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The second order abstract Cauchy problem and integrated semigroups generated by matrix pseudo-differential operators

Viorel Catană

Abstract. The main aim of this paper is to indicate how integrated semigroups and spectral properties of some classes of pseudo-differential operators can be used studying second order abstract Cauchy problems

$$u''(t) - A_p u(t) = 0,\ u(0) = u_0,\ u'(0) = u_1$$

or

$$u''(t) - A_p^q u'(t) - (aA_p^{2q} + bA_p^q + cI)u(t) = 0,\ u(0) = u_0,\ u'(0) = u_1,$$

for every initial values $(u_0, u_1) \in \mathcal{D}(A_p^M) \times \mathcal{D}(A_p^N)$. Here $A_p : \mathcal{D}(A_p) \subset L^p(\mathbf{R}^n) \to L^p(\mathbf{R}^n)$ is a closed extension of a pseudo-differential operator $A : \mathcal{S}(\mathbf{R}^n) \subset L^p(\mathbf{R}^n) \to \mathcal{S}(\mathbf{R}^n) \subset L^p(\mathbf{R}^n)$,

$$Au(x) = (2x)^{-n} \int_{\mathbf{R}^n} e^{i<x,\xi>} a(\xi)\hat{u}(\xi) d\xi$$

to the space $L^p(\mathbf{R}^n)$.

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1. Introduction

We study the second order abstract Cauchy problems (ACP_2):

$$u''(t) = A_p u(t),\ t > 0,\ u(0) = u_0,\ u'(0) = u_1, \tag{1}$$

$$\begin{aligned} &u''(t) - A_p^q u'(t) - (aA_p^{2q} + bA_p^q + cI)u(t) = 0,\ t > 0, \\ &u(0) = u_0,\ u'(0) = u_1,\ q \in \mathbf{N}^*,\ a, b, c \in \mathbf{R}, \end{aligned} \tag{2}$$

where $A_p : \mathcal{D}(A_p) \subset L^p(\mathbf{R}^n) \to L^p(\mathbf{R}^n)$ is a closed extension of a pseudo-differential operator $A : \mathcal{S}(\mathbf{R}^n) \subset L^p(\mathbf{R}^n) \to \mathcal{S}(\mathbf{R}^n) \subset L^p(\mathbf{R}^n)$ to the space $L^p(\mathbf{R}^n)$.

The most natural way to study (ACP_2) (1) or (2) is to reduce them to the first order abstract Cauchy problems (ACP_r):

$$w'(t) = B_p w(t),\ t > 0,\ w(0) = (w_0, w_1) = (u_0, u_1) \tag{3}$$

$$w'(t) = B_{p,q} w(t),\ t > 0,\ w(0) = (w_0, w_1) = (u_0, u_1) \tag{4}$$

where

$$B_p : \mathcal{D}(A_p) \times L^p(\mathbf{R}^n) \to (L^p(\mathbf{R}^n))^2,\ B_p = \begin{pmatrix} 0 & I \\ A_p & 0 \end{pmatrix},$$

$$B_{p,q} : \mathcal{D}(A_p^{2q}) \times \mathcal{D}(A_p^q) \to (L^p(\mathbf{R}^n))^2,\ B_{p,q} = \begin{pmatrix} 0 & I \\ aA_p^{2q} + bA_p^q + cI & A_p^q \end{pmatrix}.$$

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Here I is the identity operator on $L^p(\mathbf{R}^n)$, $(w_0, w_1) \in \mathcal{D}(A_p^M) \times \mathcal{D}(A_p^N)$, $M, N \in \mathbf{N}^*$ and $(L^p(\mathbf{R}^n))^2 = L^p(\mathbf{R}^n) \times L^p(\mathbf{R}^n)$ is endowed with the maximum norm (i.e. $||(w_0, w_1)|| = \max(||w_0||_p, ||w_1||_p)$, where $||\cdot||_p$ denote the norm$L^p(\mathbf{R}^n)$).

We will also refer to the first order abstract Cauchy problem (ACP):

$$u'(t) = A_p u(t),\ t > 0,\ u(0) = u_0. \tag{5}$$

Following F. Neubrauder (see [10]) we will establish the connection between (ACP$_2$) (1) or (2) and (ACP$_r$) (3) respectively (4).

In particular, we are interested in the connection between the spectral properties of the operator A_p and the fact that (ACP$_2$) or (ACP) are wellposed in Neubrauder's sense (see [3], [7], [8], [10]).

2. Preliminaries and notations

First we shall introduce a few concepts terminologies and recall some notions and results concerning pseudo-differential operators, integrated semigroups and well posed abstract Cauchy problem.

Definition 2.1. *Let $m \in \mathbf{R}$, $\rho \in (0,1]$. Then we define $S^m_{\rho,0}(\mathbf{R}^n)$ to be the set of all functions $a \in C^\infty(\mathbf{R}^n)$ such that for any multi-index $\alpha \in \mathbf{N}^n$ there is a positive constant C_α only depending on α, for which*

$$|D^\alpha a(\xi)| \le C_\alpha (1 + |\xi|)^{m - \rho|\alpha|} \quad \text{for all} \quad \xi \in \mathbf{R}^n. \tag{6}$$

We call any function $a \in S^m_{\rho,0}(\mathbf{R}^n)$ a symbol of order m and of type $(\rho, 0)$.

Definition 2.2. *Let $a \in S^m_{\rho,0}(\mathbf{R}^n)$ be a symbol. Then the pseudo-differential operator $A = a(D)$ associated with a is defined by*

$$Au(x) = (2\pi)^{-n} \int_{\mathbf{R}^n} e^{i<x,\xi>} a(\xi) \hat{u}(\xi) d\xi,\ u \in \mathcal{S}(\mathbf{R}^n),\ x \in \mathbf{R}^n. \tag{7}$$

We denote by $L^m_{\rho,0}(\mathbf{R}^n)$ the class of pseudo-differential operators of order m and of type $(\rho, 0)$.

Definition 2.3. *A symbol $a \in S^m_{\rho,0}(\mathbf{R}^n)$ is said to be elliptic iff there exist positive constants C and R such that*

$$|a(\xi)| \ge C(1 + |\xi|)^m, \quad \text{for all} \quad |\xi| \ge R. \tag{8}$$

Of course, a pseudo-differential operator $A \in L^m_{\rho,0}(\mathbf{R}^n)$ is said to be elliptic iff its symbol is elliptic.

Definition 2.4. *A symbol $a \in S^m_{\rho,0}(\mathbf{R}^n)$ is said to be strongly Carleman iff there exist positive numbers b, L and C, such that*

$$|a(\xi)| \ge C(1 + |\xi|)^b, \quad \text{for all} \quad |\xi| \ge L. \tag{9}$$

Suppose now we give $A \in L^m_{\rho,0}(\mathbf{R}^n)$. Then for each $1 \le p < \infty$, $A_p : \mathcal{S}(\mathbf{R}^n) \subset L^p(\mathbf{R}^n) \to L^p(\mathbf{R}^n)$ is a closable operator on $L^p(\mathbf{R}^n)$ and we shall denote its closure on $L^p(\mathbf{R}^n)$ by $A_p : \mathcal{D}(A_p) \subset L^p(\mathbf{R}^n) \to L^p(\mathbf{R}^n)$, $\mathcal{D}(A_p) = \{u \in L^p(\mathbf{R}^n),\ Au \in L^p(\mathbf{R}^n)\}$, $A_p u = Au$ (in the distributional sense) for all $u \in \mathcal{D}(A_p)$.

We shall call A_p the minimal operator associated with A.

Also for the same operator $A \in L^m_{\rho,0}(\mathbf{R}^n)$ and for each $1 \le p < \infty$, we can define another closed operator

$\tilde{A}_p : \mathcal{D}(\tilde{A}_p) \subset L^p(\mathbf{R}^n) \to L^p(\mathbf{R}^n)$, *where*

$$\mathcal{D}(\tilde{A}_p) = \{u \in L^p(\mathbf{R}^n);\ (\exists) f_u \in L^p(\mathbf{R}^n) \text{ s.t. } (u, A^*\varphi) = (f_u, \varphi),\ (\forall)\varphi \in \mathcal{S}(\mathbf{R}^n)\} \tag{10}$$

and $\tilde{A}_p u = f_u$. *We shall call* $\tilde{A}_p$ *the maximal operator associated with* A.

We remark that both operators A_p and $\tilde{A}_p$ are closed extensions to $L^p(\mathbf{R}^n)$ of the operator A and they really coincide (see [15]). Moreover if $A \in L^m_{\rho,0}(\mathbf{R}^n)$ is an elliptic operator of order $m \geq 0$, then $\mathcal{D}(A_p) = H^{m,p}(\mathbf{R}^n)$ the Sobolev space (see [17]).

We can now state some results concerning pseudo-differential operators, integrated semigroups, well posed Cauchy problems and the connection between them. For details and proofs see [3], [8], [10].

Theorem 2.1. *Let* $a \in S^m_{\rho,0}(\mathbf{R}^n)$ *be a strongly Carleman symbol, where* $m > 0$, $\rho \in (0,1]$. *Then for each* $1 \leq p < \infty$ *with* $\rho(a(D)_p) \neq \emptyset$ *(the resolvent set of the operator* $a(D)_p = A_p$*), we have* $\sigma(a(D)_p) = a(\mathbf{R}^n)$ *(the range of the symbol of the operator* A_p*).*

The following proposition gives sufficient criteria in order to have the resolvent set of the operator A_p nonempty.

Proposition 2.1. *Let* $a \in S^m_{\rho,0}(\mathbf{R}^n)$ *be a strongly Carleman symbol, where* $m > 0$, $\rho \in (0,1]$ *and the coercivity constant is* $b > 0$. *Then for each* $p \in (1,\infty)$ *the following assertions hold.*

(i) If $n|1/2 - 1/p|(m + 1 - \rho - b)/b < 1$, *then*

$$\rho(A_p) \neq \emptyset \Leftrightarrow a(\mathbf{R}^n) \neq \mathbf{C},$$

where $a(\mathbf{R}^n) = \{a(\xi);\ \xi \in \mathbf{R}^n\}$.

Moreover, if $b > n(m + 1 - \rho)/(n + 2)$, *then*

$$\rho(A_p) \neq \emptyset \Leftrightarrow a(\mathbf{R}^n) \neq \mathbf{C} \quad \textit{for all} \quad p \in (1,\infty);$$

(ii) If there exist constants s, C *such that* $s \leq 1$, $C > 0$,

$$n|1/2 - 1/p|/b < 1 \text{ and } \frac{|D^\alpha a(\xi)|}{|a(\xi)|} \leq C(1 + |\xi|)^{-s|\alpha|},\ (\forall)\xi \in \mathbf{R}^n,$$

$(\forall)|\alpha| \leq 1 + [n|1/2 - 1/p|]$ *(where* $[\cdot]$ *denotes the integer part of a real number), then*

$$\rho(A_p) \neq \emptyset \Leftrightarrow a(\mathbf{R}^n) \neq \mathbf{C};$$

(iii) If $b > m/2$ *and* $n(1 - \rho)|1/2 - 1/p|/(2b - m) < 1$, *then*

$$\rho(A_p) \neq \emptyset \Leftrightarrow a(\mathbf{R}^n) \neq \mathbf{C};$$

(iv) If $p = 1$ *and if there exist constants* s, C *such that* $s \in [0,1]$, $b > n + s - ns$, $|D^\alpha a(\xi)/a(\xi)| \leq C(1 + |\xi|)^{-s|\alpha|}$, $(\forall)\xi \in \mathbf{R}^n$, $(\forall)|\alpha| \leq 1 + [n|1/2 - 1/p|]$, *then*

$$\rho(A_1) \neq \emptyset \Leftrightarrow a(\mathbf{R}^n) \neq \mathbf{C}.$$

In particular, if we moreover suppose that $b > (m+1-\rho-b)(n-1)+1$, $b \geq m - \rho$, *then the same conclusion still holds.*

(v) We suppose, in particular, that $a \in S^m_{\rho,0}(\mathbf{R}^n)$ *is an elliptic symbol. Then for each* $p \in (1,\infty)$ *and* $m \geq \frac{1}{2}(n - n\rho)$ *it follows that* $\rho(A_p) \neq \emptyset \Leftrightarrow a(\mathbf{R}^n) \neq \mathbf{C}$.

Again, for $p = 1$ and $m > n + \rho - n\rho$, we obtain the same conclusion.

(vi) If $p = 2$ and $|a(\xi)| \to \infty$ when $|\xi| \to \infty$, $\xi \in \mathbf{R}^n$ (in particular, a is a strongly Carleman symbol), then

$$\rho(A_2) \neq \emptyset \Leftrightarrow a(\mathbf{R}^n) \neq \mathbf{C}. \tag{11}$$

Remark 2.1. *If $\rho = 1$ and $a \in S^m_{1,0}(\mathbf{R}^n)$ is a polynomial then the assertions (i), (ii) of Proposition 2.1 yield statements (i), (ii) of Proposition 5.3 in M. Hieber ([7]).*

N-times integrated exponentially bounded semigroups $(S(t))_{t\geq 0}$ of linear operators on a Banach space $E(N \in \mathbf{N})$ were introduced by Arendt [2] and studied by Arendt, Kelerman, Hieber, Neubrander, Thieme and many others ([1], [2], [4], [5], [6], [10], [12]).

The goal of the theory of integrated semigroups is the analysis of an abstract Cauchy problem as that in (5).

Definition 2.5. *(F. Neubrander [10]). Let $A \in \mathcal{L}(E)$ be a linear operator on a Banach space E. If there exists $N \in \mathbf{N}^*$, constants $M > 0$, $\omega \in \mathbf{R}$ and a strongly continuous family $(S(t))_{t\geq 0}$ in $\mathcal{L}(E)$ with $||S(t)|| \leq Me^{\omega t}$ for all $t \geq 0$ such that $R(\lambda, A) = (\lambda I - A)^{-1}$ exists and is given by*

$$R(\lambda, A) = \lambda^N \int_0^\infty e^{-\lambda t} S(t)dt, \tag{12}$$

for all $\lambda > \omega$, then A is called the generator of the N-times integrated semigroup $(S(t))_{t\geq 0}$ (N-t.i.s).

Definition 2.6. *(F. Neubrander [10]). Let $A \in \mathcal{L}(E)$ be a closed linear operator on a Banach space E. Then the abstract Cauchy problem (ACP) defined by (A, u_0), $u_0 \in E$, namely*

$$u'(t) = Au(t),\ u(0) = u_0,\ t > 0 \tag{13}$$

is called (N,k)-well posed iff there exist $N \in \mathbf{N}^*$, $k \in \mathbf{N}$, with $0 \leq k \leq N$ and a locally bounded function $p : [0,\infty) \to \mathbf{R}$ such that for all $u_0 \in \mathcal{D}(A^N)$, there exists a unique solution $u \in C^1([0,\infty), E)$ of the (ACP) with

$$||u(t)|| \leq p(t)||u_0||_k \quad \textit{for all} \quad t \geq 0,$$

where

$$||u_0||_k := ||u_0|| + ||Au_0|| + \ldots + ||A^k u_0||.$$

Here $||\cdot||_k$ is the graph norm of the Banach space $[\mathcal{D}(A^k)] := (\mathcal{D}(A^k), ||\cdot||_k)$.

If, in addition, we can choose $p(t) = Me^{\omega t}$, $t \geq 0$ then the (ACP) is called exponentially (N,k)-well posed.

The connection between generators of integrated semigroups and exponentially well posed (ACP) is given by the following theorem (see F. Neubrander [10], th. 4.2).

Theorem 2.2. *Let $A \in \mathcal{L}(E)$ be a linear operator on a Banach space E with nonempty resolvent set $\rho(A)$. Then the following assertions hold.*

(i) If A generates an (N-1)-t.i.s., then (ACP) is exponentially (N,N-1)-well posed.

(ii) If A is densely defined and if (ACP) is exponentially (N, N-1)-well posed then A generates an (N-1)-t.i.s.

Theorem 2.3. *(see [3], [8]). Let $a \in S^m_{\rho,0}(\mathbf{R}^n)$ be a strongly Carleman symbol, where $m > 0$, $\rho \in (0,1]$ and $1 \le p < \infty$. Then the following assertions are equivalent.*

(i) $\rho(A_p) \neq \emptyset$ and $\sup\limits_{\xi \in \mathbf{R}^n} Re\, a(\xi) \le \omega$ for some $\omega \in \mathbf{R}$;

(ii) There exists $N \in \mathbf{N}$ such that the operator A_p generates an N-t.i.s on $L^p(\mathbf{R}^n)$.

Remark 2.2. *From the proof of Theorem 2.3 the estimate: $N > n|1/2 - 1/p|(m + 1 - \rho)/b$ follows.*

Remark 2.3. *The assertions (i), (ii) in Theorem 2.3 are equivalent to (iii) (ACP) is exponentially (N+1,N)-well posed provided a is elliptic (or, hypoelliptic, of polynomial function type).*

Remark 2.4. *If $\rho = 1$ and if $a \in S^m_{1,0}$ is a polynomial, then Theorem 2.3 yields Theorem 5.1 in M. Hieber [7].*

Theorem 2.4. *(see [3], [8]). Let $1 \le p \le \infty$ and let $a \in S^m_{\rho,0}(\mathbf{R}^n)$ be a strongly Carleman symbol. Then the following assertions are equivalent.*

(i) $\sup Re\, \sigma(A_p) \in \mathbf{R} \Leftrightarrow (\sigma(A_p) \subset \{z \in \mathbf{C};\ Re\, z \le \omega\}$ for some $\omega \in \mathbf{R})$;

(ii) There exists $N \in \mathbf{N}$ such that A_p generates an N-t.i.s. on $L^p(\mathbf{R}^n)$.

Remark 2.5. *From Theorems 2.2, 2.3, 2.4 we deduce that the following assertions are equivalent:*

(i) $\rho(A_p) \neq \emptyset$ and $\sup\limits_{\xi \in \mathbf{R}^n} Re\, a(\xi) \le \omega$ for some $\omega \in \mathbf{R}$;

(ii) $\sup Re\, \sigma(A_p) \in \mathbf{R}$;

(iii) There exists $N \in \mathbf{N}$ such that A_p generates an N-t.i.s. on $L^p(\mathbf{R}^n)$.

Remark 2.6. *The previous remark is similar to Theorem 4.6 in M. Hieber [8].*

3. The incomplete second order abstract Cauchy problem

Now we turn to the (ACP$_2$) (1). We suppose that one considers those values of $1 \le p < \infty$ for which $\rho(A_p) \neq \emptyset$ (see for example Proposition 2.1 in the case of a strongly Carleman symbol).

A function $u : [0, \infty) \to \mathcal{D}(A_p)$, $u \in C^2(\mathbf{R}_+, L^p(\mathbf{R}^n))$ will be called a solution of (ACP$_2$) iff it satisfies (ACP$_2$) (1).

The following lemmas give some connections between (ACP$_2$) (1) and (ACP$_r$) (3) as well as between the operators A_p and B_p which define this problems.

Lemma 3.1. *(ACP$_2$) (1) has a unique solution $u \in C^2(\mathbf{R}_+, L^p(\mathbf{R}^n))$ if and only if (ACP$_r$) (3) has a unique solution $w = (u, u')$.*

Lemma 3.2. *Let $A_p : \mathcal{D}(A_p) \subset L^p(\mathbf{R}^n) \to L^p(\mathbf{R}^n)$ and let $B_p : \mathcal{D}(A_p) \times \times L^p(\mathbf{R}^n) \to (L^p(\mathbf{R}^n))^2$, $B_p = \begin{pmatrix} 0 & I \\ A_p & 0 \end{pmatrix}$ be the operators which define respectively (ACP$_2$) (1) and (ACP$_r$) (3). Then the following assertions hold.*

(i) B_p is a closed operators on $(L^p(\mathbf{R}^n))^2$.

(ii) The resolvent set of A_p, $\rho(A_p) \neq \emptyset$ if and only if the resolvent set of B_p, $\rho(B_p) \neq \emptyset$. Moreover, for $\lambda \in \mathbf{C}$, $\lambda^2 \in \rho(A_p)$ we have $\lambda \in \rho(B_p)$ and

$$R(\lambda, B_p) = \begin{pmatrix} \lambda R(\lambda^2, A_p) & R(\lambda^2, A_p) \\ A_p R(\lambda^2, A_p) & \lambda R(\lambda^2, A_p) \end{pmatrix}.$$

(iii) $\mathcal{D}(B_p^{2N}) = \mathcal{D}(A_p^N) \times \mathcal{D}(A_p^N)$ and for all $(u, v) \in \mathcal{D}(B_p^{2N})$:

$$||u||_N + ||v||_N \le 2||(u,v)||_{2N}^{B_p} \le 4(||u||_N + ||v||_N),$$

where $||\cdot||_N$ *denotes the graph norm of the Banach space* $[\mathcal{D}(A_p)] = (\mathcal{D}(A_p^N), ||\cdot||_N)$ *(i.e.* $||u||_N = ||u||_p + ||A_p u||_p + \ldots + ||A_p^N u||_p$, $u \in \mathcal{D}(A_p^N)$*).*

(iv) $\mathcal{D}(B_p^{2N-1}) = \mathcal{D}(A_p^N) \times \mathcal{D}(A_p^{N-1})$ *and for all* $(u,v) \in \mathcal{D}(B_p^{2N-1})$ *we have*

$$||u||_N + ||v||_{N-1} \le 2||(u,v)||_{2N-1}^{B_p} \le 4(||u||_N + ||v||_{N-1}).$$

The proofs of these lemmas are obvious and are omitted.

Definition 3.1. *([10]). Let* $A \in L^m_{\rho,0}(\mathbf{R}^n)$, $\rho \in (0,1]$, $m > 0$, *be a pseudo-differential operator such that his symbol* $a \in S^m_{\rho,0}(\mathbf{R}^n)$ *is a strongly Carleman symbol with coercivity constant,* $b > 0$. *Moreover, suppose that* $a(\mathbf{R}^n) \neq \mathbf{C}$ *and let* $1 \le p < \infty$ *satisfy one of the conditions (i)-(vi) in Proposition 2.1. Then*

(i) (ACP$_2$) (1) is $2N$*-well posed if and only if there exists a locally bounded function* $q : \mathbf{R}_+ \to \mathbf{R}_+$ *such that (ACP$_2$) has unique solution* $u \in C^2(\mathbf{R}_+, L^p(\mathbf{R}^n))$ *for all* $(u_0, u_1) \in \mathcal{D}(A_p^N) \times \mathcal{D}(A_p^N)$, $N \ge 1$, *satisfying*

$$||u(t)||_p \le q(t)(||u_0||_{N-1} + ||u_1||_{N-1}), \quad t \in \mathbf{R}_+ .$$

(ii) (ACP$_2$) (1) is $(2N+1)$*-well posed if and only if there exists a locally bounded function* $q : \mathbf{R}_+ \to \mathbf{R}_+$ *such that (ACP$_2$) has unique solution* $u \in C^2(\mathbf{R}_+, L^p(\mathbf{R}^n))$ *for all*

$(u_0, u_1) \in \mathcal{D}(A_p^{N+1}) \times \mathcal{D}(A_p^N)$ *satisfying*

$$||u(t)||_p \le q(t)(||u_0||_N + ||u_1||_{N-1}), \quad t \in \mathbf{R} .$$

If we can choose $q(t) = Ce^{\omega t}$, $C > 0$, $\omega \in \mathbf{R}$, $t \in \mathbf{R}_+$ *, then (ACP$_2$) will be called exponentially M-well posed (where* $M = 2N$ *or* $M = 2N+1$*).*

We follow now F. Neubrander's point of view and can state the following results.

Theorem 3.1. *Let* $A \in L^m_{\rho,0}(\mathbf{R}^n)$ *be a pseudo-differential operators as in definition 3.1. Let* $A_p : \mathcal{D}(A_p) \subset L^p(\mathbf{R}^n) \to L^p(\mathbf{R}^n)$ *be a closed extension of* A *to* $L^p(\mathbf{R}^n)$ *and let* $B_p : \mathcal{D}(A_p) \times L^p(\mathbf{R}^n) \to (L^p(\mathbf{R}^n))^2$, $B_p = \begin{pmatrix} 0 & I \\ A_p & 0 \end{pmatrix}$. *Then the following statements are equivalent:*

(i) (ACP$_2$) (1) M*-well posed (respectively exponentially* M*-well posed)*

(ii) (ACP$_r$) (3) $(M, M-1)$*-well posed (respectively exponentially* $(M, M-1)$*-well posed).*

Proof. Suppose (ACP$_2$) (1) is $2N$-well posed, i.e. there exists a locally bounded function $q : \mathbf{R}_+ \to \mathbf{R}_+$ such that for all $(u_0, u_1) \in \mathcal{D}(A_p^N) \times \mathcal{D}(A_p^N) = \mathcal{D}(B_p^{2N})$ there exist unique solution $u \in C^2(\mathbf{R}_+, L^p(\mathbf{R}^n))$ satisfying

$$||u(t)||_p \le q(t)(||u_0||_{N-1} + ||u_1||_{N-1}), \quad t \ge 0. \tag{14}$$

By Lemma 3.2 and (14) we obtain

$$||u(t)||_p \le q(t)(||u_0||_N + ||u_1||_{N-1}) \le 2q(t)||(u_0,u_1)||_{2N-1}^{B_p}, \quad t \ge 0. \tag{15}$$

We know by Lemma 3.1 that $u \in C^2(\mathbf{R}_+, L^p(\mathbf{R}^n))$ is a solution of (ACP$_2$) (1) iff $w = (u, u') \in C^1(\mathbf{R}_+, L^p(\mathbf{R}^n))^2)$ is a solution of (ACP$_r$) (3).

Then, combining Lemma 7.7 in [10] and Lemma 3.2 we get

$$||u'(t)||_p \leq q(t)(||u_0||_N + ||u_1||_{N-1}) \leq 2q(t)||(u_0,u_1)||_{2N-1}^{B_p}, \quad t \geq 0. \tag{16}$$

Therefore by (15), (16) we obtain

$$||(u(t),u'(t))|| = \max(||u(t)||_p, ||u'(t)||_p) \leq 2q(t)||(u_0,u_1)||_{2N-1}^{B_p},\ t \geq 0, \tag{17}$$

whence (ACP_r) (3) is $(2N, 2N-1)$-well posed. Conversely let us suppose that (ACP_r) (3) is $(2N, 2N-1)$-well posed, i.e. there exists a locally bounded function $p : \mathbf{R}_+ \to \mathbf{R}_+$ such that for all $w^0 = (u_0, u_1) \in \mathcal{D}(B_p^{2N}) = \mathcal{D}(A_p^N) \times \mathcal{D}(A_p^N)$ there exist unique solution $w = (u, v) \in C^1(\mathbf{R}_+, (L^p(\mathbf{R}^n))^2)$ satisfying

$$\begin{aligned} ||w(t)|| &= \max(||u(t)||_p, ||v(t)||_p) \leq p(t)||(u_0,u_1)||_{2N-1}^{B_p} \leq \\ &\leq 2p(t)(||u_0||_{N-1} + ||u_1||_{N-1}),\ t \geq 0. \end{aligned} \tag{18}$$

By Lemma 3.1 and (18) we see that (ACP_2) (1) has unique solution $u \in C^2(\mathbf{R}_+, L^p(\mathbf{R}^n))$ with initial values $(u_0, u_1) = w^0 \in \mathcal{D}(A_p^N) \times \mathcal{D}(A_p^N)$ such that

$$||u(t)||_p \leq 2p(t)(||u_0||_{N-1} + ||u_1||_{N-1}),\ t \geq 0. \tag{19}$$

By (19) we get that (ACP_2) (1) is $2N$-well posed. The proof in case $M = 2N+1$ requires only obvious modifications.

Theorem 3.2. *Let $A \in L^m_{\rho,0}(\mathbf{R}^n)$, A_p and B_p be as in Theorem 3.1. Then the following statements are equivalent:*

(i) (ACP_r) (3) has unique solution for all $w = (u,v) \in \mathcal{D}(B_p^M)$;

(ii) (ACP_r) (3) is $(M, M-1)$ well posed;

(iii) (ACP_2) (1) is M-well posed;

(iv) (ACP_2) (1) has unique solution for all $w = (u,v) \in \mathcal{D}(A_p^N) \times \mathcal{D}(A_p^N)$ (or for all $w = (u,v) \in \mathcal{D}(A_p^{N+1}) \times \mathcal{D}(A_p^N)$).

Proof. (i) $\Rightarrow$ (iv). This follows by Lemma 3.1.

(iv) $\Rightarrow$ (iii) We prove this for $w = (u,v) \in \mathcal{D}(A_p^N) \times \mathcal{D}(A_p^N)$. The proof for $w = (u,v) = \mathcal{D}(A_p^{N+1}) \times \mathcal{D}(A_p^N)$ requires only some modifications.

By hypothesis and by Lemma 3.2 the operator B_p has a nonempty resolvent set and by Lemma 3.1, (ACP_r) (3) has unique solution $w(t) = (u(t), u'(t))$ for all $w = (u,v) = \mathcal{D}(B_p^{2N}) = \mathcal{D}(A_p^N) \times \mathcal{D}(A_p^N)$. Hence by Theorem 3.1 (iii) (see [10]) and by Lemma 3.2, there exists a locally bounded function $p : \mathbf{R}_+ \to \mathbf{R}_+$ such that

$$\begin{aligned} &\left\| \int_0^t w(s)ds \right\| = \max\left(\left\| \int_0^t u(s)ds \right\|_p, \left\| \int_0^t u'(s)ds \right\|_p \right) = \\ &= \max\left(\left\| \int_0^t u(s)ds \right\|_p, ||u(t) - u||_p \right) \leq \\ &\leq p(t)||(u,v)||_{2N-2}^{B_p} \leq 2p(t)(||u||_{N-1} + ||v||_{N-1}) \end{aligned} \tag{20}$$

By Lemma 3.1 the first coordinate $u(t)$ of the solution $w(t) = (u(t), u'(t))$ of (ACP_r) (3) is a solution of (ACP_2) (1) and by (20) implies that (ACP_2) (1) is $2N$-well posed.

(iii) $\Rightarrow$ (ii) This is contained in Theorem 3.1.

(ii) $\Rightarrow$ (i) This follows from Definition 2.6.

Theorem 3.3. *Under the assumptions of Theorem 3.1 the following assertions are equivalent.*

(i) B_p generates an $(M-1)$-t.i.s. on $(L^p(\mathbf{R}^n))^2$;
(ii) (ACP_2) (1) is exponentially M-well posed.

Proof. By Theorem 2.2, B_p generates an $(M-1)$-t.i.s. on $(L^p(\mathbf{R}^n))^2$ iff (ACP_r) (3) is exponentially $(M, M-1)$-well posed iff (ACP_2) (1) is exponentially M-well posed. The last equivalence holds by Theorem 3.2.

Remark 3.1. *All considerations above still hold if we replace (ACP_2) (1) by (ACP_2):*

$$u''(t) = (aA_p + bI)u(t), \ u(0) = u_0, \ u'(0) = u_1 \ t > 0, \ a, b \in \mathbf{R}, \ a \neq 0. \tag{21}$$

4. The complete second order abstract Cauchy problem

In this section we study (ACP_2) (2) by reducing it to (ACP_r)

$$\begin{aligned} w'(t) &= \tilde{B}_{p,q}w(t) \\ w(0) &= (w_0, w_1) = (u_0, u_1), \end{aligned} \tag{22}$$

where

$$\tilde{B}_{p,q} : \mathcal{D}(\tilde{B}_{p,q}) \subset (L^p(\mathbf{R}^n))^2 \to (L^p(\mathbf{R}^n))^2,$$

$$\begin{aligned} \tilde{B}_{p,q} \quad = \quad & \begin{pmatrix} (\lambda_0 - A_p)^{2q} & 0 \\ 0 & (\lambda_0 - A_p)^{2q} \end{pmatrix} \times \\ & \times \begin{pmatrix} 0 & R(\lambda_0, A_p)^{2q} \\ (aA_p^{2q} + bA_p^q + cI)R(\lambda_0, A_p)^{2q} & A_p^q R(\lambda_0, A_p)^{2q} \end{pmatrix}, \end{aligned}$$

$$\mathcal{D}(\tilde{B}_{p,q}) = \big\{(u, v) \in (L^p(\mathbf{R}^n))^2;$$

$$\big[(aA_p^{2q} + bA_p^q + cI)R(\lambda_0, A_p)^{2q}u + A_p^q R(\lambda_0, A_p)^{2q}v\big] \in \mathcal{D}(A_p^{2q})\big\}$$

and $\lambda_0 \in \rho(A_p)$. Remark that $\tilde{B}_{p,q}$ is the closure of $B_{p,q}$. We see that if $u(\cdot)$ is a solution of (2) then $w(t) = (u(t), u'(t))$ is a solution of (22). Conversely, if $w(t) = (u(t), v(t))$ is a solution of (22) then the first coordinate $u(\cdot)$ is not necessarily a solution of (2), but only of

$$\begin{aligned} u''(t) - (\lambda_0 - A_p)^{2q}[A_p^q R(\lambda_0, A_p)^{2q}u'(t) + \\ +(aA_p^{2q} + bA_p^q + cI)R(\lambda_0, A_p)^{2q}u(t)] = 0, \ u(0) = u_0, u'(0) = u_1. \end{aligned} \tag{23}$$

Hence following F. Neubrander (see [10]) we can give

Definition 4.1. *Let $\lambda_0 \in \rho(A_p)$. A function $u \in C^2(\mathbf{R}_+, L^p(\mathbf{R}^n))$ with*

$$[A_p^q R(\lambda_0, A_p)^{2q}u'(t) + (aA_p^{2q} + bA_p^q + cI)R(\lambda_0, A_p)^{2q}u(t)] \in \mathcal{D}(A_p^{2q}),$$

for all $t \geq 0$ which satisfy (23) is called a **mild solution** *of (2).*

Suppose now that $\tilde{B}_{p,q}$ generates an $(N-1)$-t.i.s. on $L^p(\mathbf{R}^n))^2$. Then (22) has a unique solutions $w(t) = (u(t), u'(t))$ for all $(u_0, u_1) \in \mathcal{D}(\tilde{B}_{p,q}^N)$ which contains $\mathcal{D}(A_p^{(N+1)q}) \times \mathcal{D}(A_p^{Nq})$. Hence u is the unique mild solution of (2) and applying Theorem 4.5 (d) and Theorem 3.1 (iii) (see [10]), for suitable constants $M > 0$, $\omega \in \mathbf{R}$ we get

$$||u'(t)||_p \le \max(||u(t)||_p, ||u'(t)||_p) = ||w(t)|| \le Me^{\omega t}||(u_0,u_1)||^{B_{p,q}}_{N-1},$$

$$\begin{aligned} ||u(t)-u_0||_p &\le \max\left(\left\|\int_0^t u(s)ds\right\|_p, \left\|\int_0^t u'(s)ds\right\|_p\right) = \\ &= \left\|\int_0^t w(s)ds\right\| \le Me^{\omega t}||(u_0,u_1)||^{B_{p,q}}_{N-2}. \end{aligned}$$

In particular, if $(u_0,u_1) \in \mathcal{D}(A_p^{(N+1)q}) \times \mathcal{D}(A_p^{Nq}) \subset \mathcal{D}(\tilde{B}^N_{p,q})$ then it can be easily seen that there is a constant such that

$$\begin{aligned} ||u'(t)||_p &\le Me^{\omega t}||(u_0,u_1)||^{B_{p,q}}_{N-1} \le Ce^{\omega t}||u_0||_{Nq} + ||u_1||_{(N-1)q} \\ ||u(t)||_p &\le Me^{\omega t}||(u_0,u_1)||^{B_{p,q}}_{N-2} \le Ce^{\omega t}||u_0||_{(N-1)q} + ||u_1||_{(N-2)q} \end{aligned} \tag{24}$$

Now we can state

Proposition 4.1. *If $\tilde{B}_{p,q}$ generates an $(N-1)$-t.i.s on $(L^p(\mathbf{R}^n))^2$, then (ACP_2) (2) has unique mild solution for (at least) all initial data $(u_0,u_1) \in \mathcal{D}(A_p^{(N+1)q}) \times \mathcal{D}(A_p^{Nq})$, which depend continuously on the initial data as it can be seen in (24).*

In the sequel we want to know when the resolvent set of $\tilde{B}_{p,q}$ is nonempty and then to determine the resolvent of the operator $\tilde{B}_{p,q}$.

First of all remark that $\lambda \in \rho(\tilde{B}_{p,q})$ iff $\zeta^j_k(\lambda) \in \rho(A_p)$, $j=1,2$, $k=0,1,\ldots,q-1$, where $\zeta^j_k(\lambda)$ are the roots of the equations $\zeta^q - f_j(\lambda) = 0$ and

$$f_j(\lambda) = \{-\lambda - b \pm [(1+4a)\lambda^2 + 2b\lambda + b^2 - 4ac]^{1/2}\}/2a.$$

Hence if $\lambda \in \rho(\tilde{B}_{p,q})$ then

$$R(\lambda, \tilde{B}_{p,q}) = \begin{pmatrix} (\lambda - A_p^q)R(\lambda) & R(\lambda) \\ (aA_p^{2q} + bA_p^q + cI)R(\lambda) & \lambda R(\lambda) \end{pmatrix}, \tag{25}$$

where

$$R(\lambda) = -\prod_{j=1}^{2}\prod_{k=0}^{q-1} R(\zeta^j_k(\lambda), A_p)/a = g(\lambda)\prod_{k=0}^{q-1}(R(\zeta^1_k(\lambda), A_p) - R(\zeta^2_k(\lambda), A_p)),$$

if we set

$$g(\lambda) = -\prod_{k=0}^{q-1}(\zeta^2_k(\lambda) - \zeta^1_k(\lambda))^{-1}/a.$$

Let $H_\omega = \{\zeta^j_k(\lambda);\ \lambda \in \mathbf{C},\ \mathrm{Re}\,\lambda > \omega,\ j=1,2;\ k=0,1,\ldots,q-1\}$. Then if $\tilde{B}_{p,q}$ is the generator of an integrated semigroup it is necessary that there is a $\omega > 0$ such that $H_\omega \subset \rho(A_p)$.

Let us suppose that

$$||R(\lambda, A_p)|| \le p(|\lambda|), \quad \text{for all} \quad \lambda \in H_\omega, \tag{26}$$

where p is a polynomial.

Then by (25), (26) it can be seen that there exists a polynomial $\tilde{p}$ such that

$$||R(\lambda, \tilde{B}_{p,q})|| \leq \tilde{p}(|\lambda|), \quad \text{for all} \quad \lambda \in \mathbf{C}, \ \mathrm{Re}\,\lambda > \omega. \tag{27}$$

Combining Corollary 4.9 and Theorem 4.8 in [10] we obtain

Proposition 4.2. *Suppose that there exists $\omega > 0$ such that $R(\lambda, A_p)$ exists and satisfies (26) for every $\lambda \in H_\omega$. Then there exists $N \in \mathbf{N}^*$ such that for all $(u_0, u_1) \in \mathcal{D}(A_p^{(N+1)q}) \times \mathcal{D}(A_p^{Nq})$, (ACP_2) (2) has a unique mild solution satisfying (24).*

Remark 4.1. *In the case $q = 1$ we obtain the results given in* [10] *(section 8).*

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(Viorel Catană) Department of Mathematics I
University Politehnica of Bucharest
Splaiul Independenţei 313, 77206 Bucharest, Romania
E-mail address: catana@mathem.pub.ro