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Fixed point results for a new three steps iteration process

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Abstract. In this paper, we introduce a new three steps iteration process for approximating the fixed point of a contractive like mapping and Suzuki generalized nonexapansive mapping in the frame work of uniformly convex Banach space. Using our iteration process, we state and prove some convergence results for approximating the fixed points of Suzuki generalized nonexpansive mappings. In addition, we show that our proposed iterative scheme converges faster than some existing iterative schemes in the literature and that it is equivalent to the well known Mann iteration method in the sense of convergence. Finally, the stability (T-stable, weak w^2-stable) and data dependency results for our proposed iterative scheme are established with an analytical and numerical example given to justify our claim.

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1. Introduction

Let $(X, \|\cdot\|)$ denote a real Banach space and C be a nonempty closed and convex subset of X. A fixed point problem for a mapping $T : C \to C$ is given as: find $x \in C$ such that

$$Tx = x. \tag{1}$$

We denote the set of all fixed points of T by $F(T)$. The theory of fixed point has progressively become an invaluable area of study as many problems in mathematics, engineering, physics, economics, game theory, etc can be transformed into a fixed point problem. In general, solving fixed point problems analytically is almost impossible, thus the need for iterative solution arises. Over the years, researchers have developed several iterative schemes for approximating the solution (1) for different operators and spaces, for example, see ([12, 13, 14, 15, 16, 29, 30]). Developing a faster and more efficient iterative algorithms for solving (1) is still an active area of research. A good and reliable fixed point iteration is required at least to posses the following attributes:

(1) it should converge to a fixed point of an operator;
(2) it should be T-stable;
(3) it should be fast compare to other existing iteration in literature;
(4) it should show data dependence result.

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The Picard iterative process

$$x_{n+1} = Tx_n, \ \forall n \in \mathbb{N}, \tag{2}$$

is one of the earliest iterative process used to approximate Equation (1), where T is a contraction mapping. If T is nonexpansive, the Picard iterative process fails to approximate Equation (1) even when the existence of the fixed point is guaranteed. To overcome this limitation, researchers in this area developed different iterative processes to approximate fixed points of nonexpansive mappings and other mappings more general than nonexpansive mappings. Among many others, we have the Mann [20], Ishikawa [10], Krasnosel'skii [19], Noor [22], Abbas et al., [3], Jungck-AM [21] and so on. There are numerous papers dealing with the approximation of fixed points of nonexpansive mappings, asymptotically nonexpansive mappings, total asymptotically nonexpansive mappings in uniformly convex Banach spaces, see [1, 2, 3, 4] and the references therein.

In 2005, Suntai in [27], proposed the following iterative process: For each $u_0 \in C$, the sequence $\{u_n\}$ in C is defined by

$$\begin{cases} w_n = (1-c_n)u_n + c_nTu_n, \\ v_n = (1-a_n-b_n)u_n + a_nTw_n + b_nTu_n, \\ u_{n+1} = (1-\alpha_n-\beta_n)u_n + \alpha_nTv_n + \beta_nTw_n, \ n \geq 1, \end{cases} \tag{3}$$

where $\{\alpha_n\}, \{\beta_n\}, \{a_n\}, \{b_n\}$ and $\{c_n\}$ are sequences in $[0,1]$ such that $(\{a_n\} + \{b_n\}), (\{\alpha_n\} + \{\beta_n\})$ are in $[0,1]$.

In 2011, Sahu [25] introduced the Normal S-iteration process in Banach space and show that the rate of convergence of this iteration process is as fast as the Picard iteration process and faster than other existing iteration schemes in literature. The Normal S-iteration process is given as follows: Given a convex subset C of a normed space E and $T : C \to C$ a nonlinear mapping. For each $x_0 \in C$, the sequence $\{x_n\}$ in C is defined by

$$\begin{cases} y_n = (1-\alpha_n)x_n + \alpha_nTx_n, \\ x_{n+1} = Ty_n, \ n \geq 1, \end{cases} \tag{4}$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$.

In 2013, Karakaya et al., in [17], proposed the following iterative process: For each $q_0 \in C$, the sequence $\{q_n\}$ in C is defined by

$$\begin{cases} s_n = (1-c_n)q_n + c_nTq_n, \\ p_n = (1-a_n-b_n)s_n + a_nTs_n + b_nTq_n, \\ q_{n+1} = (1-\alpha_n-\beta_n)p_n + \alpha_nTp_n + \beta_nTs_n, \ n \geq 1, \end{cases} \tag{5}$$

where $\{\alpha_n\}, \{\beta_n\}, \{a_n\}, \{b_n\}$ and $\{c_n\}$ are sequences in $[0,1]$ such that $(\{a_n\} + \{b_n\}), (\{\alpha_n\} + \{\beta_n\})$ are in $[0,1]$. They show that the rate of convergence of this iterative scheme is faster than that of (3) with the aid of a numerical example.

In [11], Kadioglu and Yildirim introduced a Picard Normal S-iteration process and show that the rate of convergence of this iteration process is faster than the Normal S-iteration process. This iteration process is defined as follows: Given a convex subset C of a normed space E and a nonlinear mapping $T : C \to C$. For each $x_0 \in C$, the sequence $\{x_n\}$ in C is defined by

$$\begin{cases} z_n = (1-\beta_n)x_n + \beta_n T x_n, \\ y_n = (1-\alpha_n)z_n + \alpha_n T z_n, \\ x_{n+1} = T y_n, \ n \geq 1, \end{cases} \tag{6}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$.
In 2014, Gursoy and Karakay in [7] introduced a new iteration process called Picard-S iteration process, which is defined as follows: Given a convex subset C of a normed space E and a nonlinear mapping $T : C \to C$. For each $x_0 \in C$, the sequence $\{x_n\}$ in C is defined by

$$\begin{cases} z_n = (1-\alpha_n)x_n + \alpha_n T x_n, \\ y_n = (1-\beta_n)T x_n + \beta_n T z_n \\ x_{n+1} = T y_n, \ n \geq 1, \end{cases} \tag{7}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$.
They proved that this iterative process converges faster than Picard, Mann, Ishikawa, Noor, Abbas et al., and other existing iterative schemes in literature.
In 2017, Karakaya et al., in [18] introduced the following iteration process: For each $x_0 \in C$, the sequence $\{x_n\}$ in C is defined by

$$\begin{cases} z_n = T x_n, \\ y_n = (1-\alpha_n)z_n + \alpha_n T z_n \\ x_{n+1} = T y_n, \ n \geq 1, \end{cases} \tag{8}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$. They proved that this iterative process converges faster than Picard, Mann, Ishikawa, Noor, Abass et al., and other existing iterative schemes in literature.
In 2018, Ullah et al., in [31] introduced a new iteration process called M iteration process, which is given as: For each $x_0 \in C$, the sequence $\{x_n\}$ in C is defined by

$$\begin{cases} z_n = (1-\alpha_n)x_n + \alpha_n T x_n, \\ y_n = T z_n \\ x_{n+1} = T y_n, \ n \geq 1, \end{cases} \tag{9}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$.
They proved that this iterative process converges faster than Picard, Mann, Ishikawa, Noor, Abass et al., SP, CR, Normal-S process and the existing iterative schemes in literature.

Definition 1.1. Let C be a nonempty subset of a metric space (X,d). A self mapping T on C is said to be

(1) an a-contraction mapping, if for each $x, y \in C$ and $a \in (0,1)$, $\|Tx - Ty\| \leq a\|x - y\|$;
(2) Kannan mapping, if there exists $b \in (0, \frac{1}{2})$ for each $x, y \in C$, $\|Tx - Ty\| \leq b[\|x - Tx\| + b\|y - Ty\|]$;
(3) Chatterjea mapping, if there exists $c \in (0, \frac{1}{2})$ for each $x, y \in C$, $\|Tx - Ty\| \leq c[\|x - Ty\| + b\|y - Tx\|]$.

Combining the above definitions, Zamfirescu [34] introduced a class of mappings called Zamfirescu mappings and established some fixed point results for this class of mappings. This class of mappings is defined as follows:

Definition 1.2. Let X be a metric space. $T : X \to X$ is called a Zamfirescu mapping if there exist real numbers, a, b and c satisfying $0 \leq a < 1$ and $b, c \in (0, \frac{1}{2})$ such that for all $x, y \in X$, at least one of the following conditions holds:
(1) $\|Tx - Ty\| \leq a\|x - y\|$;
(2) $\|Tx - Ty\| \leq b[\|x - Tx\| + \|y - Ty\|]$;
(3) $\|Tx - Ty\| \leq c[\|x - Ty\| + \|y - Tx\|]$.

Theorem 1.1. [34] *Let X be a complete metric space and $T : X \to X$ be a Zamfirescu mapping. Then T has a unique fixed point say x^* and the Picard iterative process converges to x^*.*

In [5] Berinde introduced another class of mappings in metric space satisfying

$$\|Tx - Ty\| \leq \delta\|x - y\| + L\|x - Tx\|, \tag{10}$$

for all $x, y \in C, \delta \in (0, 1) \quad$ and $L \geq 0$.

He show that this class of mapping satisfying (10) contain the class of Zamfirescu mappings.
In [9], Imoru and Olantiwo gave the following contractive definition.

Definition 1.3. Let T be a self-mapping on a Banach space X. The mapping T is called contractive-like mapping if there exist a constant $\delta \in [0, 1)$ and a strictly increasing and continuous function $\xi : [0, \infty) \to [0, \infty)$ with $\xi(0) = 0$ such that for all $x, y \in X$,

$$\|Tx - Ty\| \leq \delta\|x - y\| + \xi(\|x - Tx\|). \tag{11}$$

They show that this class of mappings satisfying (11) is more general than those considered by Berinde [**?**], Osilike and Udomene [24] and some other contractive like mappings in literature.

Remark 1.1. If $\xi(t) = Lt$, then (11) reduces to (10).

Inspired and motivated by the above works and the ongoing research interest in this direction, our purpose of this work is to introduce a new three steps iteration process (13) and show that the proposed iterative scheme can be used to approximate the fixed point of a contractive like mapping, Suzuki generalized nonexpansive mappings and establish some convergence results for approximating the fixed points of Suzuki generalized nonexapansive mappings in the frame work of uniformly convex Banach space. In addition, we show that the proposed iterative scheme perform faster than some existing iterative schemes in literature and that it is equivalent to the well known Mann iteration method in the sense of convergence. Finally, the stability (T-stable, weak w^2-stable)and data dependency results for the proposed iterative scheme are established with an analytical and numerical example given to justify our claim.

2. Preliminaries

In this section, we give some definitions and important results which are useful in establishing our main results.

Let X be a Banach space and $S_X = \{x \in X : \|x\| \leq 1\}$ be a unit ball in X. For $\alpha \in (0,1)$ and $x, y \in S_X$ such that $x \neq y$, if $\|(1-\alpha)x + \alpha y\| < 1$, then we say X is strictly convex. If X is a strictly convex Banach space and $\|x\| = \|y\| = \|(1-\lambda)y + \lambda x\|$ $\forall x, y \in X$ and $\lambda \in (0,1)$, then $x = y$.

Definition 2.1. A Banach space X is said to be smooth if

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{12}$$

exists for all $x, y \in S_X$.

In the above definition, the norm of X is called Gateaux differentiable. For all $y \in S_X$, if the limit (12) is attained uniformly for $x \in S_X$, then the norm is said to be uniformly Gateaux differentiable or Fréchet differentiable.

Definition 2.2. A Banach space X satisfies Opial's condition [23], if for any sequence $\{x_n\} \subset X, x_n \rightharpoonup x$ implies that

$$\limsup_{n \to \infty} \|x_n - x\| < \limsup_{n \to \infty} \|x_n - y\|,$$

for all $y \in X$ such that $x \neq y$.

Definition 2.3. Let C be a subset of a normed space X. A mapping $T : C \to C$ is said to satisfy condition (A) if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ such that $f(0) = 0$ and $f(t) > 0$ $\forall$ $t \in (0, \infty)$ and that $\|x - Tx\| \geq f(d(x, F(T)))$ for all $x \in C$ where $d(x, F(T))$ denotes distance from x to $F(T)$.

Berinde [6] proposed a method to compare the fastness of two sequences.

Lemma 2.1. [6] *Let $\{a_n\}$ and $\{b_n\}$ be two sequences of real numbers converging to a and b respectively. If $\lim_{n\to\infty} \frac{|a_n - a|}{|b_n - b|} = 0$, then $\{a_n\}$ converges faster than $\{b_n\}$.*

Lemma 2.2. [6] *Suppose that for two fixed point iteration processes $\{u_n\}$ and $\{v_n\}$ both converging to the same fixed point x^*, the error estimates*

$$\|u_n - x^*\| \leq a_n \qquad n \geq 1,$$
$$\|v_n - x^*\| \leq b_n \qquad n \geq 1,$$

are available where $\{a_n\}$ and $\{b_n\}$ are two sequences of positive numbers converging to zero. If $\{a_n\}$ converges faster than $\{b_n\}$, then $\{u_n\}$ converges faster than $\{v_n\}$ to x^*.

Definition 2.4 ([6])**.** Let $T, \overline{T} : C \to C$ be two operators. We say that $\overline{T}$ is an approximate operator for T if for some $\epsilon > 0$, we have

$$\|Tx - \overline{T}x\| \leq \epsilon,$$

for all $x \in C$.

Definition 2.5. [8] Let $\{t_n\}$ be any arbitrary sequence in C. Then, an iteration procedure $x_{n+1} = f(T, x_n)$, converging to fixed point p, is said to be T-stable or stable with respect to T, if for $\epsilon_n = \|t_{n+1} - f(T, t_n)\|$, $\forall n \in \mathbb{N}$, we have $\lim_{n\to\infty} \epsilon_n = 0$ if and only if $\lim_{n\to\infty} t_n = p$.

Definition 2.6. Two sequences say $\{x_n\}$ and $\{y_n\}$ are said to be equivalence if $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Definition 2.7. [33] Let $\{t_n\}$ be an equivalent sequence of $\{x_n\}$. Then an iteration procedure $x_{n+1} = f(T, x_n)$, converging to fixed point p, is said to be weak w^2-stable with respect to T, if and only if $\lim_{n\to\infty} \|t_{n+1} - f(T, t_n)\| = 0,\ \forall n \in \mathbb{N}$, implies that $\lim_{n\to\infty} t_n = p$.

Lemma 2.3. [32] *Let $\{\Psi_n\}$ and $\{\Phi_n\}$ be nonnegative real sequences satisfying the following inequality:*

$$\Psi_{n+1} \leq (1 - \phi_n)\Psi_n + \Phi_n,$$

where $\phi_n \in (0,1)$ for all $n \in \mathbb{N}, \sum_{n=0}^{\infty} \phi_n = \infty$ and $\lim_{n\to\infty} \frac{\Phi_n}{\phi_n} = 0$, then $\lim_{n\to\infty} \Psi_n = 0$.

Lemma 2.4. [26] *Let $\{\Psi_n\}$ and $\{\Phi_n\}$ be nonnegative real sequences satisfying the following inequality:*

$$\Psi_{n+1} \leq (1 - \phi_n)\Psi_n + \phi_n\Phi_n,$$

where $\phi_n \in (0,1)$ for all $n \in \mathbb{N}, \sum_{n=0}^{\infty} \phi_n = \infty$ and $\Phi_n \geq 0$ for all $n \in \mathbb{N}$, then

$$0 \leq \limsup_{n\to\infty} \Psi_n \leq \limsup_{n\to\infty} \Phi_n.$$

Lemma 2.5. *Let X be a uniformly convex Banach space and $0 < p \leq t_n \leq q < 1\ \forall n \in \mathbb{N}$. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X such that $\limsup_{n\to\infty} \|x_n\| \leq c, \limsup_{n\to\infty} \|y_n\| \leq c$ and $\lim_{n\to\infty} \|t_n x_n + (1 - t_n)y_n\| = c$ hold for some $c \geq 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Proposition 2.6. [28] *Let C be a nonempty closed subset of a Banach space X with the Opial property and $T : C \to C$ a Suzuki generalized nonexpansive mapping. If $\{x_n\}$ converges weakly to a point z and $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$, then T(z) = z.*

Definition 2.8. Let C be a nonempty subset of a Banach space X and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is called a Fejér monotone sequence with respect to C if for all $x \in C$ and $n \geq 1$,

$$\|x_{n+1} - x\| \leq \|x_n - x\|.$$

Proposition 2.7. *Let $\{x_n\}$ be a sequence in X and C be a nonempty subset of X. Suppose that $T : C \to C$ is any nonlinear mapping and the sequence $\{x_n\}$ is Fejer monotone with respect to C, then we have the following:*
(i) *$\{x_n\}$ is bounded.*
(ii) *The sequence $\{\|x_n - x^*\|\}$ is decreasing and converges for all $x^* \in F(T)$.*

Lemma 2.8 ([28])**.** *Let C be a nonempty subset of a Banach space X. Let $T : C \to C$ be a Suzuki generalized nonexpansive mapping and $F(T) \neq \emptyset$, then T is quasi-nonexpansive.*

3. Rate of convergence, stability and data dependency

In this section, we establish the rate of convergence, stability and data dependency results for the iterative process (13). In addition, we show that the proposed iterative scheme perform faster than some existing iterative schemes in literature and that it is equivalent to the well known Mann iteration method in the sense of convergence.

We define our iterative process as follows: For each $x_0 \in C$, the sequence $\{x_n\}$ in C is defined by

$$\begin{cases} z_n = (1-\gamma_n)x_n + \gamma_n T x_n, \\ y_n = (1-\alpha_n-\beta_n)z_n + \alpha_n T z_n + \beta_n T x_n, \\ x_{n+1} = T y_n, \ n \geq 1, \end{cases} \tag{13}$$

where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0,1]$, satisfying $(\alpha_n+\beta_n) \in [0,1]$.

Remark 3.1. (1) If $\gamma_n = \alpha_n = \beta_n = 0$, then iteration (13) reduces to (2).
(2) If $\gamma_n = \alpha_n = 0, (\gamma_n = \beta_n = 0)$ then iteration (13) reduces to (4).
(3) If $\beta_n = 0$, then iteration (13) reduces to (6).
(4) If $\beta_n + \alpha_n = 1$, then iteration (13) reduces to (7).
(5) If $\gamma_n = 1$ and $\beta_n = 0$, then iteration (13) reduces to (8).
(6) If $\alpha_n = 1$ and $\beta_n = 0$, then iteration (13) reduces to (9).

Theorem 3.1. *Let C be a nonempty closed and convex subset of a uniformly convex Banach space X. Let T be a mapping satisfying (11) and $\{x_n\}$ be defined by the iteration process (13) with sequences $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ in $[0,1]$ such that $(\alpha_n + \beta_n) \in [0,1]$ satisfying $\sum_{n=0}^{\infty}(a_n + b_n) = \infty$. Then $\{x_n\}$ converges strongly to a unique fixed point of T.*

Proof. To start with, we will establish that $\lim_{n\to} x_n = x^*$. Using (13), we have

$$\begin{aligned} \|z_n - x^*\| &\leq (1-\gamma_n)\|x_n - x^*\| + \gamma_n\|Tx_n - x^*\| \\ &= (1-\gamma_n)\|x_n - x^*\| + \gamma_n\|Tx^* - Tx_n\| \\ &\leq (1-\gamma_n)\|x_n - x^*\| + \gamma_n\delta\|x^* - x_n\| + \gamma_n\xi(\|x^* - Tx^*\|) \\ &= (1-(1-\delta)\gamma_n)\|x_n - x^*\|. \end{aligned} \tag{14}$$

Using (13) and (14), we also have

$$\begin{aligned} \|y_n - x^*\| &\leq (1-\alpha_n-\beta_n)\|z_n - x^*\| + \alpha_n\|Tz_n - x^*\| + \beta_n\|Tx_n - x^*\| \\ &= (\alpha_n\delta + 1 - \alpha_n - \beta_n)\|z_n - x^*\| + \beta_n\delta\|x_n - x^*\| \\ &\leq [(\alpha_n\delta + 1 - \alpha_n - \beta_n)(1-(1-\delta)\gamma_n) + \beta_n\delta]\|x_n - x^*\| \\ &= [1 - \alpha_n - \beta_n + \alpha_n\delta + \beta_n\delta - (1-\delta)\gamma_n(1 + \alpha_n\delta - \alpha_n - \beta_n)]\|x_n - x^*\| \\ &\leq [1 - \alpha_n - \beta_n + \alpha_n\delta + \beta_n\delta]\|x_n - x^*\| \\ &= (1-(1-\delta)(\alpha_n+\beta_n))\|x_n - x^*\|. \end{aligned} \tag{15}$$

Using (13) and (15), we have

$$\begin{aligned} \|x_{n+1} - x^*\| &\leq \|Ty_n - x^*\| \\ &\leq \delta\|y_n - x^*\| \\ &\leq \delta(1-(1-\delta)(\alpha_n+\beta_n))\|x_n - x^*\|. \end{aligned} \tag{16}$$

From (16), we have

$$\begin{aligned}
\|x_{n+1} - x^*\| &\leq \delta(1 - (1-\delta)(\alpha_n + \beta_n))\|x_n - x^*\| \\
\|x_n - x^*\| &\leq \delta(1 - (1-\delta)(\alpha_{n-1} + \beta_{n-1}))\|x_{n-1} - x^*\| \\
&\vdots \\
\|x_1 - x^*\| &\leq \delta(1 - (1-\delta)(\alpha_0 + \beta_0))\|x_n - x^*\|. \qquad (17)
\end{aligned}$$

From (17), we have that

$$\|x_{n+1} - x^*\| \leq \|x_0 - x^*\|\delta^{n+1} \prod_{m=0}^{n} (1 - (1-\delta)(\alpha_m + \beta_m)). \qquad (18)$$

Since $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \delta$ and $(\alpha_n + \beta_n)$ are in $[0, 1]$, we have $(1 - (1-\delta)(\alpha_n + \beta_n)) \in (0, 1)$. We recall the inequality $1 - x \leq e^{-x}$ for all $x \in [0, 1]$, thus from (18), we have

$$\|x_{n+1} - x^*\| \leq \frac{\delta^{n+1}\|x_0 - x^*\|}{e^{(1-\delta)\sum_{m=0}^{n}(\alpha_m+\beta_m)}}.$$

Taking the limit of both sides of the above inequalities, we have $\lim_{n\to\infty} \|x_n - x^*\| = 0$.

We now establish that x^* is unique. Let $x^*, x_2^* \in F(T)$, such that $x^* \neq x_2^*$, using the definition of T, we have

$$\begin{aligned}
\|x^* - x_2^*\| = \|Tx^* - Tx_2^*\| &\leq \delta\|x^* - x_2^*\| \leq \|x^* - x_2^*\| \\
\Rightarrow \|x^* - x_2^*\| &\leq \|x^* - x_2^*\|.
\end{aligned}$$

Clearly, we have that $\|x^* - x_2^*\| = \|x^* - x_2^*\|$, if not we get a contradiction $\|x^* - x_2^*\| < \|x^* - x_2^*\|$. Hence, we have that $x^* = x_2^*$. Thus the proof is complete. □

Remark 3.2. We note that using similar approach as in Theorem 3.1 with some conditions, it is easy to see that iterative process (3), (4), (5), (6), (7), (8) and (9) converges to a unique fixed point of T.

Theorem 3.2. *Let C be a nonempty closed and convex subset of a uniformly convex Banach space X and T be a mapping satisfying (11). Let $\{x_n\}$ be defined by the iteration process (13) with the sequences $\{\gamma_n\}, \{\alpha_n\}\{\beta_n\}$,
$\{a_n\}, \{b_n\}, \{c_n\}$ in $[0, 1]$ such that $(a_n + b_n) \in [0, 1]$, satisfying $\alpha + \beta \leq \alpha_n + \beta_n \leq 1$, for all $n \in \mathbb{N}$ and some $\alpha, \beta \in (0, 1)$. Then $\{x_n\}$ converges faster to x^* than iteration processes (3) and (5).*

Proof. From (18) in Theorem 3.1, and using the assumption, we have that

$$\begin{aligned}
\|x_{n+1} - x^*\| &\leq \|x_0 - x^*\|\delta^{n+1} \prod_{m=0}^{n} (1 - (1-\delta)(\alpha_m + \beta_m)) \\
&= \|x_0 - x^*\|\delta^{n+1}[(1 - (1-\delta)(\alpha_m + \beta_m))]^{n+1} \\
&\leq \|x_0 - x^*\|\delta^{n+1}[(1 - (1-\delta)(\alpha + \beta))]^{n+1}.
\end{aligned}$$

Using similar argument as in Theorem 3.1, and our assumption, it is easy to see that the iteration process (3) takes the form

$$\begin{aligned}\|u_{n+1}-x^*\| &\leq \|u_0-x^*\|\prod_{m=0}^{n}(1-(1-\delta)(\alpha_m+\beta_m))\\ &= \|u_0-x^*\|[(1-(1-\delta)(\alpha_m+\beta_m))]^{n+1}\\ &\leq \|u_0-x^*\|[(1-(1-\delta)(\alpha+\beta))]^{n+1}.\end{aligned}$$

Using similar argument as in Theorem 3.1, and our assumption, it is easy to see that the iteration process (5) takes the form

$$\begin{aligned}\|q_{n+1}-x^*\| &\leq \|q_0-x^*\|\prod_{m=0}^{n}(1-(1-\delta)(\alpha_m+\beta_m))\\ &= \|q_0-x^*\|[(1-(1-\delta)(\alpha_m+\beta_m))]^{n+1}\\ &\leq \|q_0-x^*\|[(1-(1-\delta)(\alpha+\beta))]^{n+1}.\end{aligned}$$

Now, let

$$\begin{aligned}a_n &= \|x_0-x^*\|\delta^{n+1}[(1-(1-\delta)(\alpha+\beta))]^{n+1}\\ b_n &= \|u_0-x^*\|[(1-(1-\delta)(\alpha+\beta))]^{n+1}\\ c_n &= \|q_0-x^*\|[(1-(1-\delta)(\alpha+\beta))]^{n+1}\end{aligned}$$

and

$$\begin{aligned}\Psi_n = \frac{a_n}{b_n} &= \frac{\delta^{n+1}\|x_0-x^*\|[(1-(1-\delta)(\alpha+\beta))]^{n+1}}{\|u_0-x^*\|[(1-(1-\delta)(\alpha+\beta))]^{n+1}} \to 0 \text{ as } n\to 0,\\ \phi_n = \frac{a_n}{c_n} &= \frac{\delta^{n+1}\|x_0-x^*\|[(1-(1-\delta)(\alpha+\beta))]^{n+1}}{\|q_0-x^*\|[(1-(1-\delta)(\alpha+\beta))]^{n+1}} \to 0 \text{ as } n\to 0.\end{aligned}$$

Thus, the proof is complete. □

Theorem 3.3. *Let C be a nonempty closed and convex subset of a uniformly convex Banach space X and T be a mapping satisfying (11). Let $\{u_n\}$ be the Mann iteration defined in* [20] *and $\{x_n\}$ be defined by the iteration process* (13) *with $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \in [0,1]$ and $\sum_{n=0}^{\infty}(\alpha_n+\beta_n) = \sum_{n=0}^{\infty}\alpha_n = \infty$. Then the following are equivalent:*
(1) *the Mann iteration converges to the fixed point x^* of T;*
(2) *Our iteration scheme (13) converges to the fixed point x^* of T.*

Proof. We start by showing that (1) ⇒ (2). Suppose that the Mann iterative process converges to the fixed point x^*, that is $\lim_{n\to\infty}\|u_n-x^*\| = 0$. Using (13), Mann iteration and the fact that $x^* = Tx^*$, we have

$$\begin{aligned}\|x_{n+1}-u_{n+1}\| &\leq (1-\alpha_n)\|Ty_n-u_n\| + \alpha_n\|Ty_n-Tu_n\|\\ &\leq (1-\alpha_n)\|Tu_n-Ty_n\| + (1-\alpha_n)\|Tu_n-u_n\| + \alpha_n\|Ty_n-Tu_n\|\\ &= \|Tu_n-Ty_n\| + (1-\alpha_n)\|Tu_n-u_n\|\\ &\leq \delta\|u_n-y_n\| + \xi(\|u_n-Tu_n\|) + (1-\alpha_n)\|Tu_n-u_n\|. \qquad (19)\end{aligned}$$

Now, observe that

$$\begin{aligned}\|u_n - y_n\| &\leq (1-\alpha_n-\beta_n)\|u_n - z_n\| + \alpha_n\|u_n - Tz_n\| + \beta_n\|u_n - Tx_n\| \\ &\leq (1-\alpha_n-\beta_n)\|u_n - z_n\| + \alpha_n\|u_n - Tu_n\| + \alpha_n\|Tu_n - Tz_n\| \\ &+ \beta_n\|u_n - Tu_n\| + \beta_n\|Tu_n - Tx_n\| \\ &\leq (1-\alpha_n-\beta_n+\alpha_n\delta)\|u_n - z_n\| + (\alpha_n+\beta_n)\|u_n - Tu_n\| \\ &+ (\alpha_n+\beta_n)\xi(\|u_n - Tu_n\|) + \beta_n\delta\|u_n - x_n\|,\end{aligned}$$

also,

$$\begin{aligned}\|u_n - z_n\| &\leq (1-\gamma_n)\|u_n - x_n\| + \gamma_n\|u_n - Tx_n\| \\ &\leq (1-\gamma_n)\|u_n - x_n\| + \gamma_n\|u_n - Tu_n\| + \gamma_n\|Tu_n - Tx_n\| \\ &\leq (1-(1-\delta)\gamma))\|u_n - x_n\| + \gamma_n\|u_n - Tu_n\| + \gamma_n\xi(\|u_n - Tu_n\|).\end{aligned}$$

We then have

$$\begin{aligned}&\|u_n - y_n\| \leq \\ &\leq [(1-\alpha_n-\beta_n+\alpha_n\delta)(1-(1-\delta)\gamma_n) + \beta_n\delta]\|u_n - x_n\| + [(1-\alpha_n-\beta_n+\alpha_n\delta)\gamma_n \\ &+ (\alpha_n+\beta_n)]\xi(\|u_n - Tu_n\|) + [(1-\alpha_n-\beta_n+\alpha_n\delta)\gamma_n + (\alpha_n+\beta_n)]\|u_n - Tu_n\| \\ &\leq (1-(1-\delta)(\alpha_n+\beta_n))\|u_n - x_n\| + [(1-\alpha_n-\beta_n+\alpha_n\delta)\gamma_n \\ &+ (\alpha_n+\beta_n)]\xi(\|u_n - Tu_n\|) + [(1-\alpha_n-\beta_n+\alpha_n\delta)\gamma_n + (\alpha_n+\beta_n)]\|u_n - Tu_n\|.\end{aligned}$$

Substituting the above into (19), we have

$$\begin{aligned}\|x_{n+1} - u_{n+1}\| &\leq (1-(1-\delta)(\alpha_n+\beta_n))\|u_n - x_n\| + [(1-\alpha_n-\beta_n+\alpha_n\delta)\gamma_n \\ &+ (\alpha_n+\beta_n) + 1]\xi(\|u_n - Tu_n\|) + [(1-\alpha_n-\beta_n+\alpha_n\delta)\gamma_n + (\alpha_n+\beta_n) \\ &+ (1-\alpha_n)]\|u_n - Tu_n\| \\ &= (1-(1-\delta)(\alpha_n+\beta_n))\|u_n - x_n\| + A_n. \qquad (20)\end{aligned}$$

where

$$\begin{aligned}A_n &= [(1-\alpha_n-\beta_n+\alpha_n\delta)\gamma_n + (\alpha_n+\beta_n) + 1]\xi(\|u_n - Tu_n\|) \\ &+ [(1-\alpha_n-\beta_n+\alpha_n\delta)\gamma_n + (\alpha_n+\beta_n) + (1-\alpha_n)]\|u_n - Tu_n\|.\end{aligned}$$

Note that

$$\|Tu_n - u_n\| \leq \|Tu_n - Tx^*\| + \|x^* - u_n\| \leq (1+\delta)\|u_n - x^*\| \to 0 \quad \text{as} \quad n \to \infty.$$

More so, $\lim_{n\to\infty}\|Tu_n - u_n\| = \xi(\lim_{n\to\infty}\|Tu_n - u_n\|) = 0$. It follows that $A_n \to 0$ as $n \to \infty$. From (20), using Lemma 2.3, we have that $\|x_n - u_n\| \to 0$ as $n \to \infty$. Therefore, we have

$$\|x_n - x^*\| \leq \|x_n - u_n\| + \|u_n - x^*\| \to 0 \text{ as } n \to \infty.$$

Hence $\lim_{n\to\infty}\|x_n - x^*\| = 0$. That is iteration (13) converges to the fixed point x^* of T.

We now show that (2) $\Rightarrow$ (1). Suppose that iterative process (13) converges to the fixed point x^*, that is $\lim_{n\to\infty}\|x_n - x^*\| = 0$. Using (13), Mann iteration and the fact that $x^* = Tx^*$, we have

$$\begin{aligned}\|x_{n+1} - u_{n+1}\| &\le (1-\alpha_n)\|Ty_n - u_n\| + \alpha_n\|Ty_n - Tu_n\| \\ &\le (1-\alpha_n)\|Ty_n - y_n\| + (1-\alpha_n)\|y_n - u_n\| + \alpha_n\delta\|y_n - u_n\| \\ &+ \alpha_n\xi(\|y_n - Ty_n\|) \\ &\le (1-(1-\delta)\alpha_n)\|y_n - u_n\| + (1-\alpha_n)\|Ty_n - y_n\| + \alpha_n\xi(\|y_n - Ty_n\|).\end{aligned} \tag{21}$$

Now, observe that

$$\begin{aligned}\|y_n - u_n\| &\le (1-\alpha_n-\beta_n)\|z_n - u_n\| + \alpha_n\|Tz_n - u_n\| + \beta_n\|Tx_n - u_n\| \\ &\le (1-\alpha_n-\beta_n)\|z_n - u_n\| + \alpha_n\|Tz_n - z_n\| + \alpha_n\|z_n - u_n\| \\ &+ \beta_n\|Tx_n - x_n\| + \beta_n\|x_n - u_n\| \\ &= (1-\beta_n)\|z_n - u_n\| + \alpha_n\|Tz_n - z_n\| + \beta_n\|Tx_n - x_n\| + \beta_n\|x_n - u_n\|,\end{aligned}$$

$$\begin{aligned}\|z_n - u_n\| &\le (1-\gamma_n)\|x_n - u_n\| + \gamma_n\|Tx_n - u_n\| \\ &\le (1-\gamma_n)\|x_n - u_n\| + \gamma_n\|Tx_n - x_n\| + \gamma_n\|x_n - u_n\| \\ &= \|x_n - u_n\| + \gamma_n\|Tx_n - x_n\|.\end{aligned}$$

Therefore, we have

$$\|y_n - u_n\| \le \|x_n - u_n\| + ((1-\beta_n)\gamma_n + \beta_n)\|Tx_n - x_n\| + \alpha_n\|Tz_n - z_n\|.$$

Substituting the above into (21), we have

$$\begin{aligned}\|x_{n+1} - u_{n+1}\| &\le (1-(1-\delta)\alpha_n)\|x_n - u_n\| + (1-(1-\delta)\alpha_n)((1-\beta_n)\gamma_n \\ &+ \beta_n)\|Tx_n - x_n\| + (1-(1-\delta)\alpha_n)\alpha_n\|Tz_n - z_n\| \\ &+ (1-\alpha_n)\|Ty_n - y_n\| + \alpha_n\xi(\|Ty_n - y_n\|) \\ &= (1-(1-\delta)\alpha_n)\|x_n - u_n\| + A_n,\end{aligned} \tag{22}$$

where

$$\begin{aligned}A_n &= (1-(1-\delta)\alpha_n)((1-\beta_n)\gamma_n + \beta_n)\|Tx_n - x_n\| \\ &+ (1-(1-\delta)\alpha_n)\alpha_n\|Tz_n - z_n\| + (1-\alpha_n)\|Ty_n - y_n\| + \alpha_n\xi(\|Ty_n - y_n\|).\end{aligned}$$

Note that

$$\begin{aligned}\|Tx_n - x_n\| &\le \|Tx_n - Tx^*\| + \|x^* - x_n\| \le (1+\delta)\|x_n - x^*\| \to 0 \quad \text{as} \quad n\to\infty, \\ \|Ty_n - y_n\| &\le \|Ty_n - Tx^*\| + \|x^* - y_n\| \le (1+\delta)\|y_n - x^*\| \\ &\le (1+\delta)(1-(1-\delta)(\alpha_n+\beta_n))\|x_n - x^*\| \to 0 \quad \text{as} \quad n\to\infty, \\ \|Tz_n - z_n\| &\le \|Tz_n - Tx^*\| + \|x^* - z_n\| \le (1+\delta)\|z_n - x^*\| \\ &\le (1+\delta)(1-(1-\delta)\gamma_n)\|x_n - x^*\| \to 0 \quad \text{as} \quad n\to\infty.\end{aligned}$$

More so, $\lim_{n\to\infty}\|Ty_n - y_n\| = \xi(\lim_{n\to\infty}\|Ty_n - y_n\|) = 0$. It follows that $A_n \to 0$ as $n\to\infty$. From (22), using Lemma 2.3, we have that $\|x_n - u_n\| \to 0$ as $n\to\infty$. Therefore, we have

$$\|u_n - x^*\| \le \|x_n - u_n\| + \|x_n - x^*\| \to 0 \text{ as } n\to\infty.$$

Hence $\lim_{n\to\infty}\|u_n - x^*\| = 0$. That is the Mann iteration converges to the fixed point x^* of T. □

Theorem 3.4. *Let C be a nonempty closed and convex subset of a uniformly convex Banach space X and T be a mapping satisfying (11). Let $\{x_n\}$ be defined by the iteration process (13) such that $\sum_{n=0}^{\infty}(\alpha_n + \beta_n) = \infty$. Then the iterative scheme (13) is $T-$ stable.*

Proof. Let $\{t_n\} \subset X$ be any arbitrary sequence in C and suppose that the sequence generated by (13) is $x_{n+1} = f(T, x_n)$ converging to a unique fixed point x^* and that $\epsilon_n = \|t_{n+1} - f(T, t_n)\|$. To show that T is stable, we need to prove that $\lim_{n\to\infty} \epsilon_n = 0$ if and only if $\lim_{n\to\infty} t_n = x^*$.
Suppose that $\lim_{n\to\infty} \epsilon_n = 0$. Using triangular inequality and (16), we have that

$$\begin{aligned}
\|t_{n+1} - x^*\| &\leq \|t_{n+1} - f(T, t_n)\| + \|f(T, t_n) - x^*\| \\
&= \epsilon_n + \|T((1 - \alpha_n - \beta_n)(1 - \gamma_n)t_n \\
&+ (1 - \alpha_n - \beta_n)\gamma_n Tt_n + \alpha_n T((1 - \gamma_n)t_n + \gamma_n Tt_n) + \beta_n Tt_n) - x^*\| \\
&\leq \epsilon_n + \delta(1 - (1 - \delta)(\alpha_n + \beta_n))\|t_n - x^*\| \\
&\leq \epsilon_n + (1 - (1 - \delta)(\alpha_n + \beta_n))\|t_n - x^*\|.
\end{aligned}$$

Let $\Psi_n = \|t_n - x^*\|, \phi_n = (1 - \delta)(\alpha_n + \beta_n) \in (0, 1)$ and $\Phi_n = \epsilon_n$. By our hypothesis that, $\lim_{n\to\infty} \epsilon_n = 0$, it follows that $\lim_{n\to\infty} \frac{\epsilon_n}{(1-\delta)(\alpha_n+\beta_n)} = \lim_{n\to\infty} \frac{\Phi_n}{\phi_n} = 0$. Using Lemma (2.3), we have that $\lim_{n\to\infty} t_n = x^*$.
Conversely, suppose that $\lim_{n\to\infty} t_n = x^*$. We have that

$$\begin{aligned}
\epsilon_n &= \|t_{n+1} - f(T, t_n)\| \\
&\leq \|t_{n+1} - x^*\| + \|x^* - f(T, t_n)\| \\
&\leq \|t_{n+1} - x^*\| + (1 - (1 - \delta)(\alpha_n + \beta_n))\|t_n - x^*\|.
\end{aligned}$$

Using our hypothesis that $\lim_{n\to\infty} t_n = x^*$, we then have that $\lim_{n\to\infty} \epsilon_n = 0$.
Hence, iteration (13) is stable with respect to T. □

Theorem 3.5. *Let C be a nonempty closed and convex subset of a uniformly convex Banach space X and T be a mapping satisfying (11). Let $\{x_n\}$ be defined by the iteration process (13). Then the iteration (13) is weak w^2-stable with respect to T.*

Proof. Let $\{p_n\} \subset C$ be an equivalent sequence of $\{x_n\}$ and suppose that $\epsilon_n = \|p_{n+1} - Tr_n\|$, where $r_n = (1 - \alpha_n - \beta_n)q_n + \alpha_n Tq_n + \beta_n Tp_n$ and $q_n = (1 - \gamma_n)p_n + \gamma_n Tp_n$. Let $\lim_{n\to\infty} \epsilon_n = 0$, using triangular inequality and (13), we have

$$\begin{aligned}
\|p_{n+1} - x^*\| &\leq \|p_{n+1} - x_{n+1}\| + \|x_{n+1} - x^*\| \\
&\leq \|p_{n+1} - Tr_n\| + \|Tr_n - Ty_n\| + \|x_{n+1} - x^*\| \\
&= \epsilon_n + \|Ty_n - Tr_n\| + \|x_{n+1} - x^*\| \\
&\leq \epsilon_n + \delta\|y_n - r_n\| + \xi(\|y_n - Ty_n\|) + \|x_{n+1} - x^*\|.
\end{aligned}$$

Now, observe that

$$\begin{aligned}
\|y_n - r_n\| &=\leq (1 - \alpha_n - \beta_n)\|z_n - q_n\| + \alpha_n\|Tz_n - Tq_n\| + \beta_n\|Tx_n - Tp_n\| \\
&\leq (1 - \alpha_n - \beta_n)\|z_n - q_n\| + \alpha_n\delta\|z_n - q_n\| + \alpha_n\xi(\|z_n - Tz_n\|) \\
&+ \beta_n\delta\|x_n - p_n\| + \beta_n\xi(\|x_n - Tx_n\|) \\
&= (1 - \alpha_n - \beta_n + \alpha_n\delta)\|z_n - q_n\| + \alpha_n\xi(\|z_n - Tz_n\|) \\
&+ \beta_n\delta\|x_n - p_n\| + \beta_n\xi(\|x_n - Tx_n\|)
\end{aligned}$$

also, we have

$$\begin{aligned}\|z_n - q_n\| &\leq (1-\gamma_n)\|x_n - p_n\| + \gamma_n\|Tx_n - Tp_n\| \\ &\leq (1-\gamma_n)\|x_n - p_n\| + \gamma_n\delta\|x_n - p_n\| + \gamma_n\xi(\|x_n - Tx_n\|) \\ &= (1-(1-\delta)\gamma_n)\|x_n - p_n\| + \gamma_n\xi(\|x_n - Tx_n\|).\end{aligned}$$

We then have that

$$\begin{aligned}\|y_n - r_n\| \leq\ & (1-\alpha_n-\beta_n+\alpha_n\delta)(1-(1-\delta)\gamma_n)\|x_n - p_n\| \\ &+ (1-\alpha_n - b_n + \alpha_n\delta)\gamma_n\xi(\|x_n - Tx_n\|) + \alpha_n\xi(\|z_n - Tz_n\|) \\ &+ \beta_n\delta\|x_n - p_n\| + \beta_n\xi(\|x_n - Tx_n\|).\end{aligned}$$

Therefore, we have that

$$\begin{aligned}\|p_{n+1} - x^*\| \leq\ & \epsilon_n + [\delta(1-\alpha_n-\beta_n+\alpha_n\delta)(1-(1-\delta)\gamma_n) + \beta_n\delta^2]\|x_n - p_n\| \\ &+ \delta(1-\alpha_n-\beta_n+\alpha_n\delta)\gamma_n\xi(\|x_n - Tx_n\|) \\ &+ \alpha_n\delta\xi(\|z_n - Tz_n\|) + \beta_n\delta\xi(\|x_n - Tx_n\|) + \xi(\|y_n - Ty_n\|) + \|x_{n+1} - x^*\|.\end{aligned}$$

We have established in Theorem 3.1 that $\lim_{n\to\infty}\|x_n - x^*\| = 0$, consequently $\lim_{n\to\infty}\|x_{n+1} - x^*\| = 0$ and since $\{x_n\}$ and $\{p_n\}$ are equivalent, we have $\lim_{n\to\infty}\|x_n - p_n\| = 0$. It also follows that

$$\begin{aligned}\|x_n - Tx_n\| &\leq \|x_n - x^*\| + \|Tx^* - Tx_n\| \\ &\leq (1+\delta)\|x_n - x^*\| \to 0, \quad \text{as} \quad n\to\infty.\end{aligned}$$

Using similar approach, we can show that $\lim_{n\to\infty}\|z_n - Tz_n\| = \lim_{n\to\infty}\|y_n - Ty_n\| = 0$.

Since, $\lim_{n\to\infty}\|y_n - Ty_n\| = \xi(\lim_{n\to\infty}\|y_n - Ty_n\|) = 0$. Similar argument holds for others. Thus, we have $\lim_{n\to\infty}\|p_{n+1} - x^*\| = 0$ and consequently $\lim_{n\to\infty}\|p_n - x^*\| = 0$. Thus, $\{x_n\}$ is weak w^2-stable with respect to T. □

Example 3.1. Let $C = [0,1]$ and $Tx = \frac{x}{4}$. Clearly, zero is the fixed point of T. We need to show that T satisfy (11). To do this, with $\delta = \frac{1}{4}$ and for any increasing function ξ with $\xi(0) = 0$. We have

$$\begin{aligned}\|Tx - Ty\| - \delta\|x-y\| - \xi(\|x - Tx\|) &= \frac{1}{4}|x-y| - \frac{1}{4}|x-y| - \xi(|x - \frac{x}{4}|) \\ &= -\xi(\frac{3x}{4}) \leq 0.\end{aligned}$$

Let $\alpha_n = \beta_n = \gamma_n = \frac{1}{n+2}$ and $x_0 \in [0,1]$. It follows that

$$\begin{aligned}z_n &= \left(1 - \frac{1}{n+2} + \frac{1}{4(n+2)}\right)x_n = \left(1 - \frac{3}{4(n+2)}\right)x_n \\ y_n &= \left(1 - \frac{9}{4(n+2)} + \frac{21}{4^2(n+2)^2}\right)x_n \\ x_{n+1} &= \frac{1}{4}\left(1 - \frac{9}{4(n+2)} + \frac{21}{4^2(n+2)^2}\right)x_n = \left(1 - \left(\frac{3}{4} + \frac{9}{4^2(n+2)} - \frac{21}{4^3(n+2)^2}\right)\right)x_n.\end{aligned}$$

Let $t_n = \frac{3}{4} + \frac{9}{4^2(n+2)} - \frac{21}{4^3(n+2)^2}$. Clearly, $t_n \in (0,1)$ for all $n \in \mathbb{N}$ and $\sum_{n=0}^{\infty} t_n = \infty$. Using Lemma 2.3, we have $\lim_{n\to\infty} x_n = 0$. Let $p_n = \frac{1}{n+3}$. We now establish that 13 is stable, with respect to T.

$$\begin{aligned}\epsilon_n &= |p_{n+1} - f(T, p_n)| \\ &= \left|p_{n+1} - \left(\frac{1}{4} - \frac{9}{4^2(n+2)} + \frac{21}{4^3(n+2)^2}\right)p_n\right| \\ &= \left|\frac{1}{n+4} - \frac{1}{4(n+3)} + \frac{9}{4^2(n+2)(n+3)} - \frac{21}{4^3(n+2)^2(n+3)}\right|.\end{aligned}$$

Clearly, $\lim_{n\to\infty} \epsilon_n = 0$.

Theorem 3.6. *Let $\overline{T}$ be an approximate operator of a mapping T satisfying (11). Let $\{x_n\}$ be an iterative sequence generated by (13) for T and define an iterative scheme $\{\overline{x}_n\}$ as follows:*

$$\begin{cases}\overline{z}_n = (1-\gamma_n)\overline{x}_n + \gamma_n \overline{T}\overline{x}_n \\ \overline{y}_n = (1-\alpha_n-\beta_n)\overline{z}_n + \alpha_n \overline{T}\overline{z}_n + \beta_n \overline{T}\overline{x}_n \\ \overline{x}_{n+1} = \overline{T}\overline{y}_n \;\; n \geq 1,\end{cases} \tag{23}$$

where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0,1)$ and $\frac{1}{2} \leq (\alpha_n + \beta_n)$ for all $n \in \mathbb{N}$ such that $\sum_{n=0}^{\infty}(\alpha_n+\beta_n) = \infty$. If $Tx^* = x^*$ and $\overline{T}\overline{x}^* = \overline{x}^*$ such that $\lim_{n\to\infty} \overline{x}_n = \overline{x}^*$, then we have

$$d(x^*, \overline{x}^*) \leq \frac{5\epsilon}{1-k},$$

where $\epsilon > 0$ is a fixed number.

Proof. Using (13) and (23), we have

$$\begin{aligned}\|z_n - \overline{z}_n\| &\leq (1-\gamma_n)\|x_n - \overline{x}_n\| + \gamma_n\|Tx_n - \overline{T}\overline{x}_n\| \\ &\leq (1-\gamma_n)\|x_n - \overline{x}_n\| + \gamma_n\|Tx_n - T\overline{x}_n\| + \gamma_n\|T\overline{x}_n - \overline{T}\overline{x}_n\| \\ &\leq (1-\gamma_n)\|x_n - \overline{x}_n\| + \gamma_n\delta\|x_n - \overline{x}_n\| + \gamma_n\xi(\|x_n - Tx_n\|) + \gamma_n\epsilon \\ &= (1-(1-\delta)\gamma_n)\|x_n - \overline{x}_n\| + \gamma_n\xi(\|x_n - Tx_n\|) + \gamma_n\epsilon.\end{aligned} \tag{24}$$

Using (13), (23) and (24), we have

$$\begin{aligned}\|y_n - \overline{y}_n\| &\leq (1-\alpha_n-\beta_n)\|z_n - \overline{z}_n\| + \alpha_n\|Tz_n - \overline{T}\overline{z}_n\| + \beta_n\|Tx_n - \overline{T}\overline{x}_n\| \\ &\leq (1-\alpha_n-\beta_n)\|z_n - \overline{z}_n\| + \alpha_n\|Tz_n - T\overline{z}_n\| + \alpha_n\|T\overline{z}_n - \overline{T}\overline{z}_n\| \\ &+ \beta_n\|Tx_n - T\overline{x}_n\| + \beta_n\|T\overline{x}_n - \overline{T}\overline{x}_n\| \\ &\leq (\alpha_n\delta + 1 - \alpha_n - \beta_n)\|z_n - \overline{z}_n\| + \beta_n\xi(\|x_n - Tx_n\|) + \alpha_n\xi(\|z_n - Tz_n\|) \\ &\quad + \beta_n\delta\|x_n - \overline{x}_n\| + \alpha_n\epsilon + \beta_n\epsilon \\ &\leq [(\alpha_n\delta + 1 - \alpha_n - \beta_n)(1-(1-\delta)\gamma_n) + \beta_n\delta]\|x_n - \overline{x}_n\| \\ &+ [(\alpha_n\delta + 1 - \alpha_n - \beta_n)\gamma_n + \beta_n]\xi(\|x_n - Tx_n\|) \\ &\quad + \alpha_n\xi(\|z_n - Tz_n\|) + (\alpha_n\delta + 1 - \alpha_n - \beta_n)\gamma_n\epsilon + \alpha_n\epsilon + \beta_n\epsilon.\end{aligned} \tag{25}$$

Using (13), (23) and (25), we have

$$\begin{aligned}
&\|x_{n+1} - \overline{x}_{n+1}\| \leq \\
&\leq \|Ty_n - \overline{T}\overline{y}_n\| \\
&\leq \|Ty_n - T\overline{y}_n\| + \|T\overline{y}_n - \overline{T}\overline{y}_n\| \\
&\leq \delta\|y_n - \overline{y}_n\| + \xi(\|y_n - Ty_n\|) + \epsilon \\
&\leq \delta\{[(\alpha_n\delta + 1 - \alpha_n - \beta_n)(1 - (1-\delta)\gamma_n) + \beta_n\delta]\|x_n - \overline{x}_n\| \\
&+ [(\alpha_n\delta + 1 - \alpha_n - \beta_n)\gamma_n + \beta_n]\xi(\|x_n - Tx_n\|) \\
&+ \alpha_n\xi(\|z_n - Tz_n\|) + (\alpha_n\delta + 1 - \alpha_n - \beta_n)\gamma_n\epsilon + \alpha_n\epsilon + \beta_n\epsilon\} + \xi(\|y_n - Ty_n\|) + \epsilon \\
&\leq (1 - (1-\delta)(\alpha_n + \beta_n))\|x_n - \overline{x}_n\| + [(\alpha_n\delta + 1 - \alpha_n - \beta_n)\gamma_n + \beta_n]\xi(\|x_n - Tx_n\|) \\
&+ \alpha_n\xi(\|z_n - Tz_n\|) + (\alpha_n + \beta_n)\epsilon + \xi(\|y_n - Ty_n\|) + 2\epsilon.
\end{aligned}$$

Using our assumption that $\frac{1}{2} \leq (\alpha_n + \beta_n)$, we have

$$1 - (\alpha_n + \beta_n) \leq (\alpha_n + \beta_n)$$
$$\Rightarrow 1 = 1 - (\alpha_n + \beta_n) + (\alpha_n + \beta_n) \leq (\alpha_n + \beta_n) + (\alpha_n + \beta_n) = 2(\alpha_n + \beta_n).$$

Therefore, we have that

$$\begin{aligned}
\|x_{n+1} - \overline{x}_{n+1}\| &\leq (1 - (1-\delta)(\alpha_n + \beta_n))\|x_n - \overline{x}_n\| + 2(\alpha_n + \beta_n)\xi(\|x_n - Tx_n\|) \\
&+ 2(\alpha_n + \beta_n)\alpha_n\xi(\|z_n - Tz_n\|) + (\alpha_n + \beta_n)\epsilon \\
&+ 2(\alpha_n + \beta_n)\xi(\|y_n - Ty_n\|) + 4(\alpha_n + \beta_n)\epsilon \\
&= 1 - (1-\delta)(\alpha_n + \beta_n))\|x_n - \overline{x}_n\| + (\alpha_n + \beta_n)(1-\delta)\times \\
&\times \frac{2\xi(\|x_n - Tx_n\|) + 2\alpha_n\xi(\|z_n - Tz_n\|) + 2\xi(\|y_n - Ty_n\|) + 5\epsilon}{(1-\delta)}
\end{aligned}$$

Let

$$\Psi_n = \|x_n - \overline{x}_n\|, \phi_n = (1-\delta)(\alpha_n + \beta_n)$$

and

$$\Phi_n = \frac{2\xi(\|x_n - Tx_n\|) + 2\alpha_n\xi(\|z_n - Tz_n\|) + 2\xi(\|y_n - Ty_n\|) + 5\epsilon}{1-\delta}.$$

From Theorem 3.1, we have that $\lim_{n\to\infty}\|x_n - x^*\| = 0$. Also, observe that

$$\begin{aligned}
\|x_n - Tx_n\| &\leq \|x_n - x^*\| + \|Tx^* - T_n\| \\
&\leq (1+\delta)\|x_n - x^*\| \to 0 \quad \text{as} \quad n \to \infty.
\end{aligned}$$

Using similar approach, we have that $\lim_{n\to\infty}\|y_n - Ty_n\| = \lim_{n\to\infty}\|z_n - Tz_n\| = 0$. More so, we have that

$$\lim_{n\to\infty}\xi(\|y_n - Ty_n\|) = \xi(\lim_{n\to\infty}\|y_n - Ty_n\|) = 0.$$

The same argument holds for others. Using Lemma 2.4, we have that

$$0 \leq \limsup_{n\to\infty}\|x_n - \overline{x}_n\| \leq \limsup_{n\to\infty}\frac{5\epsilon}{1-\delta}. \tag{26}$$

Using our hypothesis that $\lim_{n\to\infty}\overline{x}_n = \overline{x}^*$, (26) and from Theorem 3.1, we conclude that

$$\|x^* - \overline{x}^*\| \leq \frac{5\epsilon}{1-\delta}.$$

Hence, the proof is complete. □

4. Convergence result

In this section, we establish some convergence results for Suzuki generalized nonexpansive mapping. We recall from [28] that, a mapping $T: C \to C$ is said to be Suzuki generalized nonexpansive if for all $x, y \in C$, we have

$$\frac{1}{2}\|x - Tx\| \leq \|x - y\| \Rightarrow \|Tx - Ty\| \leq \|x - y\|.$$

Let C be a nonempty closed and convex subset of a uniformly convex Banach space X and $T: C \to C$ be a Suzuki generalized nonexpansive mapping. Let $\{x_n\}$ be the sequence defined in (13).

We first state and prove the following lemmas which will be needed in the proof of our main theorems. Now, observe that for all $x^* \in F(T)$, we have

$$\begin{aligned}
&\frac{1}{2}\|x^* - Tx^*\| = \frac{1}{2}\|x^* - x^*\| \leq \|x^* - z_n\|,\\
&\frac{1}{2}\|x^* - Tx^*\| = \frac{1}{2}\|x^* - x^*\| \leq \|x^* - x_n\| \quad \text{and}\\
&\frac{1}{2}\|x^* - Tx^*\| = \frac{1}{2}\|x^* - x^*\| \leq \|x^* - y_n\|,
\end{aligned} \tag{27}$$

which implies that

$$\begin{aligned}
&\|Tx^* - Tz_n\| \leq \|x^* - z_n\|\\
&\|Tx^* - Tx_n\| \leq \|x^* - x_n\| \quad \text{and}\\
&\|Tx^* - Ty_n\| \leq \|x^* - y_n\|.
\end{aligned} \tag{28}$$

Lemma 4.1. *Let C be a nonempty closed and convex subset of a uniformly convex Banach space X. Let $T: C \to C$ be a Suzuki generalized nonexpansive mapping with $F(T) \neq \emptyset$. Suppose that $\{x_n\}$ is defined by (13), where $\{\beta_n\}, \{\gamma_n\}$ and $\{\alpha_n\}$ are sequences in $[0,1]$ with $(\alpha_n + \beta_n) \in [0,1]$. Then the following hold:*

(i) *$\{x_n\}$ is bounded.*

(ii) *$\lim_{n\to\infty} \|x_n - x^*\|$ exists for all $x^* \in F(T)$.*

Proof. Using (13) and Proposition 2.8, we have

$$\begin{aligned}
\|z_n - x^*\| &\leq (1 - \gamma_n)\|x_n - x^*\| + \gamma_n\|Tx_n - x^*\|\\
&\leq (1 - \gamma_n)\|x_n - x^*\| + \gamma_n\|x_n - x^*\|\\
&\leq \|x_n - x^*\|.
\end{aligned} \tag{29}$$

Using (13), (29) and Proposition 2.8, we have

$$\begin{aligned}
\|y_n - x^*\| &\leq (1 - \alpha_n - \beta_n)\|z_n - x^*\| + \alpha_n\|Tz_n - x^*\| + \beta_n\|Tx_n - x^*\|\\
&\leq (1 - \alpha_n - \beta_n)\|z_n - x^*\| + \alpha_n\|z_n - x^*\| + \beta_n\|x_n - x^*\|\\
&= (1 - \beta_n)\|z_n - x^*\| + \beta_n\|x_n - x^*\|\\
&\leq (1 - \beta_n)\|x_n - x^*\| + \beta_n\|x_n - x^*\|\\
&= \|x_n - x^*\|.
\end{aligned} \tag{30}$$

Using (13), (30) and Proposition 2.8, we have

$$\begin{aligned}\|x_{n+1} - x^*\| &= \|Ty_n - x^*\| \\ &\leq \|y_n - x^*\| \\ &\leq \|x_n - x^*\|. \end{aligned} \tag{31}$$

This shows that $\{\|x_n - x^*\|\}$ is bounded and non-decreasing for all $x^* \in F(T)$. Thus, $\lim_{n\to\infty} \|x_n - x^*\|$ exists. □

Lemma 4.2. *Let C be a nonempty closed and convex subset of a uniformly convex Banach space X. Let $T : C \to C$ be a Suzuki generalized nonexpansive mapping with $F(T) \neq \emptyset$. Suppose that $\{x_n\}$ is defined by (13), where $\{\beta_n\}, \{\gamma_n\}$ and $\{\alpha_n\}$ are sequences in $[0, 1]$, with $(\alpha_n + \beta_n) \in [0, 1]$, then $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$.*

Proof. Since $F(T) \neq \emptyset$, then we can find $x^* \in F(T)$. We have established in Lemma 4.1 that $\{x_n\}$ is bounded and $\lim_{n\to\infty} \|x_n - x^*\|$ exists. Suppose that $\lim_{n\to\infty} \|x_n - x^*\| = c$. If we take c = 0, then we are done. Thus, we consider the case where $c > 0$. From (29), we have $\|z_n - x^*\| \leq \|x_n - x^*\|$, it then follows that

$$\limsup_{n\to\infty} \|z_n - x^*\| \leq c. \tag{32}$$

Also, using Proposition 2.8, we have $\|Tx_n - x^*\| \leq \|x_n - x^*\|$, it then follows that

$$\limsup_{n\to\infty} \|Tx_n - x^*\| \leq c. \tag{33}$$

Using (30) and (31), we have

$$\begin{aligned}\|x_{n+1} - x^*\| &\leq \|y_n - x^*\| \\ &\leq (1 - \beta_n)\|z_n - x^*\| + \beta_n\|x_n - x^*\|.\end{aligned}$$

Taking the $\liminf_{n\to\infty}$ of both sides and rearranging the inequalities, we have

$$c \leq \liminf_{n\to\infty} \|z_n - x^*\|. \tag{34}$$

From (32) and (34), we obtain that $\lim_{n\to\infty} \|z_n - x^*\| = c$. That is,

$$\lim_{n\to\infty} \|(1 - \gamma_n)x_n + \gamma_n Tx_n - x^*\| = c.$$

Thus, by Lemma 2.5, we have

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0.$$

□

Theorem 4.3. *Let X be a uniformly convex Banach space which satisfies the Opial's condition and C a nonempty closed convex subset of X. Let $T : C \to C$ be a Suzuki generalized nonexpansive mapping with $F(T) \neq \emptyset$ and $\{x_n\}$ be a sequence defined by iteration (13). Then $\{x_n\}$ converges weakly to a fixed point of T.*

Proof. In Lemma 4.1, we established that $\lim_{n\to\infty} \|x_n - x^*\|$ exists and that $\{x_n\}$ is bounded. Now, since X is uniformly convex, we can find a subsequence say $\{x_{n_i}\}$ of $\{x_n\}$ that converges weakly in C. We now establish that $\{x_n\}$ has a unique weak subsequential limit in $F(T)$. Let u and v be weak limits of the subsequences $\{x_{n_k}\}$ and $\{x_{n_j}\}$ of $\{x_n\}$ respectively. By Theorem 4.2, we have that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$ and $I - T$ is demiclosed with respect to zero by Proposition 2.6, we therefore have that $Tu = u$. Using similar approach, we can show that $v = Tv$. In what follows, we establish uniqueness. From Lemma 4.1, we have that $\lim_{n\to\infty} \|x_n - v\|$ exists. Now, suppose that $u \neq v$, then by Opial's condition,

$$\begin{aligned}\lim_{n\to\infty} \|x_n - u\| &= \lim_{k\to\infty} \|x_{n_k} - u\| < \lim_{k\to\infty} \|x_{n_k} - v\| \\ &= \lim_{n\to\infty} \|x_n - v\| = \lim_{j\to\infty} \|x_{n_j} - v\| \\ &< \lim_{j\to\infty} \|x_{n_j} - u\| = \lim_{n\to\infty} \|x_n - u\|.\end{aligned}$$

This is a contradiction, so $u = v$. Hence, $\{x_n\}$ converges weakly to a fixed point of $F(T)$ and this completes the proof. □

Theorem 4.4. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X. Let T be a Suzuki generalized nonexpansive mapping on C, $\{x_n\}$ defined by (13) and $F(T) \neq \emptyset$. Then $\{x_n\}$ converges strongly to a point of $F(T)$ if and only if $\liminf_{n\to\infty} d(x_n, F(T)) = 0$ where $d(x, F(T)) = \inf \|x - x^*\| : x^* \in F(T)$.*

Proof. Suppose that $\{x_n\}$ converges to a fixed point, say x^* of T.
Then $\lim_{n\to\infty} d(x_n, x^*) = 0$, and since $0 \leq d(x_n, F(T)) \leq d(x_n, x^*)$, it follows that $\lim_{n\to\infty} d(x_n, F(T)) = 0$. Therefore, $\liminf_{n\to\infty} d(x_n, F(T)) = 0$.
Conversely, suppose that $\liminf_{n\to\infty} d(x_n, F(T)) = 0$. From Lemma 4.1, we have that $\lim_{n\to\infty} \|x_n - x^*\|$ exists and that $\lim_{n\to\infty} d(x_n, F(T))$ exists for all $x^* \in F(T)$. By our hypothesis, $\liminf_{n\to\infty} d(x_n, F(T)) = 0$, so for any give $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$, such that for all $n \geq n_0$, we have $d(x_n, F(T)) \leq \epsilon$. We now show that $\{x_n\}$ is a Cauchy sequence in C. Since, $\lim_{n\to\infty} d(x_n, F(T)) = 0$, for any give $\epsilon > 0$, there exist $n_0 \in \mathbb{N}$ such that for $n, m \geq n_0$, we have

$$\begin{aligned}d(x_m, F(T)) &\leq \frac{\epsilon}{2}, \\ d(x_n, F(T)) &\leq \frac{\epsilon}{2}.\end{aligned}$$

Therefore, we have

$$\begin{aligned}\|x_m - x_n\| &\leq \|x_m - x^*\| + \|x_n - x^*\| \\ &\leq d(x_m, F(T)) + d(x_n, F(T)) \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.\end{aligned}$$

Hence, $\{x_n\}$ is a Cauchy sequence in C. Since C is closed, then there exists a point $x_1 \in C$ such that $\lim_{n\to\infty} x_n = x_1$. Since $\lim_{n\to\infty} d(x_n, F(T)) = 0$, it follows that $\lim_{n\to\infty} d(x_1, F(T)) = 0$. Since, $F(T)$ is closed, $x_1 \in F(T)$. □

Theorem 4.5. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X. Let T be a Suzuki generalized nonexpansive mapping, $\{x_n\}$ defined by (13) and $F(T) \neq \emptyset$. Let T satisfy condition (A), then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. From Lemma 4.1, we have $\lim_{n\to\infty} \|x_n - F(T)\|$ exists and by Theorem 4.2, we have $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. Using the fact that

$$0 \leq \lim_{n\to\infty} f(d(x, F(T)) \leq \lim_{n\to\infty} \|x_n - Tx_n\| = 0 \;\forall x \in C,$$

we have that $\lim_{n\to\infty} f(d(x_n, F(T))) = 0$. Since f is nondecreasing with $f(0) = 0$ and $f(t) > 0$ for $t \in (0,\infty)$, it then follows that $\lim_{n\to\infty} d(x_n, F(T)) = 0$. Hence, by Theorem 4.4 $\{x_n\}$ converges strongly to $x^* \in F(T)$. □

5. Numerical examples

In this section, we present an example of Suzuki generalized nonexpansive mapping, which is not a nonexpansive mapping. Using this example, we compare our iterative process with two other iterative processes in the literature.

Example 5.1. Define a mapping $T : [0,1] \to [0,1]$ as

$$Tx = \begin{cases} 1 - x \text{ if } x \in [0, 1/5), \\ \frac{x+4}{5} \text{ if } x \in [1/5, 1]. \end{cases} \tag{35}$$

Then T is a Suzuki generalized nonexpansive mapping but not nonexpansive mapping.

Proof. To establish this, we consider the following cases:

Case 1: Let $x \in [0, \frac{1}{5})$, as such we have that $\frac{1}{2}\|x - Tx\| = \frac{1-2x}{2} \in (\frac{3}{10}, \frac{1}{2}]$. By definition, for $\frac{1}{2}\|x - Tx\| \le \|x - y\|$, we must have that $y \ge \frac{1}{2}$, that is $y \in [\frac{1}{2}, 1]$. And so, we obtain that

$$\|Tx - Ty\| = \left|\frac{5x + y - 1}{5}\right| < \frac{1}{5}$$

and

$$\|x - y\| = |x - y| > \left|\frac{1}{5} - \frac{1}{2}\right| = \frac{3}{10}.$$

Thus, we have that $\frac{1}{2}\|x - Tx\| \le \|x - y\| \Rightarrow \|Tx - Ty\| \le \|x - y\|$.

Case 2: Let $x \in [\frac{1}{5}, 1]$, as such we have that $\frac{1}{2}\|x - Tx\| = \frac{2-2x}{5} \in [0, \frac{4}{5}]$. By definition, for $\frac{1}{2}\|x - Tx\| \le \|x - y\|$, we must have that $\frac{2-2x}{5} \le |x - y|$. Due to $|x - y|$, we have two possibilities.

Case 2a: If $x < y$, we have that $\frac{2-2x}{5} < y - x$, as such we must have that $\frac{2+3x}{5} \le y \Rightarrow y \in [\frac{13}{25}, 1] \subset [\frac{1}{5}, 1]$. And so, we obtain that

$$\|Tx - Ty\| = \left|\frac{x+4}{5} - \frac{y+4}{5}\right| = \frac{1}{5}|x - y| \le \|x - y\|.$$

Thus, we have that $\frac{1}{2}\|x - Tx\| \le \|x - y\| \Rightarrow \|Tx - Ty\| \le \|x - y\|$.

Case 2b: If $x \ge y$, we have that $\frac{2-2x}{5} \le x-y$, as such we must have that $y \le \frac{7x-2}{5} \Rightarrow y \in [\frac{-3}{25}, 1]$. We only need to consider the case in which $y \in [0, 1]$. For $y \le \frac{7x-2}{5}$, we obtain that $x \ge \frac{5y+2}{7}$, which implies that $x \in [\frac{2}{7}, 1]$, as such we going to consider $x \in [\frac{2}{7}, 1]$ and $y \in [0, 1]$. For $x \in [\frac{2}{7}, 1]$ and $y \in [\frac{1}{5}, 1]$ have been considered in case 2a. So, we consider $x \in [\frac{2}{7}, 1]$ and $y \in [0, \frac{1}{5})$. To start with suppose $x \in [\frac{2}{7}, \frac{2}{5}]$ and $y \in [0, \frac{1}{5})$, we therefore have that

$$\|Tx - Ty\| = \left|\frac{x+4}{5} - (1 - y)\right| = \left|\frac{x + 5y - 1}{5}\right| \le \frac{2}{25}$$

and

$$\|x-y\| = |x-y| > \left|\frac{2}{7} - \frac{1}{5}\right| = \frac{3}{35}.$$

Thus, we have that $\frac{1}{2}\|x - Tx\| \leq \|x - y\| \Rightarrow \|Tx - Ty\| \leq \|x - y\|$.
Also for $x \in [\frac{2}{5}, 1]$ and $y \in [0, \frac{1}{5})$, we therefore have that

$$\|Tx - Ty\| = \left|\frac{x+4}{5} - (1-y)\right| = \left|\frac{x+5y-1}{5}\right| \leq \frac{1}{5}$$

and

$$\|x-y\| = |x-y| > \left|\frac{2}{5} - \frac{1}{5}\right| = \frac{1}{5}.$$

Thus, we have that $\frac{1}{2}\|x-Tx\| \leq \|x-y\| \Rightarrow \|Tx-Ty\| \leq \|x-y\|$. Hence T is a Suzuki generalized nonexpansive mapping. However to show that T is not nonexpansive, we take $x = \frac{3}{16}$ and $y = \frac{1}{5}$, we then have that

$$\|Tx - Ty\| = \left|1 - \frac{3}{16} - \frac{21}{25}\right| = \frac{11}{400} > \frac{1}{80} = \left|\frac{3}{16} - \frac{1}{5}\right| = \|x - y\|.$$

Thus T is not a nonexpansive mapping.
In what follows, we numerically compare our new iteration process with two existing iteration processes. We take $\alpha_n = \beta_n = a_n = b_n = \frac{1}{\sqrt{5n+1}}, \gamma_n = c_n = \frac{1}{\sqrt{n+1}}$ and $x_0 = 0.8$. The comparison of the iterative schemes are shown below.

Step	Our Algorithm	Karakaya et al.	Suntai
0	0.8	0.8	0.8
1	0.9921276	0.9824707	0.9512980
2	0.9995176	0.9969754	0.9776478
3	0.9999633	0.9992771	0.9875051
4	0.9999968	0.9997894	0.9922617
5	0.9999997	0.9999297	0.9948881
6	1.0000000	0.9999741	0.9964674
7	1.0000000	0.9999897	0.9974752
8	1.0000000	0.9999956	0.9981475
9	1.0000000	0.9999980	0.9986116
10	1.0000000	0.9999991	0.9989411
11	1.0000000	0.9999996	0.9991802
12	1.0000000	0.9999998	0.9993572
13	1.0000000	0.9999999	0.9994904
14	1.0000000	1.0000000	0.9995920
15	1.0000000	1.0000000	0.9996705

The comparison shows that, our newly proposed iterative scheme converges faster than the other iterative schemes. We have shown that our newly proposed iterative process is more efficient and converges faster than some iterative processes in literature. □

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