Annals of the University of Craiova, Mathematics and Computer Science Series
Volume 46(2), 2019, Pages 320–327
ISSN: 1223-6934

On cyclic $(\phi - \psi)$-Kannan and $(\phi - \psi)$-Chatterjea contractions in metric spaces

Mohammad Al-Khaleel and Sharifa Al-Sharif

Abstract. We use in this paper the concept of cyclic $(\phi - \psi)$-Kannan and $(\phi - \psi)$-Chatterjea contractions to study new extensions of the Kannan and Chatterjea fixed point theorems. We give some generalized versions of the fixed point results proved in the literature. The analysis and theory are illustrated by some examples.

2010 Mathematics Subject Classification. 47H10,46T99,54H25.
Key words and phrases. Cyclic contraction, Kannan contraction, Chatterjea contraction, fixed point theory.

1. Introduction and preliminaries

In [1], Kannan successfully proved that if X is complete, then every what is so-called Kannan contraction T has a unique fixed point which extends the well-known Banach's contraction principle [2]. The definition that was introduced by Kannan is stated below.

Definition 1.1 (See [1]). A mapping $T : X \to X$, where (X, d) is a metric space, is said to be a Kannan contraction if there exists $\alpha \in \left[0, \frac{1}{2}\right)$ such that for all $x, y \in X$, the inequality

$$d\left(Tx, Ty\right) \le \alpha \left[d\left(x, Tx\right) + d\left(y, Ty\right)\right],$$

holds.

Another definition which is a sort of dual of Kannan contraction, is presented by Chatterjea [3] as follows.

Definition 1.2 (See [3]). A mapping $T : X \to X$, where (X, d) is a metric space, is said to be a Chatterjea contraction if there exists $\alpha \in \left[0, \frac{1}{2}\right)$ such that for all $x, y \in X$, the inequality

$$d\left(Tx, Ty\right) \le \alpha \left[d\left(x, Ty\right) + d\left(y, Tx\right)\right],$$

holds.

Chatterjea [3] also proved using his new definition that if X is complete, then every Chatterjea contraction has a unique fixed point. In 1972, Zamfirescu [4] introduced a very interesting fixed point theorem which combines the contractive conditions of Banach, of Kannan, and of Chatterjea.

Received April 23, 2018. Accepted March 30, 2019.

Theorem 1.1 (See [4]). *Let (X,d) be a complete metric space and $T : X \to X$ a map for which there exist the real numbers α, β, and γ satisfying $0 \le \alpha < 1$, $0 \le \beta, \gamma < \frac{1}{2}$, such that for $x, y \in X$ at least one of the following is true.*
(i) $d(Tx, Ty) \le \alpha d(x, y)$,
(ii) $d(Tx, Ty) \le \beta [d(x, Tx) + d(y, Ty)]$,
(iii) $d(Tx, Ty) \le \gamma [d(x, Ty) + d(y, Tx)]$.
Then T has a unique fixed point p and the Picard iteration $\{x_n\}_{n=0}^{\infty}$ defined by $x_{n+1} = Tx_n$, $n = 0, 1, 2, \ldots$ converges to p for any $x_0 \in X$.

The cyclical extensions for these fixed point theorems were obtained at a later time, by considering non-empty closed subsets $\{A_i\}_{i=1}^{p}$ of a complete metric space X and a cyclical operator $T : \bigcup_{i=1}^{p} A_i \to \bigcup_{i=1}^{p} A_i$, i.e., satisfies $T(A_i) \subseteq A_{i+1}$ for all $i \in \{1, 2, \ldots, p\}$. In [5], Rus presented the cyclical extension for the Kannan's theorem, and Petric in [6] presented cyclical extensions for Chatterjea and Zamfirescu theorems using fixed point structure arguments. The cyclic contractive mappings type has been widely considered by many researchers, for a recent work one can see for example [7, 8] and references therein. Redefining the concept of Chatterjea contraction was introduced by Choudhury in [9] as follows.

Definition 1.3 (See [9]). A mapping $T : X \to X$, where (X, d) is a metric space, is said to be a weak Chatterjea contraction if for all $x, y \in X$, the inequality

$$d(Tx, Ty) \le \frac{1}{2} [d(x, Ty) + d(y, Tx)] - \psi(d(x, Ty), d(y, Tx)),$$

holds, where $\psi : [0, \infty)^2 \to [0, \infty)$ is a continuous function such that $\psi(x, y) = 0$ if and only if $x = y = 0$.

Choudhury [9] proved the following theorem.

Theorem 1.2 (See [9]). *If (X, d) is a complete metric space, then every weak Chatterjea contraction T has a unique fixed point.*

The concept of a control function in terms of altering distances was addressed by Khan *et. al.* [10] which lead to a new category of fixed point problems. Altering distances have been used in metric fixed point theory in many papers, see for example [11]-[13] and references therein. We define in what follows, an altering distance function which will be used throughout the paper.

Definition 1.4. The function $\phi : [0, \infty) \to [0, \infty)$ is called an altering distance function, if the following properties are satisfied.
(i) ϕ is continuous and nondecreasing,
(ii) $\phi(t) = 0$ if and only if $t = 0$.

In this paper, we study new extensions of the Kannan and Chatterjea fixed point theorems. We give some generalized versions of the fixed point results proved in the literature. In particular, we present some generalized versions of fixed point theorems of cyclic nonlinear contractions type by the use of the continuous function ψ given in Definition 1.3 and the altering distance function ϕ given in Definition 1.4. The analysis and theory are illustrated by some examples.

2. Main results

We begin this section by giving definitions of what we call a cyclic $(\phi-\psi)$-Kannan contraction and a cyclic $(\phi-\psi)$-Chatterjea contraction.

Definition 2.1. Let $\{A_i\}_{i=1}^{p}$ be non-empty closed subsets of a metric space (X,d), and suppose $T:\bigcup_{i=1}^{p}A_i\to\bigcup_{i=1}^{p}A_i$ is a cyclical operator. Then T is said to be a cyclic $(\phi-\psi)$-Kannan contraction if there exists constants α,β with $0\le\beta<1$ and $0<\alpha+\beta\le 1$, such that for any $x\in A_i, y\in A_{i+1}, i=1,2,\dots,p$, we have

$$\phi\left(d\left(Tx,Ty\right)\right)\le\phi\left(\alpha d\left(x,Tx\right)+\beta d\left(y,Ty\right)\right)-\psi\left(d\left(x,Tx\right),d\left(y,Ty\right)\right),$$

and T is said to be a cyclic $(\phi-\psi)$-Chatterjea contraction if there exists constants α,β with $0\le\alpha\le\frac{1}{2}$ and $0<\alpha+\beta\le 1$, such that for any $x\in A_i, y\in A_{i+1}, i=1,2,\dots,p$, we have

$$\phi\left(d\left(Tx,Ty\right)\right)\le\phi\left(\alpha d\left(x,Ty\right)+\beta d\left(y,Tx\right)\right)-\psi\left(d\left(x,Ty\right),d\left(y,Tx\right)\right),$$

where $\phi:[0,\infty)\to[0,\infty)$ is an altering distance function, and $\psi:[0,\infty)^2\to[0,\infty)$ is a continuous function with $\psi(t,s)=0$ if and only if $t=s=0$.

Theorem 2.1. *Let $\{A_i\}_{i=1}^{p}$ be non-empty closed subsets of a complete metric space (X,d) and $T:\bigcup_{i=1}^{p}A_i\to\bigcup_{i=1}^{p}A_i$ be at least one of the following.*

- *a cyclic $(\phi-\psi)$-Kannan contraction,*
- *a cyclic $(\phi-\psi)$-Chatterjea contraction,*

then T has a unique fixed point $z\in\bigcap_{i=1}^{p}A_i$.

Proof. Take $x_0\in X$ and consider the sequence given by $x_{n+1}=Tx_n, n\ge 0$. If there exists $n_0\in N$ such that $x_{n_0+1}=x_{n_0}$, then the point of existence of the fixed point is proved. So, suppose that $x_{n+1}\ne x_n$ for any $n=0,1,\dots$. Then there exists $i_n\in\{1,\dots,p\}$ such that $x_{n-1}\in A_{i_n}$ and $x_n\in A_{i_{n+1}}$. Now, assume first that T is a cyclic $(\phi-\psi)$-Kannan contraction. Then, we have

$$\begin{aligned}\phi\left(d\left(x_n,x_{n+1}\right)\right) &= \phi\left(d\left(Tx_{n-1},Tx_n\right)\right)\\ &\le \phi\left(\alpha d\left(x_{n-1},Tx_{n-1}\right)+\beta d\left(x_n,Tx_n\right)\right)\\ &\qquad -\psi\left(d\left(x_{n-1},Tx_{n-1}\right),d\left(x_n,Tx_n\right)\right)\\ &\le \phi\left(\alpha d\left(x_{n-1},x_n\right)+\beta d\left(x_n,x_{n+1}\right)\right)\\ &\qquad -\psi\left(d\left(x_{n-1},x_n\right),d\left(x_n,x_{n+1}\right)\right)\\ &\le \phi\left(\alpha d\left(x_{n-1},x_n\right)+\beta d\left(x_n,x_{n+1}\right)\right).\end{aligned}$$

Since ϕ is a nondecreasing function, we get

$$d\left(x_n,x_{n+1}\right)\le\alpha d\left(x_{n-1},x_n\right)+\beta d\left(x_n,x_{n+1}\right),$$

which implies

$$d\left(x_n,x_{n+1}\right)\le\frac{\alpha}{1-\beta}d\left(x_{n-1},x_n\right),\forall n. \tag{1}$$

Since $0 < \alpha + \beta \leq 1$, we get that $d(x_n, x_{n+1})$ is a nonincreasing sequence of nonnegative real numbers. Hence, there is $r \geq 0$ such that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = r.$$

Using the continuity of ϕ and ψ, we get

$$\begin{aligned} \phi(r) &\leq \phi((\alpha+\beta)r) - \psi(r,r) \\ &\leq \phi(r) - \psi(r,r), \end{aligned}$$

which implies that $\psi(r,r) = 0$, and hence, $r = 0$.

Similarly, if T is a cyclic $(\phi - \psi)$-Chatterjea contraction, then we have

$$\begin{aligned} \phi(d(x_n, x_{n+1})) &= \phi(d(Tx_{n-1}, Tx_n)) \\ &\leq \phi(\alpha d(x_{n-1}, Tx_n) + \beta d(x_n, Tx_{n-1})) \\ &\qquad -\psi(d(x_{n-1}, Tx_n), d(x_n, Tx_{n-1})) \\ &\leq \phi(\alpha d(x_{n-1}, x_{n+1}) + \beta d(x_n, x_n)) \\ &\qquad -\psi(d(x_{n-1}, x_{n+1}), d(x_n, x_n)) \\ &\leq \phi(\alpha d(x_{n-1}, x_{n+1})). \end{aligned}$$

Since, ϕ is a nondecreasing function, we get

$$d(x_n, x_{n+1}) \leq \alpha d(x_{n-1}, x_{n+1}), \tag{2}$$

and by triangular inequality, we have

$$\begin{aligned} d(x_n, x_{n+1}) &\leq \alpha d(x_{n-1}, x_{n+1}) \\ &\leq \alpha [d(x_{n-1}, x_n) + d(x_n, x_{n+1})], \end{aligned}$$

which implies

$$d(x_n, x_{n+1}) \leq \frac{\alpha}{1-\alpha} d(x_{n-1}, x_n). \tag{3}$$

Since $0 \leq \alpha \leq \frac{1}{2}$, we get that $\{d(x_n, x_{n+1})\}$ is a nonincreasing sequence of nonnegative real numbers. Hence, there is $r \geq 0$ such that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = r.$$

Now, if $\alpha = 0$, then clearly, $r = 0$, and if $0 < \alpha < \frac{1}{2}$, then $\frac{\alpha}{1-\alpha} < 1$, and by induction, we have

$$d(x_n, x_{n+1}) \leq \left(\frac{\alpha}{1-\alpha}\right)^n d(x_0, x_1),$$

and hence, $r = 0$. Now, if $\alpha = \frac{1}{2}$, then from (2), we have

$$d(x_{n-1}, x_{n+1}) \geq 2d(x_n, x_{n+1}),$$

and hence,

$$\lim_{n\to\infty} d(x_{n-1}, x_{n+1}) \geq 2r,$$

but,

$$d(x_{n-1}, x_{n+1}) \leq d(x_{n-1}, x_n) + d(x_n, x_{n+1}),$$

and as $n \to \infty$, we have

$$\lim_{n\to\infty} d(x_{n-1}, x_{n+1}) \leq 2r.$$

Therefore, $\lim_{n\to\infty} d(x_{n-1}, x_{n+1}) = 2r$. Using the continuity of ϕ and ψ, we get

$$\begin{aligned}\phi(r) &\leq \phi\left(\frac{1}{2}2r\right) - \psi(2r, 0)\\ &= \phi(r) - \psi(2r, 0),\end{aligned}$$

which implies that $\psi(2r, 0) = 0$, and hence, $r = 0$.

In the sequel, we show that $\{x_n\}$ is a Cauchy sequence in (X, d). To do so, we need to prove first, the claim that for every $\epsilon > 0$, there exists $n \in \mathbb{N}$ such that if $p, q \geq n$ with $p - q \equiv 1\,(m)$, then $d(x_p, x_q) < \epsilon$. Suppose the contrary case, i.e., there exists $\epsilon > 0$ such that for any $n \in \mathbb{N}$, we can find $p_n > q_n \geq n$ with $p_n - q_n \equiv 1\,(m)$ satisfying $d(x_{p_n}, x_{q_n}) \geq \epsilon$. Now, we take $n > 2m$. Then corresponding to $q_n \geq n$, we can choose p_n in such a way that it is the smallest integer with $p_n > q_n$ satisfying $p_n - q_n \equiv 1\,(m)$ and $d(x_{p_n}, x_{q_n}) \geq \epsilon$. Therefore, $d\left(x_{q_n}, x_{p_{n-m}}\right) < \epsilon$. Using the triangular inequality,

$$\epsilon \leq d(x_{p_n}, x_{q_n}) \leq d\left(x_{q_n}, x_{p_{n-m}}\right) + \sum_{i=1}^{m} d\left(x_{p_{n-i}}, x_{p_{n-i+1}}\right) < \epsilon + \sum_{i=1}^{m} d\left(x_{p_{n-i}}, x_{p_{n-i+1}}\right).$$

Letting $n \to \infty$ in the last inequality, and taking into account that $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$, we obtain $\lim_{n\to\infty} d(x_{p_n}, x_{q_n}) = \epsilon$. Again, by triangle inequality, we have

$$\begin{aligned}\epsilon &\leq d(x_{q_n}, x_{p_n})\\ &\leq d(x_{q_n}, x_{q_{n+1}}) + d(x_{q_{n+1}}, x_{p_{n+1}}) + d(x_{p_{n+1}}, x_{p_n})\\ &\leq d(x_{q_n}, x_{q_{n+1}}) + d(x_{q_{n+1}}, x_{q_n}) + d(x_{q_n}, x_{p_n}) + d(x_{p_n}, x_{p_{n+1}}) + d(x_{p_{n+1}}, x_{p_n})\\ &\leq 2d(x_{q_n}, x_{q_{n+1}}) + d(x_{q_n}, x_{p_n}) + 2d(x_{p_n}, x_{p_{n+1}}).\end{aligned}$$

Taking the limit as $n \to \infty$, and taking into account that $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$, we get $\lim_{n\to\infty} d\left(x_{q_{n+1}}, x_{p_{n+1}}\right) = \epsilon$. Since x_{p_n} and x_{q_n} lie in different adjacently labelled sets A_i and A_{i+1} for certain $1 \leq i \leq m$, assuming that T is a cyclic $(\phi - \psi)$-Kannan contraction, we have

$$\begin{aligned}\phi\left(d\left(x_{q_{n+1}}, x_{p_{n+1}}\right)\right) &= \phi(d(Tx_{q_n}, Tx_{p_n}))\\ &\leq \phi(\alpha d(x_{q_n}, Tx_{q_n}) + \beta d(x_{p_n}, Tx_{p_n}))\\ &\quad -\psi(d(x_{q_n}, Tx_{q_n}), d(x_{p_n}, Tx_{p_n})).\end{aligned}$$

Letting $n \to \infty$ in the last inequality, we obtain

$$\phi(\epsilon) \leq \phi(0) - \psi(0, 0) = 0.$$

Therefore, we get $\epsilon = 0$ which is a contradiction.

Similarly, assuming that T is a cyclic $(\phi - \psi)$-Chatterjea contraction, we have

$$\begin{aligned}\phi\left(d\left(x_{q_{n+1}}, x_{p_{n+1}}\right)\right) &= \phi(d(Tx_{q_n}, Tx_{p_n}))\\ &\leq \phi(\alpha d(x_{q_n}, Tx_{p_n}) + \beta d(x_{p_n}, Tx_{q_n}))\\ &\quad -\psi(d(x_{q_n}, Tx_{p_n}), d(x_{p_n}, Tx_{q_n})).\end{aligned}$$

Letting $n \to \infty$ in the last inequality, we obtain

$$\phi(\epsilon) \leq \phi((\alpha + \beta)\epsilon) - \psi(\epsilon, \epsilon).$$

Therefore, since $0 < \alpha + \beta \leq 1$, we get $\psi(\epsilon, \epsilon) = 0$, and hence, $\epsilon = 0$, which is a contradiction.

From the above proved claim for both cases, i.e., the case when T is a cyclic $(\phi-\psi)$-Kannan contraction and the case when T is a cyclic $(\phi-\psi)$-Chatterjea contraction, and for arbitrary $\epsilon>0$, we can find $n_0\in\mathbb{N}$ such that if $p,q>n_0$ with $p-q=1(m)$, then $d(x_p,x_q)<\epsilon$. Since $\lim\limits_{n\to\infty} d(x_n,x_{n+1})=0$, we can find $n_1\in\mathbb{N}$ such that

$$d(x_n,x_{n+1})\le\frac{\epsilon}{m},\text{ for } n>n_1.$$

Now, for $r,s>\max\{n_0,n_1\}$ and $s>r$, there exists $k\in\{1,2,\dots,m\}$ such that $s-r=k(m)$. Therefore, $s-r+j=1(m)$ for $j=m-k+1$. So, we have

$$d(x_r,x_s)\le d(x_r,x_{s+j})+d(x_{s+j},x_{s+j-1})+\cdots+d(x_{s+1},x_s).$$

This implies

$$d(x_r,x_s)\le\epsilon+\frac{\epsilon}{m}\sum_{j=1}^{m}1=2\epsilon.$$

Thus, $\{x_n\}$ is a Cauchy sequence in $\bigcup\limits_{i=1}^{p}A_i$. Consequently, $\{x_n\}$ converges to some $z\in\bigcup\limits_{i=1}^{p}A_i$. However, in view of cyclical condition, the sequence $\{x_n\}$ has an infinite number of terms in each A_i, for $i=1,2,\dots,p$. Therefore, $z\in\bigcap\limits_{i=1}^{p}A_i$.

Now, we will prove that z is a fixed point of T. Suppose $z\in A_i$, $Tz\in A_{i+1}$, and we take a subsequence x_{n_k} of $\{x_n\}$ with $x_{n_k}\in A_{i-1}$. Then, assuming that T is a cyclic $(\phi-\psi)$-Kannan contraction, we have

$$\begin{aligned}\phi\left(d\left(x_{n_{k+1}},Tz\right)\right)&=\phi\left(d\left(Tx_{n_k},Tz\right)\right)\\&\le\phi\left(\alpha d\left(x_{n_k},Tx_{n_k}\right)+\beta d\left(z,Tz\right)\right)\\&\qquad-\psi\left(d\left(x_{n_k},Tx_{n_k}\right),d\left(z,Tz\right)\right)\\&\le\phi\left(\alpha d\left(x_{n_k},Tx_{n_k}\right)+\beta d\left(z,Tz\right)\right).\end{aligned}$$

Letting $k\to\infty$, we have

$$\phi\left(d\left(z,Tz\right)\right)\le\phi\left(\alpha d\left(z,z\right)+\beta d\left(z,Tz\right)\right),$$

and since ϕ is a nondecreasing function, we get

$$d\left(z,Tz\right)\le\beta d\left(z,Tz\right).$$

Thus, since $0\le\beta<1$, we have $d\left(z,Tz\right)=0$, and hence, $z=Tz$.

Similarly, assuming that T is a cyclic $(\phi-\psi)$-Chatterjea contraction, then we have

$$\begin{aligned}\phi\left(d\left(x_{n_{k+1}},Tz\right)\right)&=\phi\left(d\left(Tx_{n_k},Tz\right)\right)\\&\le\phi\left(\alpha d\left(x_{n_k},Tz\right)+\beta d\left(z,Tx_{n_k}\right)\right)-\psi\left(d\left(x_{n_k},Tz\right),d\left(z,Tx_{n_k}\right)\right)\\&\le\phi\left(\alpha d\left(x_{n_k},Tz\right)+\beta d\left(z,Tx_{n_k}\right)\right).\end{aligned}$$

Letting $k\to\infty$, we have

$$\phi\left(d\left(z,Tz\right)\right)\le\phi\left(\alpha d\left(z,Tz\right)+\beta d\left(z,z\right)\right),$$

since ϕ is a nondecreasing function, we get

$$d\left(z,Tz\right)\le\alpha d\left(z,Tz\right).$$

Thus, since $0\le\alpha\le\frac{1}{2}$, we have $d\left(z,Tz\right)=0$, and hence, $z=Tz$. $\square$

Remark 2.1. Results in Theorem 2.1 are generalized versions of the fixed point theorems of cyclic nonlinear contractions in the sense that many other theorems in the literature are special cases of them. For example, if one takes $\phi(t) = t$, $\psi(t) = 0$, and $\alpha = \beta$ in Theorem 2.1, then we get a well-known theorem given by Theorem 3 in [6]. Also, if one takes $\alpha = \beta = \frac{1}{2}$ in Theorem 2.1, then we have the two theorems given by Theorem 2.1 and Theorem 2.2 in [14], and many others.

3. Examples and applications

We give below two examples in order to validate the proved result.

Example 3.1. Let X be a complete metric space, m positive integer, $A_1, \ldots, A_m$ non-empty closed subsets of X, and $X = \bigcup_{i=1}^{m} A_i$. Let $T : X \to X$ be an operator such that

a) $X = \bigcup_{i=1}^{m} A_i$ is a cyclic representation of X with respect to T.

b) for any $x \in A_i$, $y \in A_{i+1}$, $i = 1, 2, \ldots, m$, where $A_{m+1} = A_1$ and $\rho : [0, \infty) \to [0, \infty)$ is a Lebesgue integrable mapping satisfies $\int_0^t \rho(s)\ ds > 0$ for $t > 0$, we have one of the following:

$$\int_0^{d(Tx,Ty)} \rho(t)\ dt \leq \int_0^{\alpha d(x,Tx)+\beta d(y,Ty)} \rho(t)\ dt,$$

or

$$\int_0^{d(Tx,Ty)} \rho(t)\ dt \leq \int_0^{\alpha d(Tx,y)+\beta d(Ty,x)} \rho(t)\ dt.$$

Then T has a unique fixed point $z \in \bigcap_{i=1}^{m} A_i$.

In order to see this, one might let $\phi : [0, \infty) \to [0, \infty)$ be defined as $\phi(t) = \int_0^t \rho(s)\ ds > 0$. Then ϕ is alternating distance function, and by taking $\psi(t) = 0$, we get the result.

Example 3.2. Let $X = [-1, 1] \subseteq \mathbb{R}$ with $d(x, y) = |x - y|$. Let $T : [-1, 1] \to [-1, 1]$ be given by

$$T(x) = \begin{cases} -\frac{1}{2}xe^{-\frac{1}{|x|}}, & x \in (0, 1], \\ 0, & x = 0, \\ -\frac{1}{3}xe^{-\frac{1}{|x|}}, & x \in [-1, 0). \end{cases}$$

By taking $\psi(t) = 0$, $\phi(t) = t$, and $x \in [0, 1]$, $y \in [-1, 0]$, we have

$$\begin{array}{rcl} |Tx - Ty| & = & |-\frac{1}{2}xe^{-\frac{1}{|x|}} + \frac{1}{3}ye^{-\frac{1}{|y|}}| \\ & \leq & \frac{1}{2}|x| + \frac{1}{3}|y| \\ & \leq & \frac{1}{2}|x + \frac{1}{2}xe^{-\frac{1}{|x|}}| + \frac{1}{3}|y + \frac{1}{3}ye^{-\frac{1}{|y|}}| \\ & = & \frac{1}{2}|Tx - x| + \frac{1}{3}|Ty - y|, \end{array}$$

which implies that T has a unique fixed point in $[-1, 0] \cap [0, 1]$ which is $z = 0$.

References

[1] R. Kannan, Some results on fixed points, *Bull. Calcutta Math. Soc.* **10** (1968), 71–76.

[2] S. Banach, Surles operations dans les ensembles et leur application aux equation sitegrales, *Fund. Math.* **3** (1922), 133–181.
[3] S. Chatterjea, Fixed point theorems, *C. R. Acad. Bulgare Sci.* **25** (1972), 727–730.
[4] T. Zamfirescu, Fixed point theorems in metric spaces, *Arch. Math.* (Basel) **23** (1972), 292–298.
[5] I. Rus, Cyclic representations and fixed points, *Annals of the Tiberiu Popoviciu Seminar of Functional Equations, Approximation and Convexity* **3** (2005), 171–178.
[6] M. Petric, Some results concerning cyclical contractive mappings, *General Mathematics* **18** (2010), nr. 4, 213–226,.
[7] S. Radenović, A note on fixed point theory for cyclic φ-contractions, *Fixed Point Theory and Applications* **2015** (2015), Art. 189. DOI: 10.1186/s13663-015-0437-8
[8] S. Radenović, T. Došenović, T. Aleksić-Lampert, Z. Golubović, A note on some recent fixed point results for cyclic contractions in b-metric spaces and an application to integral equations, *Applied Mathematics and Computation* **273** (2016), 155–164.
[9] B.S. Choudhury, Unique fixed point theorem fo weak C-contractive mappings, *Kathmandu Univ. J. Sci. Eng. Tech.* **5** (2009), no. 1, 6–13.
[10] M. Khan, M. Swaleh, S. Sessa, Fixed point theorem by altering distances between points, *Bull. Austral. Math. soc.* **30** (1984), no. 1, 1–9.
[11] K.R. Sastry, G. Babu, Some fixed point theorems by altering distances between the points, *Indian Journal of Pure and Applied Mathematics* **30** (1999), no. 6, 641–647,.
[12] K. Sastry, S. Naidu, G.R. Babu, G.A. Naidu, Generalization of common fixed point theorems for weakly commuting map by altering distances, *Tamkang Journal of Mathematics* **31** (2000), no. 3, 243–250.
[13] S. Naidu, Some fixed point theorems in metric spaces by altering distances, *Czechoslovak Mathematical Journal* **53** (2003), no. 1, 205–212.
[14] M. Al-Khaleel, A. Awad, Sh. AlShareef, Some results for cyclic nonlinear contractive mappings in metric spaces, *Acta Math. Acad. Paed. Ny.* **29** (2013), 9–18.

(Mohammad Al-Khaleel) Department of Applied Mathematics and Sciences, Khalifa University, Abu Dhabi, UAE
Department of Mathematics, Yarmouk University, Irbid, Jordan
E-mail address: `mohammad.alkhaleel@kustar.ac.ae`

(Sharifa Al-Sharif) Department of Mathematics, Yarmouk University, Irbid, Jordan