M\}} |\nabla g_2^\epsilon|^{p(x)}.
\end{aligned}
$$

Moreover, since $b(t,x)$ and $(\nabla u^\epsilon)_{\epsilon\geq 0}$ are, respectively, bounded in $L^{p'_-}(0,T;W_0^{1,p(\cdot)}(\Omega))$ and $L^{p_-}(0,T;W_0^{1,p(\cdot)}(\Omega))$, and as meas$\{h \leq |v^\epsilon| < M\}$ converges uniformly to zero as $h \to \infty$ with respect to ϵ, then, thanks to the equi-integrability of $|\nabla g_2^\epsilon|^{p(x)}$, we can pass to the limit in (I_2) as $\epsilon \to 0$ and $h \to +\infty$ respectively, and using Lebesgue dominated convergence theorem, we easily get $I_2 = \omega(\epsilon, h)$. It remains to estimate I_3, let us remark that, since $(\nabla u^\epsilon \chi_{|v^\epsilon|\leq M})$ is bounded in $L^{p_-}(0,T;W_0^{1,p(\cdot)}(\Omega))$, (16) implies that $(a(t,x,\nabla u^\epsilon)\chi_{\{|v^\epsilon|\leq M\}})_{\epsilon>0}$ is bounded in $L^{p'(\cdot)}(Q)$. The a.e. convergence of v^ϵ to v as $\epsilon \to 0$, implies that $|\nabla T_k(v)|\chi_{\{|v^\epsilon|\leq k\}}$ strongly converges to zero in $L^{p_-}(0,T;W_0^{1,p(\cdot)}(\Omega))$. So that by the Lebesgue dominated convergence theorem

$$
\limsup_{\epsilon\to 0} \int\int_{\{|v^\epsilon|>k\}} a(t,x,\nabla u^\epsilon \chi_{\{|v^\epsilon|\leq M\}})\nabla T_k(v)dxdt = 0
$$

and we readily have that

$$
\begin{aligned}
I_3 &= \int_{\{|v^\epsilon\}|>k} a(t,x\nabla u^\epsilon \chi_{\{|v^\epsilon|\leq M\}})\nabla T_k(v)_\mu dxdt \\
&= \int_{\{|v^\epsilon|>k\}} a(t,x,\nabla u^\epsilon \chi_{\{|v^\epsilon|\leq M\}}) \cdot \nabla(T_k(v)_\mu - T_k(v))dxdt \\
&= \omega(\epsilon) + \int_{\{|v^\epsilon|>k\}} a(t,x,\nabla u^\epsilon \chi_{\{|v^\epsilon|\leq M\}})\nabla(T_k(v)_\mu - T_k(v))dxdt.
\end{aligned}
$$

Recall that $(a(t,x,\nabla u^\epsilon \chi_{\{|v^\epsilon|\leq M}))_{\epsilon>0}$ is bounded in $L^{p'(\cdot)}(Q)$ and thanks to the strong convergence of $T_k(v)_\mu$ to $T_k(v)$ in X, by Lebesgue Dominated Convergence theorem

$$
\int_{\{|v^\epsilon|>k\}} a(t,x,\nabla u^\epsilon \chi_{\{|v^\epsilon|\leq M\}})\nabla(T_k(v)_\mu - T_k(v))dxdt = \omega(\epsilon,\mu).
$$

We can conclude that $I_3 = \omega(\epsilon, \mu)$. On the other hand, using (45), according to the fact that I_2 and I_3 converges to zero, then

$$
\int_Q a(t,x,\nabla u^\epsilon) \cdot \nabla w^\epsilon dxdt = \int_{\{|v^\epsilon|\leq k\}} a(t,x,\nabla u^\epsilon) \cdot \nabla(v^\epsilon - T_k(v)_\mu)dxdt + \omega(\epsilon,\mu,h).
$$

Moreover, (44) and (45) together with (43) yields

$$\int_Q (v^\epsilon)_t w^\epsilon dxdt + \int_{\{|v^\epsilon|\le k\}} a(t,x,\nabla u^\epsilon)\cdot\nabla(v^\epsilon - (T_k(v))_\mu)dxdt = \omega(\epsilon,\mu,h). \tag{46}$$

While, for the first term of (46), using the Lemma 2.1 in [38], we have

$$\int_Q (v^\epsilon)_t w^\epsilon dxdt \ge \omega(\epsilon,\mu,h).$$

Hence (46) becomes

$$\int_{\{|v^\epsilon|\le k\}} a(t,x,\nabla u^\epsilon)\cdot\nabla(v^\epsilon - (T_k(v))_\mu)dxdt \le \omega(\epsilon,\mu,h). \tag{47}$$

While, since $\nabla T_k(v)_\mu \to \nabla T_k(v)$ strongly in $(L^{p(\cdot)}(Q))^N$ as $\mu \to +\infty$ and $g_2^\epsilon \to g_2$ strongly in $L^{p_-}(0,T;W_0^{1,p(\cdot)}(\Omega))$, thanks to (47), we easily obtain

$$\int_Q a(t,x,\nabla(g_2^\epsilon + T_k(v^\epsilon))\chi_{\{|v^\epsilon|\le k\}}\nabla(u^\epsilon - T_k(v)))dxdt.$$

Moreover, again thanks to the fact that $\nabla T_k(v)_\mu \to \nabla T_k(v)$ strongly in $(L^{p(\cdot)}(Q))^N$ as $\mu \to +\infty$, and from (47),

$$\int_Q a(t,x,\nabla u^\epsilon \chi_{\{|v^\epsilon|\le k\}})\nabla(T_k(v^\epsilon) - T_k(v))dxdt \le \omega(\epsilon,\mu,h).$$

Therefore, passing to the limit in (45) as ϵ tends to zero, μ and h tends to infinity respectively, we deduce that

$$\limsup_{\epsilon\to 0}\int_Q a(t,x,\nabla u^\epsilon \chi_{\{|v^\epsilon|\le k\}})\cdot\nabla(T_k(v^\epsilon) - T_k(v)) \le 0.$$

Now, let k be such that $\chi_{\{|v^\epsilon|\le k\}} \to \chi_{\{|v|\le k\}}$ a.e. and $g_2^n \to g_2$ strongly in the space $L^{p_-}(0,T;W_0^{1,p(\cdot)}(\Omega))$, then using (16) [6, Lemma 3.2], we get

$$a(t,x,\nabla(g_2^n + T_k(v)\chi_{\{|v^\epsilon|\le k\}})) \to a(t,x,\nabla(g_2 + T_k(v)\chi_{\{|v|\le k\}})) \text{ in } (L^{p(\cdot)}(Q))^N \tag{48}$$

and from (48) we obtain

$$\begin{aligned}&\int_Q (a(t,x,\nabla(g_2^n + T_k(v^\epsilon))) - a(t,x,\nabla(g_2 + T_k(v))))\cdot\nabla(T_k(v^\epsilon) - T_k(v))dxdt\\ &\le -\int_0^T\int_\Omega a(t,x,\nabla(g_2 + T_k(v)))\cdot\nabla(T_k(v^\epsilon) - T_k(v))dxdt + \omega(\epsilon,\mu,h).\end{aligned} \tag{49}$$

Using the weak convergence of $\nabla T_k(v^\epsilon)$ to $\nabla T_k(v)$ in $(L^{p(\cdot)}(Q))^N$, we conclude that

$$\limsup_{\epsilon\to 0}\int_Q a(t,x,\nabla(g_2^\epsilon + T_k(v))\chi_{\{|v^\epsilon|\le k\}})\nabla(T_k(v^\epsilon) - T_k(v))dxdt = 0.$$

In the same time, we can pass to the limit in (49) as ϵ tends to zero, μ and h tends to infinity respectively, to deduce that

$\limsup_{\epsilon\to 0}\int_Q [a(t,x,\nabla u^\epsilon \chi_{\{|v^\epsilon|\le k\}}) - a(t,x,\nabla(g_2^\epsilon + T_k(v))\chi_{\{|v^\epsilon|\le k\}})]\cdot(\nabla u^\epsilon - \nabla(g_2^\epsilon + T_k(v))) = 0.$

Using that $\chi_{\{|v^\epsilon|\le k\}}$ almost everywhere converges to $\chi_{\{|v^\epsilon|\le k\}}$ and that g_2^ϵ strongly converges to g_2 in $L^{p_-}(0,T;W_0^{1,p(\cdot)}(\Omega))$, then thanks to the standard monotonicity argument which relies on (17) (see [12, Lemma 5]) we readily have from (50),

$$\nabla u^\epsilon \chi_{\{|v^\epsilon|\le k\}} \to \nabla(g_2 + T_k(v))\chi_{\{|v^\epsilon|\le k\}} = \nabla u \chi_{\{|v^\epsilon|\le k\}} \text{ a.e. in } Q,$$

which means that

$$a(t,x,\nabla u^\epsilon\chi_{\{|v^\epsilon|\le k\}})\nabla u^\epsilon \to a(t,x,\nabla u\chi_{\{|v^\epsilon|\le k\}})\nabla u \text{ strongly in } L^1(Q) \text{ and a.e. in } Q.$$

Finally, collecting together all these facts with (15), we obtain the equi-integrability of the sequence $|\nabla u^\epsilon|^{p(x)}\chi_{\{|v^\epsilon|\le k\}}$ in Q, we can write as consequences of Vitali's theorem and since g_2^ϵ strongly converges in $L^{p_-}(0,T;W_0^{1,p(\cdot)}(\Omega))$ yields

$$T_k(u^\epsilon - g_2^\epsilon) \to T_k(u-g_2) \text{ strongly in } L^{p_-}(0,T;W_0^{1,p(\cdot)}(\Omega)).$$

Now, we have to check that

$$\nabla T_k(u^\epsilon - g_2^\epsilon) \to \nabla T_k(u^\epsilon - g_2^\epsilon) \text{ in } (L^{p(\cdot)}(Q))^N.$$

We need the following lemmas.

Lemma 4.2. [26, Theorem 1.4] *Let $v, v_n \in L^{p(\cdot)}(Q)$, $n = 1, 2, \cdots$. Then the following statements are equivalent*

$$\begin{cases} (1) & \underset{n\to\infty}{lim} |v_n - v|_{\rho(\cdot)} = 0; \\ (2) & \underset{n\to\infty}{lim} (v_n - v) = 0; \\ (3) & v_n \textit{ converges to } v \textit{ in } Q \textit{ in measure and } \underset{n\to\infty}{lim} \rho_{p(\cdot)}(v_n) = \rho_{p(\cdot)}(v). \end{cases}$$

Lemma 4.3. *Lebesgue Generalized Convergence Theorem: Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of measurable functions and f a measurable function such that $f_n \to f$ a.e. in Q. let $(g_n)_{n\in\mathbb{N}} \subset L^1(Q)$ such that $|f_n| \le g_n$ a.e. in Q and $g_n \to g$ in $L^1(Q)$. Then*

$$\int_Q f_n dxdt \to \int_Q f dxdt.$$

Now, set $f^\epsilon = |\nabla T_k(u^\epsilon)|^{p(x)}$, $f = |\nabla T_k(u)|^{p(x)}$, $g^\epsilon = a(t,x,\nabla u^\epsilon\chi_{\{|v^\epsilon|\le k\}})\cdot\nabla u^\epsilon$ and $g = a(t,x,\nabla u\chi_{\{|v^\epsilon|\le k\}})\cdot\nabla u$, f^ϵ is a sequence of measurable functions, f is a measurable function and according to the almost convergence of $\nabla T_k(u_n)$ to $\nabla T_k(u)$ in Ω,

$$f^\epsilon \to f \text{ a.e. in Q.}$$

Using $a(x,\nabla T_k(u^\epsilon))\cdot\nabla T_k(u^\epsilon) \to a(x,\nabla T_k(u))\cdot\nabla T_k(u)$ strongly in $L^1(\Omega)$ and a.e. in Ω, we have $(g^\epsilon)_{\epsilon>0} \subset L^1(Q)$, $g^\epsilon \to g$ a.e. in Q, $g^\epsilon \to g$ in $L^1(Q)$, and $|f^\epsilon| \le Cg^\epsilon$. Then, by Lemma 4.3 we have

$$\int_Q f^\epsilon dxdt \to \int\int_Q f dxdt,$$

which is equivalent to

$$\int_Q |\nabla T_k(u^\epsilon)|^{p(x)} dxdt \to \int\int_Q |\nabla T_k(u)|^{p(x)} dxdt.$$

We deduce from (2) that the sequence $(\nabla T_k(u^\epsilon))_{\epsilon>0}$ converges to $\nabla T_k(u)$ in Q in measure. Then, by Lemma 4.2, we deduce that

$$\lim_{\epsilon\to 0}\int_Q |\nabla T_k(u^\epsilon) - \nabla T_k(u)|^{p(x)} dxdt = 0,$$

which is equivalent to

$$\nabla T_k(u^\epsilon) \to \nabla T_k(u) \text{ in } (L^{p(\cdot)}(Q))^N.$$

Finally, we are able to prove that problem (18) has a renormalized solution. Let $S \in W^{2,\infty}(\mathbb{R})$ be such that S' has a compact support, and let $\varphi \in C_c^\infty(Q)$; then the approximating solutions u^ϵ and $u^\epsilon - g_2^\epsilon$ satisfy

$$\begin{aligned}
&-\int_\Omega S(u_0^\epsilon)\varphi(0)dx - \int_0^T \langle \varphi_t, S(u^\epsilon - g_2^\epsilon)\rangle + \int_Q S'(u^\epsilon - g_2^\epsilon)a(t,x,\nabla u^\epsilon)\nabla\varphi dxdt \\
&+ \int_Q S''(u^\epsilon - g_2^\epsilon)a(t,x,\nabla u^\epsilon)\nabla(u^\epsilon - g_2^\epsilon)\varphi dxdt = \int_Q S'(u^\epsilon - g_2^\epsilon)f^\epsilon\varphi dxdt \\
&+ \int_Q S'(u^\epsilon - g_2^\epsilon)F^\epsilon \cdot \nabla\varphi dxdt + \int_Q S''(u^\epsilon - g_2^\epsilon)F^\epsilon \cdot \nabla(u^\epsilon - g_2^\epsilon)\varphi dxdt \\
&+ \int_Q S'(u^\epsilon - g_2^\epsilon)G_1^\epsilon\nabla\varphi dxdt + \int_Q S''(u^\epsilon - g_2^\epsilon)G_1^\epsilon\nabla(u^\epsilon - g_2^\epsilon)\varphi dxdt.
\end{aligned} \tag{50}$$

We consider the first term in the left-hand side of (50). Since S is continuous, Proposition 4.1 implies that $S(u^\epsilon - g_2^\epsilon)$ converges to $S(u - g_2)$ a.e. in Q and weakly$_*$ in $L^\infty(Q)$. Then $(S(u^\epsilon - g_2^\epsilon))_t$ converges to $(S(u - g_2))_t$ in $D'(Q)$ as $\epsilon \to 0$, that is

$$\int_Q (S(u^\epsilon - g_2^\epsilon))_t\varphi dxdt \to \int_Q (S(u - g_2))_t\varphi dxdt.$$

As supp $S' \subset [-M, M]$ for some $M > 0$, we have

$$\begin{cases}
S'(u^\epsilon - g_2^\epsilon)a(t,x,\nabla u^\epsilon) = S'(u^\epsilon - g_2^\epsilon)a(t,x,\nabla T_M(u^\epsilon(u^\epsilon - g_2^\epsilon) + \nabla g_2^\epsilon)) \\
S''(u^\epsilon - g_2^\epsilon)a(t,x,\nabla u^\epsilon)\cdot\nabla(u^\epsilon - g_2^\epsilon) \\
\qquad\qquad = S''(u^\epsilon - g_2^\epsilon)a(t,x,\nabla T_M(u^\epsilon - g_2^\epsilon) + \nabla g_2^\epsilon)\nabla T_M(u^\epsilon - g_2^\epsilon).
\end{cases}$$

Using Proposition 4.1, the strong convergence of g_2^ϵ to g_2 in $L^{p_-}(0,T;W_0^{1,p(\cdot)}(\Omega))$ and assumption (16), we have

$$\begin{cases}
S'(u^\epsilon - g_2^\epsilon)a(t,x,\nabla T_M(u^\epsilon - g_2^\epsilon) + \nabla g_2^\epsilon) \\
\qquad\qquad \to S'(u - g_2)a(t,x,\nabla T_M(u - g_2) + \nabla g_2) \text{ in } (L^{p'(\cdot)}(Q))^N \\
S''(u^\epsilon - g_2^\epsilon)a(t,x,\nabla T_M(u^\epsilon - g_2^\epsilon) + \nabla g_2^\epsilon)\nabla T_M(u^\epsilon - g_2^\epsilon) \\
\qquad\qquad \to S''(u - g_2)a(t,x,\nabla T_M(u - g_2) + \nabla g_2)\nabla T_M(u - g_2) \text{ in } L^1(Q).
\end{cases}$$

The pointwise convergence of $S'(u^\epsilon - g_2^\epsilon)$ to $S'(u - g_2)$ and the strong convergence of f^ϵ to f in $L^1(Q)$ yields

$$f^\epsilon S'(u^\epsilon - g_2^\epsilon) \to fS'(u - g_2) \text{ strongly in } L^1(Q) \text{ as } \epsilon \to 0.$$

Finally, we recall that $\nabla S'(u^\epsilon - g_2^\epsilon) \rightharpoonup \nabla S'(u - g_2)$ weakly in $(L^{p(\cdot)}(Q))^N$. Then the term $S''(u^\epsilon - g_2^\epsilon)\ F \cdot \nabla(u^\epsilon - g_2^\epsilon)$ which is equal to $F \cdot \nabla S'(u^\epsilon - g_2^\epsilon)$ satisfies

$$S''(u^\epsilon - g_2^\epsilon)F \cdot \nabla(u^\epsilon - g_2^\epsilon) \rightharpoonup F \cdot \nabla S'(u - g_2) \text{ in } L^1(Q) \text{ as } \epsilon \to 0.$$

We can identifies the term $F \cdot \nabla S'(u - g_2)$ with $S''(u - g_2)\ F \cdot \nabla(u - g_2)$. As a consequence of the last convergence results, we are in position to pass to the limit as $\epsilon \to 0$ in (50), and to conclude that u satisfies Definition 3.2. It remains to show that $S(u - g_2)$ satisfies the initial condition (31). To this end, we take in mind the last convergence results of the terms of equation (50), which imply that

$$(S(u^\epsilon - g_2^\epsilon))_t \text{ is bounded in } X^* + L^1(Q).$$

While $S(u^\epsilon - g_2^\epsilon)$ strongly converges in X, we deduce [38, Theorem 1.1] that $S(u^\epsilon - g_2^\epsilon)$ being bounded in $L^\infty(Q)$ and

$$S(u^\epsilon - g_2^\epsilon) \to S(u - g_2) \text{ strongly in } C([0,T]; L^1(Q)).$$

It follows that

$$S(u^{\epsilon}-g_2^{\epsilon})(0)\to S(u_0) \text{ strongly in } L^1(Q).$$

Hence (31) is fulfilled. Thus, the proof of existence of renormalized solution u of problem (18) is complete.

Now, we try to stress the fact that the notion of renormalized solution, as in the elliptic case, should be the right one to get uniqueness. As we said before, if the datum μ belongs to $\mathcal{M}_0(Q)$, the renormalized solution turns out to be unique (see [22]); the same happens for general Sobolev spaces with diffuse measure as data and $u_0\in L^1(\Omega)$ as initial condition by choosing an appropriate test function motivated by [16]. Let S_n be defined as in Definition 3.2. We take $T_k(S_n(u_1-g_2)-S_n(u_2-g_2))$ as a test function in both the equation solved by u_1 and u_2, we subtract them to obtain $\mathcal{J}_0+\mathcal{J}_1=\mathcal{J}_2+\mathcal{J}_3+\mathcal{J}_4+\mathcal{J}_5+\mathcal{J}_6+\mathcal{J}_7$, where

$$\begin{cases}\mathcal{J}_0=\int_0^T\int_\Omega(S_n(u_1-g_2)-S_n(u_2-g_2))_tT_k(S_n(u_1-g_2)-S_n(u_2-g_2)),\\ \mathcal{J}_1=\int_0^T\int_\Omega[S_n'(u_1-g_2)a(t,x,\nabla u_1)-S_n'(u_2-g_2)a(t,x,\nabla u_2)]\\ \qquad\qquad\nabla T_k(S_n(u_1-g_2)-S_n(u_2-g_2)),\\ \mathcal{J}_2=-\int_Q[S_n''(u_1-g_2)a(t,x,\nabla u_1)\nabla(u_1-g_2)-S_n''(u_2-g_2)a(t,x,\nabla u_2)\nabla(u_2-g_2)]\\ \qquad\cdot[T_k(S_n(u_1-g_2)-S_n(u_2-g_2))],\\ \mathcal{J}_3=\int_Q f(S_n'(u_1-g_2)-S_n'(u-g_2))T_k(S_n(u_1-g_2)-S_n(u_2-g_2)),\\ \mathcal{J}_4=\int_Q F(S_n'(u_1-g_2)-S_n'(u_2-g_2))\nabla T_k(S_n(u_1-g_2)-S_n(u_2-g_2)),\\ \mathcal{J}_5=\int_Q[S_n''(u_1-g_2)F\nabla(u_1-g_2)-S_n''(u_2-g_2)F\nabla(u_2-g_2)]\\ \qquad\cdot T_k(S_n(u_1-g_2)-S_n(u_2-g_2)),\\ \mathcal{J}_6=\int_Q[G_1(S_n'(u_1-g_2)-S_n'(u_2-g_2))\nabla T_k(S_n(u_1-g_2)-S_n(u_2-g_2))],\\ \mathcal{J}_7=\int_Q[S_n''(u_1-g_2)G_1\nabla(u_1-g_2)-S_n''(u_2-g_2)G\nabla(u_2-g_2)]\\ \qquad\cdot T_k(S_n(u_1-g_2)-S_n(u_2-g_2)).\end{cases}$$

We estimate $\mathcal{J}_i$, $i=1,\dots,7$ one by one. Recalling $\Theta_k(r)$, J_0 can be written as

$$\mathcal{J}_0=\int_\Omega\Theta_k(S_n(u_1-g_2)-S_n(u_2-g_2))(T)dx \\ -\int_\Omega\Theta_k(S_n(u_1-g_2)-S_n(u_2-g_2))(T)dx.$$

Due to the same initial condition for u_1-g_2 and u_2-g_2, and properties of Θ_k,

$$\mathcal{J}_0=\int_\Omega\Theta_k(S_n(u_1-g_2)-S_n(u_2-g_2))(0)dx\geq 0.$$

We deal with $\mathcal{J}_1$ splitting it as bellow.

$$\begin{aligned}\mathcal{J}_1=&\int_{\{|S_n(u_1-g_2)-S_n(u_2-g_2)|\le k\}\cap\{|u_1-g_1|\le n,|u_2-g_2|\le n\}}[a(t,x,\nabla u_1)-a(t,x,\nabla u_2)]\cdot(\nabla u_1-\nabla u_2)\\ &+\int_{\{|S_n(u_1-g_2)-S_n(u_2-g_2)|\le k\}\cap\{|u_1-g_2|\le n,|u_2-g_2|>n\}}\\ &\quad[S_n'(u_1-g_2)a(t,x,\nabla u_1)-S_n'(u_2-g_2)a(t,x,\nabla u_2)]\cdot\nabla(S_n(u_1-g_2)-S_n(u_2-g_2))]\\ &+\int_{\{|S_n(u_1-g_2)-S_n(u_2-g_2)|\le k\}\cap\{|u_2-g_2|>n\}}\\ &\quad[S_n'(u_1-g_2)a(t,x,\nabla u_1)-S_n'(u_2-g_2)a(t,x,\nabla u_2)]\cdot\nabla(S_n(u_1-g_2)-S_n(u_2-g_2))]\\ &:=\mathcal{J}_1^1+\mathcal{J}_1^2+\mathcal{J}_1^3.\end{aligned}$$

Next, as $\{|S_n(u_1-g_2)-S_n(u_2-g_2)|\leq k, |u_1-g_2|>n\}\subset\{|u_1-g_2|>n, |u_2-g_2|>n-k\}$ and using the fact that $S_n'(t)=0$ if $|t|>n+1$ and $|S_n'(t)|\leq 1$, we have

$$\begin{aligned}|\mathcal{J}_1^3|\leq&\int_{\{n\leq|u_2-g_2|\leq n+1\}}|a(t,x,\nabla u_1)||\nabla(u_1-g_2)|dxdt\\&+\int_{\{n\leq|u_1-g_2|\leq n+1\}\cap\{n-k\leq|u_2-g_2|\leq n+1\}}|a(t,x,\nabla u_1)||\nabla(u_2-g_2)|dxdt\\&+\int_{\{n\leq|u_1-g_2|\leq n+1\}\cap\{n-k\leq|u_2-g_2|\leq n+1\}}|a(t,x,\nabla u_2)||\nabla(u_1-g_2)|dxdt\\&+\int_{\{n-k\leq|u_2-g_2|\leq n+1\}}|a(t,x,\nabla u_2)||\nabla(u_2-g_2)|dxdt.\end{aligned}\tag{51}$$

We deduce from the first integral in the right- hand side of (51),

$$\begin{aligned}&\int_{\{n\leq|u_1-g_2|\leq n+1\}}|a(t,x,\nabla u_1)||\nabla(u_1-g_2)|dxdt\\&\leq\int_{\{n\leq|u_1-g_2|\leq n+1\}}\beta(b(t,x)+|\nabla u_1|^{p(x)-1})|\nabla(u_1-g_2)|dxdt\\&\leq\int_{\{n\leq|u_1-g_2|\leq n+1\}}\beta b(t,x)|\nabla(u_1-g_2)|\\&\quad+\int_{\{n\leq|u_1-g_2|\leq n+1\}}\beta|\nabla u_1|^{p(x)-1}|\nabla(u_1-g_2)|dxdt\\&\leq\int_{\{|u_1-g_2|\leq n+1\}}\frac{C}{p'_-}|b(t,x)|^{p'(x)}dxdt+\int_{\{|u_1-g_2|\leq n+1\}}\frac{C}{p_-}|\nabla(u_1-g_2)|^{p(x)}dxdt\\&\quad+\int_{\{|u_1-g_2|\leq n+1\}}\frac{C}{p'_-}|\nabla u_1|^{p(x)}dxdt+\int_{\{|u_1-g_2|\leq n+1\}}\frac{C}{p_-}|\nabla(u_1-g_2)|^{p(x)}dxdt.\end{aligned}$$

Since $b(t,x)$ is bounded in $L^{p'_-}(0,T;W_0^{1,p(\cdot)}(\Omega))$ and meas$\{n\leq|u_1-g_2|\leq n+1\}$ converges uniformly to zero as $n\to\infty$, we deduce from conditions (29) and (33) that

$$\lim_{n\to+\infty}\int\int_{|u_1-g_2|\leq n+1}|a(t,x,\nabla u_1)||\nabla(u_1-g_2)|dxdt=0.$$

Similarly, we prove that all the other integrals in the right-hand side of (51) converge to zero as $n\to+\infty$. Thus $\mathcal{J}_1^3$ converges to zero. Changing the roles of u_1-g_2 and u_2-g_2, we may get the similar arguments for $\mathcal{J}_1^2$. Furthermore, $\mathcal{J}_1^2$ converges to zero. An application of Fatou's Lemma gives

$$\liminf_{n\to+\infty}\mathcal{J}_1\geq\int_{\{|u_1-u_2|\leq k\}}[a(t,x,\nabla u_1)-a(t,x,\nabla u_2)](\nabla u_1-\nabla u_2)dxdt.$$

Now, we can pass to the study of the limit of $\mathcal{J}_2$. We have

$$\begin{aligned}\mathcal{J}_2=&\int_0^T\int_\Omega[S_n''(u_1-g_2)a(t,x,\nabla u_1)\nabla(u_1-g_2)]T_k(S_n(u_1-g_2)-S_n(u_2-g_2))dxdt\\&+\int_0^T\int_\Omega[S_n''(u_2-g_2)a(t,x,\nabla u_2)\nabla(u_2-g_2)]T_k(S_n(u_1-g_2)-S_n(u_2-g_2))dxdt\\&=\mathcal{J}_2^1+\mathcal{J}_2^2.\end{aligned}$$

By symmetry between $\mathcal{J}_2^1$ and $\mathcal{J}_2^2$, it is enough to prove that J_2^1 tends to zero. Since $|S_n''(s)| \leq 1$ and $S_n''(s) \neq 0$ only if $|s| \in [n, n+1]$, using (16) we can write

$$\begin{aligned}
|\mathcal{J}_2^1| &\leq k \int\int_{\{n \leq |u_1 - g_2| \leq n+1\}} |a(t,x,\nabla u_1)||\nabla(u_1 - g_2)| \\
&\leq k \int_{\{n \leq |u_1 - g_2| \leq n+1\}} \beta(b(t,x) + |\nabla u_1|^{p(x)-1})||\nabla(u_1 - g_2)|dxdt \\
&\leq k \int_{\Omega} \beta(b(t,x) + |\nabla u_1|^{p(x)-1})|\nabla(u_1 - g_2)|\chi_{\{n \leq |u_1 - g_2| \leq n+1\}} dxdt \\
&\to 0 \text{ as } n \to +\infty.
\end{aligned}$$

We conclude that $\lim_{n \to +\infty} \mathcal{J}_2 = 0$. Let us recall that by definition of S_n, we have that S_n' converge to 1 for every s in $\mathbb{R}$. Then

$$f(S_n'(u_1 - g_2) - S_n'(u_2 - g_2)) \to 0 \text{ strongly in } L^1(Q) \text{ as } n \to +\infty.$$

Using the dominated convergence Theorem, we deduce that $\lim_{n \to +\infty} \mathcal{J}_3 = 0$. Let us study the limit of $\mathcal{J}_6$, we have $S_n'(u_1 - g_2) - S_n'(u_2 - g_2) = 0$ in $\{|u_1 - g_2| \leq n, |u_2 - g_2| \leq n\} \cup \{|u_1| > n+1, |u_2| > n+1\}$, then $\mathcal{J}_6 = \mathcal{J}_6^1 + \mathcal{J}_6^2 + \mathcal{J}_6^3$, where

$$\begin{aligned}
\mathcal{J}_6^1 = \int_{\{|S_n(u_1-g_2) - S_n(u_2-g_2)| \leq k\} \cap \{|u_1 - g_2| \leq n, |u_2 - g_2| > n\}} &[G_1(S_n'(u_1 - g_2) - S_n'(u_2 - g_2)) \\
&\cdot \nabla(S_n(u_1 - g_2) - S_n(u_2 - g_2))]
\end{aligned}$$

Recalling that $S_n(t) = t$ if $|t| \leq n$, S_n is nondecreasing and Supp $S_n' \subset [-n-1, n+1]$,

$$|\mathcal{J}_6^1| \leq \int_{\{n-k \leq |u_1 - g_2| \leq n\}} |G_1||\nabla(u_1 - g_2)|dxdt + \int_{\{n \leq |u_2 - g_2| \leq n+1\}} |G_1||\nabla(u_2 - g_2)|dxdt.$$

So that, using Hölder's inequality, we get

$$\begin{aligned}
&|\mathcal{J}_6^1| \leq C\|G_1\|_{p'(x)} \\
&\times (\max(\int_{\{n-k \leq |u_1 - g_2 \leq n|\}} |\nabla u_1 - \nabla g_2|^{p(x)})^{\frac{1}{p-}}, (\int_{\{n-k \leq |u_1 - g_2| \leq n\}} |\nabla u_1 - \nabla g_2|^{p(x)} dxdt)^{\frac{1}{p+}}) \\
&+ \max(\int_{\{n \leq |u_2 - g_2| \leq n+1\}} |\nabla u_2 - \nabla g_2|^{p(x)} dxdt)^{\frac{1}{p-}}, (\int_{\{n \leq |u_2 - g_2| \leq n+1\}} |\nabla u_2 - \nabla g_2|^{p(x)} dxdt)^{\frac{1}{p+}}))
\end{aligned}$$

Thus by (29) we get that $(\mathcal{J}_6^1)$ converges to 0 as $n \to \infty$. The same is true for $(\mathcal{J}_6^2)$

$$\begin{aligned}
\mathcal{J}_6^2 = \int_{\{|S_n(u_1-g_2) - S_n(u_2-g_2)| \leq k\} \cap \{n \leq |u_1 - g_2| \leq n+1\}} &[G_1(S_n'(u_1 - g_2) - S_n'(u_2 - g_2)) \\
&\cdot \nabla(S_n(u_1 - g_2) - S_n(u_2 - g_2))dxdt].
\end{aligned}$$

Since $|S_n(t)| > n - k$ implies $|t| > n - k$, we have

$$|\mathcal{J}_6^2| \leq \int_{\{n \leq |u_1 - g_2| \leq n+1\}} |G_1||\nabla(u_1 - g_2)| + \int_{\{n-k \leq |u_2 - g_2| \leq n+1\}} |G_1||\nabla(u_2 - g_2)|dxdt.$$

So that using Hölder's inequality and (29), we get that $(\mathcal{J}_6^2)$ converges to zero. The term $(\mathcal{J}_6^3)$ can be dealt with the same way using that $S_n'(t) = 0$ if $|t| > n+1$. Hence we deduce $\lim_{n \to +\infty} \mathcal{J}_6 = 0$. As regards $(\mathcal{J}_7)$, note that using the properties of S_n'' and

(16), we can split the integral as follows.

$$
\begin{aligned}
|\mathcal{J}_7| = &\int_Q S_n''(u_1-g_2)G_1\cdot\nabla(u_1-g_2)T_k(S_n(u_1-g_2)-S_n(u_2-g_2))dxdt \\
&-\int_Q S_n''(u_2-g_2)G_1\cdot\nabla(u_2-g_2)T_k(S_n(u_1-g_2)-S_n(u_2-g_2))dxdt.
\end{aligned} \tag{52}
$$

We denote $(\mathcal{J}_7^1,\mathcal{J}_7^2)$ the two integrals of (52). Using the properties of S_n and S_n'' (recall that $S_n''(s)=-\text{sgn}(s)\chi_{\{n\le|s|\le n+1\}}$) we have

$$
\begin{aligned}
|\mathcal{J}_7^1| &\le k\int_{\{n\le|u_1-g_2|\le n+1\}}|G_1||\nabla(u_1-g_2)|dxdt\le Ck\|G_1\|_{L^{p'(x)}(Q)}\\
&\times\max(\int_{\{n\le|u_1-g_2|\le n+1\}}|\nabla u_1-\nabla g_2|^{p(x)})^{\frac{1}{p_-}},(\int_{\{n\le|u_1-g_2|\le n+1\}}|\nabla u_1-\nabla g_2|^{p(x)})^{\frac{1}{p_+}}).
\end{aligned}
$$

Applying Hölder inequality and using property (33), we easily get that $(\mathcal{J}_7^1)$ converges to zero as n tends to infinity. Similarly, we have

$$
\begin{aligned}
|\mathcal{J}_7^2| &\le Ck\|G_1\|_{L^{p'(x)}(Q)}\\
&\times\max(\int_{\{n\le|u_2-g_2|\le n+1\}}|\nabla u_2-\nabla g_2|^{p(x)})^{\frac{1}{p_-}},(\int_{\{n\le|u_2-g_2|\le n+1\}}|\nabla u_2-\nabla g_2|^{p(x)})^{\frac{1}{p_+}}).
\end{aligned}
$$

Again Hölder inequality together with (29) allow to deduce that $(\mathcal{J}_7^2)$ converges to zero as well. So that we finally get that $\lim\limits_{n\to+\infty}\mathcal{J}_7=0$. Similarly we have $\lim\limits_{n\to+\infty}\mathcal{J}_4=0$ and $\lim\limits_{n\to+\infty}\mathcal{J}_5=0$. Putting together $(\mathcal{J}_1)-(\mathcal{J}_6)$ and $(\mathcal{J}_7)$, we obtain $\lim\limits_{n\to\infty}\sum_{i=0}^{1}\mathcal{J}_i=\lim\limits_{n\to\infty}\sum_{i=2}^{7}\mathcal{J}_i$, as n tends to infinity. Then

$$
\int_{\{|u_1-u_2|\le k\}}[a(t,x,\nabla u_1)-a(t,x,\nabla u_2)](\nabla u_1-\nabla u_2)dxdt\le 0
$$

letting k tends to infinity (recall that u_1 and u_2 are finite a.e. in Q), we deduce that

$$
\int_Q[a(t,x,\nabla u_1)-a(t,x,\nabla u_2)](\nabla u_1-\nabla u_2)dxdt\le 0.
$$

The strict monotonicity assumption (17) implies that $\nabla u_1=\nabla u_2$ a.e. in Q. Then, let $\zeta_n=T_1(T_{n+1}(u_1-g_2)-T_{n+1}(u_2-g_2))$. We have $\zeta_n\in L^{p_-}(0,T;W_0^{1,p(\cdot)}(\Omega))$ and since $\nabla(u_1-g_2)=\nabla(u_2-g_2)$ a.e. in Q,

$$
\nabla\zeta_n=\begin{cases}0 \text{ on } \{|u_1-g_2|\le n+1,|u_2-_2|\le n+1\}\cup\{|u_1-g_2|>n+1,|u_2-g_2|>n+1\}\\ 1_{\{u_1-g_2-T_{n+1}(u_2-g_2)|\le 1\}}\nabla(u_1-g_2) \text{ on } \{|u_1-g_2|\le n+1,|u_2-g_2|>n+1\}\\ -1_{\{u_2-g_2-T_{n+1}(u_1-g_2)|\le 1\}}\nabla(u_2-g_2) \text{ on } \{|u_1-g_2|>n+1,|u_2-g_2|\le n+1\}.\end{cases}
$$

But, if $|s|>n+1,|t|\le n+1$ and $|t-T_{n+1}(s)|\le 1$, then $n\le|t|\le n+1$, and

$$
\begin{aligned}
\int_Q|\nabla\zeta_n|^{p(x)}dxdt\le&\int_{\{n\le|u_1-g_2|\le n+1\}}|\nabla(u_1-g_2)|^{p(x)}dxdt\\
&+\int_{\{n\le|u_2-g_2|\le n+1\}}|\nabla(u_2-g_2)|^{p(x)}dxdt\to 0 \text{ as } n\to+\infty.
\end{aligned}
$$

Then, $\zeta_n\to 0$ in $L^{p_-}(0,T;W_0^{1,p(\cdot)}(\Omega))$, and thus in $\mathcal{D}'(Q)$ as $n\to+\infty$. Since $\zeta_n\to T_1(u_1-g_1)-(u_2-g_2))$ a.e. in S as $n\to+\infty$ and remains bounded by 1, we also have $\zeta_n\to T_1((u_1-g_2)-(u_2-g_2))$ in $\mathcal{D}'(Q)$. Hence, $T_1((u_1-g_2)-(u_2-g_2))=0$

i.e., $u_1 - g_2 = u_2 - g_2$ on Q. Therefore $u_1 = u_2$. Thus, we obtain the uniqueness of the renormalized solution to (18). □

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