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A topological result for a class of anisotropic difference equations

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Abstract. In the present paper, we establish a new topological existence result derived from the Leray-Schauder degree and show the existence of a nontrivial homoclinic solution for a class of non-homogeneous anisotropic difference equation settled in the variable exponent sequence space $l^{p(k)}(\mathbb{Z})$.

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1. Introduction

We study the following difference equation

$$-\triangle(a(k-1,\triangle u(k-1))) + V(k)|u(k)|^{p(k)-2}u(k) = f(k,u(k)), \ k \in \mathbb{Z} \quad (1)$$

$$\lim_{|k|\to\infty} u(k) = 0 \quad (2)$$

where $\triangle u(k) = u(k+1) - u(k)$ is the forward difference operators; $V(k)$, $k \in \mathbb{Z}$, is a sequence of real numbers and $a(k,\cdot), f(k,\cdot) : \mathbb{Z}\times\mathbb{R} \to \mathbb{R}$ are continuous functions satisfying some assumptions. Since we impose the boundary condition (2), we are looking for homoclinic solutions of problem (1)-(2).

In case $a(k,t) = |t|^{p(k)-2}t$, $(k,t) \in \mathbb{Z}\times\mathbb{R}$, equation (1) becomes the $p(\cdot)$-Laplace difference equation of type

$$-\triangle^2_{p(k-1)}u(k-1) + V(k)|u(k)|^{p(k)-2}u(k) = f(k,u(k)),$$

where $\triangle^2_{p(\cdot)}$ stands for the $p(\cdot)$-Laplace difference operator defined as

$$-\triangle^2_{p(k-1)}u(k-1) = |\triangle u(k)|^{p(k)-2}\triangle u(k) - |\triangle u(k-1)|^{p(k-1)-2}\triangle u(k-1), \ \ k \in \mathbb{Z}.$$

Additionally, when $a(k,t) = |t|^{p(k)-2}t$, $(k,t) \in \mathbb{Z}\times\mathbb{R}$, equation (1) is the discrete counterpart of the following nonlinear differential equation

$$-(|u'|^{p(x)-2}u')' + V(x)|u|^{p(x)-2}u = f(x,u), \ x \in I \subset \mathbb{R}. \quad (3)$$

Equations of the form (3), as well as their multi-dimensional versions, appear in many applications, such as fluid dynamics and nonlinear elasticity, to name a few (see, *e.g.*, [12, 33] and references therein). In the case when $p(x) = 2$, (3) becomes the stationary nonlinear Schrödinger equation (NLS). It has an enormous number of applications, for instance, in nonlinear optics [27] and condensed matter physics [22].

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As in the case of equation (3), equation (1) reduces to the stationary discrete nonlinear Schrödinger equation (DNLS) when $a(k,t) = |t|^{p(k)-2}t$, and $p(k) = 2$. As its continuous counterpart, DNLS has many applications in various areas of physics (see, *e.g.*, [1, 2, 16, 18, 21]). On the other hand, there is a number of rigorous results about this equation. Here we only mention papers [31, 32, 37, 38] in which the existence of solutions satisfying (2) is studied by means of critical point theory and variational methods [3]. There has been an intensive study for the Dirichlet problems of discrete p-Laplacian and $p(k)$-Laplacian equations we refer to the papers [4, 13, 29, 30, 35], where the main tools applied are critical point theory and variational methods. However, to the best of our knowledge [5, 26], where $a(k,t) = |t|^{p(k)-2}t$, and [17] where $V(k) = 1$, are the only papers dealing with problems (1)-(2).
In [26], for example, the authors deal with the difference non-homogeneous equations of type

$$-\triangle^2_{p(k-1)}u(k-1) + V(k)|u(k)|^{q(k)-2}u(k) = f(k,u(k)),\ k \in \mathbb{Z} \tag{4}$$

$$\lim_{|k|\to\infty} u(k) = 0 \tag{5}$$

where $p(\cdot), q(\cdot) : \mathbb{Z} \to (1,\infty)$ and $V(\cdot) : \mathbb{Z} \to \mathbb{R}$ are T-periodic functions; $f(k,t) : \mathbb{Z}\times\mathbb{R} \to \mathbb{R}$ is a function such that it is continuous in t and T-periodic in k. Applying variational methods, i.e. the mountain-pass lemma, they obtain nontrivial homoclinic solutions for problem (4)-(5).

The main result of the present paper concerns the existence of a nontrivial homoclinic solution of problem (1)-(2). To do this, we employ a topological method, i.e., Theorem 4.1, derived from the Leray-Schauder degree, see, e.g., the papers [6, 9, 14, 23, 25, 28] and monographs [7, 10, 15, 36], in which important topological results were obtained. To the author best knowledge, the results of the present paper are new and original.

2. Preliminaries

For each $p(k) : \mathbb{Z} \to (1,\infty)$ such that

$$p^- := \inf_{k\in\mathbb{Z}} p(k) \quad \text{and} \quad p^+ := \sup_{k\in\mathbb{Z}} p(k),$$

let us introduce the following variable exponent sequence space

$$l^{p(k)}(\mathbb{Z}) = \left\{ u = (u(k)) : u : \mathbb{Z} \to \mathbb{R},\ \sum_{k\in\mathbb{Z}} |u|^{p(k)} < \infty \right\}.$$

Equipped with the Luxembourg norm

$$\|u\|_{l^{p(k)}(\mathbb{Z})} = \|u\|_{p(k)} = \inf\left\{ \lambda > 0 : \sum_{k\in\mathbb{Z}} \left|\frac{u}{\lambda}\right|^{p(k)} \leq 1 \right\},$$

$(l^{p(k)}(\mathbb{Z}), \|\cdot\|_{p(k)})$ is a separable Banach space which is reflexive provided $p^+ < +\infty$ ([12]).

The functional $\rho_{p(k)}: l^{p(k)}(\mathbb{Z}) \to \mathbb{R}$ defined by $\rho_{p(k)}(u) := \sum_{k\in\mathbb{Z}} |u(k)|^{p(k)}$ is a convex modular due to the fact that $\varphi(k,t) = |t|^{p(k)}$, $(k,t) \in \mathbb{Z}\times\mathbb{R}$, is a convex function.

Proposition 2.1. *([12, 17, 24]) If $u \in l^{p(k)}(\mathbb{Z})$ and $p^+ < +\infty$, then the following statements hold:*

(i) $\|u\|_{p(k)} < 1 (= 1, > 1) \Longleftrightarrow \rho_{p(k)}(u) < 1 (= 1, > 1)$;
(ii) $\|u\|_{p(k)} = a \Longleftrightarrow \rho_{p(k)}(\frac{u}{a}) = 1$ $(u \neq 0)$;
(iii) $\|u\|_{p(k)} \leq 1 \Longrightarrow \|u\|_{p(k)}^{p^+} \leq \rho_{p(k)}(u) \leq \|u\|_{p(k)}^{p^-}$;
(iv) $\|u\|_{p(k)} \geq 1 \Longrightarrow \|u\|_{p(k)}^{p^-} \leq \rho_{p(k)}(u) \leq \|u\|_{p(k)}^{p^+}$;
(v) $\rho_{p(k)}(u_n - u) \to 0$ $(\to \infty) \Longleftrightarrow \|u_n - u\|_{p(k)} \to 0$ $(\to \infty)$;
(vi) $u_n \to u$ *in* $l^{p(k)} \Longleftrightarrow \lim_{n\to\infty} u_n(k) = u(k)$ *for all* $k \in \mathbb{Z}$ *and* $\lim_{n\to\infty} \rho_{p(k)}(u_n) = \rho_{p(k)}(u)$.

Furthermore, we introduce the space in which we study problem (1)-(2)

$$E = \{u \in l^{p(k)}(\mathbb{Z}) : (V^{1/p(k)}u) \in l^{p(k)}(\mathbb{Z})\} \tag{6}$$

endowed with the norm

$$\|u\|_E = \inf\left\{\eta > 0 : \sum_{n\in\mathbb{Z}} V(k)\left|\frac{u}{\eta}\right|^{p(k)} \leq 1\right\} = \|V^{1/p(k)}u\|_E.$$

Here $V^{1/p(k)}u$ stands for the sequence with elements $V(k)^{1/p(k)}u(k)$. Moreover, since E is isomorphic to $l^{p(k)}(\mathbb{Z})$ via the operator of multiplication by $V(k)$, it is a reflexive Banach space.

Let us define the functional $\rho_E : E \to \mathbb{R}$ by

$$\rho_E(u) := \sum_{k\in\mathbb{Z}} V(k)|u|^{p(k)}.$$

Then $\rho_E(\cdot)$ and $\|\cdot\|_E$ possess properties similar to those listed in Proposition 2.1.

Proposition 2.2 (Young inequality)**.** *Let $p(k) > 1$ for all $k \in \mathbb{Z}$ and $p^{-1}(k) + (p'(k))^{-1} = 1$ for all $k \in \mathbb{Z}$. Then for all $a, b \in \mathbb{R}$ we have*

$$|a||b| \leq p(k)^{-1}|a|^{p(k)} + p'(k)^{-1}|b|^{p'(k)}.$$

Proposition 2.3 (Discrete Hölder inequality)**.** *Given the functions $r_1(k), r_2(k) : \mathbb{Z} \to (1,\infty)$ define $s(k) : \mathbb{Z} \to (1,\infty)$ by*

$$\frac{1}{s(k)} = \frac{1}{r_1(k)} + \frac{1}{r_2(k)}.$$

Then there exists a constant $K > 0$ such that for all $u \in l^{r_1(k)}(\mathbb{Z})$ and $v \in l^{r_2(k)}(\mathbb{Z})$, $uv \in l^{s(k)}(\mathbb{Z})$ and

$$\sum_{k\in\mathbb{Z}} |uv|^{s(k)} \leq K\|u\|_{r_1(k)}\|v\|_{r_2(k)}.$$

Proof. If $\|u\|_{r_1(k)} = 0$ or $\|v\|_{r_2(k)} = 0$, then $uv \equiv 0$, hence the proof is clear. Therefore, we may assume that these quantities are positive. First, we consider the case $s(k) = 1$.

Let us choose $a = \frac{|u|}{\|u\|_{r_1(k)}}$ and $b = \frac{|v|}{\|v\|_{r_2(k)}}$ in Young inequality. Then, by Proposition 2.1, it leads

$$\begin{aligned}\sum_{k\in\mathbb{Z}} \frac{|uv|}{\|u\|_{r_1(k)}\|v\|_{r_2(k)}} &\leq \sum_{k\in\mathbb{Z}} \frac{1}{r_1(k)} \left|\frac{u}{\|u\|_{r_1(k)}}\right|^{r_1(k)} + \sum_{k\in\mathbb{Z}} \frac{1}{r_2(k)} \left|\frac{v}{\|v\|_{r_2(k)}}\right|^{r_2(k)} \\ &\leq \frac{1}{r_1^-}\rho_{r_1(k)}\left(\frac{u}{\|u\|_{r_1(k)}}\right) + \frac{1}{r_2^-}\rho_{r_2(k)}\left(\frac{v}{\|v\|_{r_2(k)}}\right)\end{aligned}$$

or

$$\sum_{k\in\mathbb{Z}} |uv| \leq \left(\frac{1}{r_1^-} + \frac{1}{r_2^-}\right)\|u\|_{r_1(k)}\|v\|_{r_2(k)}, \quad \frac{1}{r_1^-} + \frac{1}{r_2^-} = K. \tag{7}$$

Let us proceed with the case $s(k) \neq 1$. The rest of the proof is inspired by the proof of Corollary 2.28 in [11]. For the same reason above, we may assume that $\|u\|_{r_1(k)}$ and $\|v\|_{r_2(k)}$ are positive; further, by homogeneity we may assume that $\|u\|_{r_1(k)} \leq 1$ and $\|v\|_{r_2(k)} \leq 1$. Then by Proposition 2.1, $\rho_{r_1(k)}(u) \leq 1$ and $\rho_{r_2(k)}(v) \leq 1$. Let us consider the relations $u \in l^{r_1(k)}(\mathbb{Z}) \Rightarrow |u|^{s(k)} \in l^{r_1(k)/s(k)}(\mathbb{Z})$ and $v \in l^{r_2(k)}(\mathbb{Z}) \Rightarrow |v|^{s(k)} \in l^{r_2(k)/s(k)}(\mathbb{Z})$. We may suppose that

$$\||u|^{s(k)}\|_{r_1(k)/s(k)} \leq 1 \text{ and } \||v|^{s(k)}\|_{r_2(k)/s(k)} \leq 1. \tag{8}$$

Indeed, since $\rho_{r_1(k)}(u) \leq 1$, it reads

$$\rho_{r_1(k)/s(k)}(|u|^{s(k)}) = \sum_{k\in\mathbb{Z}} ||u|^{s(k)}|^{r_1(k)/s(k)} = \sum_{k\in\mathbb{Z}} |u|^{r_1(k)} \leq 1,$$

which means that $\||u|^{s(k)}\|_{r_1(k)/s(k)} \leq 1$. Similarly, $\||v|^{s(k)}\|_{r_2(k)/s(k)} \leq 1$ holds as well. Then, since $\frac{1}{r_1(k)/s(k)} + \frac{1}{r_2(k)/s(k)} = 1$, we can use (7) for functions $|u|^{s(k)} \in l^{r_1(k)/s(k)}(\mathbb{Z})$ and $|v|^{s(k)} \in l^{r_2(k)/s(k)}(\mathbb{Z})$, that is,

$$\begin{aligned}\rho_{s(k)}(uv) &= \sum_{k\in\mathbb{Z}} |u|^{s(k)}|v|^{s(k)} \\ &\leq K(s, r_1, r_2)\||u|^{s(k)}\|_{r_1(k)/s(k)}\||v|^{s(k)}\|_{r_2(k)/s(k)} \\ &\leq K(s, r_1, r_2).\end{aligned}$$

Therefore, we have $uv \in l^{s(k)}(\mathbb{Z})$ and

$$\|uv\|_{s(k)} \leq K(s, r_1, r_2) = K(s, r_1, r_2)\|u\|_{r_1(k)}\|v\|_{r_2(k)},$$

or

$$\sum_{k\in\mathbb{Z}} |uv|^{s(k)} \leq K(s, r_1, r_2)\|u\|_{r_1(k)}\|v\|_{r_2(k)}. \tag{9}$$

□

Corollary 2.4. *Assume that the functions $r_i(k) : \mathbb{Z} \to (1, \infty)$ satisfy*

$$\sum_{i=1}^{m} \frac{1}{r_i(k)} = 1.$$

Then there exists a constant $K = K(r_i) > 0$ such that for all $u_i \in l^{r_i(k)}(\mathbb{Z})$, $1 \le i \le m$

$$\sum_{i=1}^{m} |u_1 u_2 \cdots u_m| \le K \|u_1\|_{r_1(k)} \|u_2\|_{r_2(k)} \cdots \|u_m\|_{r_m(k)}.$$

Proof. By using Proposition 2.3 along with the induction method the result follows at once. □

Proposition 2.5. *Assume that $1 \le r_1(k) \le r_2(k) \le +\infty$ for all $k \in \mathbb{Z}$. Then, the embedding $l^{r_1(k)}(\mathbb{Z}) \hookrightarrow l^{r_2(k)}(\mathbb{Z})$ is continuous and*

$$\|u\|_{r_2(k)} \le C\|u\|_{r_1(k)}, \quad C > 0.$$

Proof. Let $u \in l^{r_1(k)}(\mathbb{Z})$. Then $\sum_{k\in\mathbb{Z}} |u|^{r_1(k)} < \infty$ which means that there exists $N \in \mathbb{N}$ such that $|u(k)|^{r_1(k)} \le 1$, that is $|u(k)| \le 1$, whenever $k > |N|$. Therefore, considering that $1 \le r_1(k) \le r_2(k)$ for all $k \in \mathbb{Z}$, it follows

$$|u(k)|^{r_2(k)} \le |u(k)|^{r_1(k)} \text{ whenever } |k| > N,$$

and hence $\sum_{k\in\mathbb{Z}} |u|^{r_2(k)} < \infty$. So we have $l^{r_1(k)}(\mathbb{Z}) \subset l^{r_2(k)}(\mathbb{Z})$. Let $u \in l^{r_1(k)}(\mathbb{Z})$ such that $\|u\|_{r_1(k)} < 1$. Thus, by Proposition 2.1, $\sum_{k\in\mathbb{Z}} |u|^{r_1(k)} < 1$. Therefore, $|u|^{r_1(k)} < 1$ for all $k \in \mathbb{Z}$ and as a result $|u(k)|^{r_2(k)} \le |u(k)|^{r_1(k)}$ for all $k \in \mathbb{Z}$. Then

$$\sum_{k\in\mathbb{Z}} |u(k)|^{r_2(k)} \le \sum_{k\in\mathbb{Z}} |u(k)|^{r_1(k)}.$$

Using by Proposition 2.1 once more, it follows

$$\|u\|_{r_2(k)} \le C\|u\|_{r_1(k)},$$

where $C = C(r_2^+)$ is a positive real number. Therefore, the embedding operator is bounded, that is, $l^{r_1(k)}(\mathbb{Z}) \hookrightarrow l^{r_2(k)}(\mathbb{Z})$ continuously. □

Proposition 2.6. *If (V1) holds the embedding $E \hookrightarrow l^{p(k)}(\mathbb{Z})$ is compact.*

Proof. Using (V1), Proposition 2.1, and the same arguments applied in the proof of Proposition 2.5, we obtain that $E \subset l^{p(k)}(\mathbb{Z})$, and for any $u \in E$ with $\|u\|_E < 1$, we have

$$\|u\|_{p(k)} \le V_0^{-1/p^+} \|u\|_E,$$

which means that the embedding operator is bounded, that is, $E \hookrightarrow l^{p(k)}(\mathbb{Z})$ continuously.

Now, consider a sequence $(u_n) \subset E$ such that $u_n \rightharpoonup 0$ in E. Then, there exists a constant $M > 0$ such that $\rho_E(u_n) \le M$ for all n, and $u_n(k) \to 0$ for all $k \in \mathbb{Z}$ as $n \to +\infty$. Let $\varepsilon > 0$. By assumption (V1) there exists $N \in \mathbb{N}$ such that $V(k) \ge \frac{M}{\varepsilon}$ whenever $|k| > N$. Therefore,

$$\sum_{n\in\mathbb{Z}} |u_n|^{p(k)} \le \sum_{|k|\le N} |u_n|^{p(k)} + \frac{\varepsilon}{M} \sum_{|k|>N} V(k)|u_n|^{p(k)} \le \sum_{|k|\le N} |u_n|^{p(k)} + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary and the finite sum in the right-hand side converges to 0, Proposition 2.1 (v) allows us to have $u_n \to 0$ in $l^{p(k)}(\mathbb{Z})$ as $n \to +\infty$. □

Proposition 2.7. *Assume that X is a compact subset of $l^{p(k)}(\mathbb{Z})$. Then, given any $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that for all $u \in X$ the following convergence of infinite sum holds*

$$\sum_{|k|>N} |u|^{p(k)} < \varepsilon.$$

Proof. Assume by contradiction that there exists a sequence $(u_n) \subseteq X$ and a real number $\varepsilon > 0$ such that

$$\sum_{|k|>n} |u_n|^{p(k)} > \varepsilon, \quad \forall n \in \mathbb{N}.$$

Considering that X is a compact subset of $l^{p(k)}(\mathbb{Z})$, there exists a subsequence, not relabelled, (u_n) such that $u_n \to u$ in $l^{p(k)}(\mathbb{Z})$ for some $u \in X$. Thus, there is a $N \in \mathbb{N}$ satisfying

$$\sum_{|k|>N} |u|^{p(k)} < \frac{\varepsilon}{2}.$$

Since $u_n \to u$ in $l^{p(k)}(\mathbb{Z})$, $\|u_n - u\|_{p(k)} < \frac{\varepsilon}{2}$ whenever $n \geq N$. Hence, by Proposition 2.1, we have $\sum_{|k|>n} |u_n - u|^{p(k)} < \frac{\varepsilon}{2}$ for $n \geq N$. Therefore, we obtain

$$\varepsilon < \sum_{|k|>n} |u_n|^{p(k)} \leq \sum_{|k|>n} |u_n - u|^{p(k)} + \sum_{|k|>n} |u|^{p(k)} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon, \quad n \geq N$$

which is impossible. □

We suppose that $a(k,t) : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ is a continuous function in t and there exists a function $A : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ such that $a(k,t) = \frac{\partial}{\partial t} A(k,t)$ for all $k \in \mathbb{Z}$. Through this paper we assume that

$$p(k), q(k) : \mathbb{Z} \to [2, \infty) \text{ such that } q^+ < p^- \leq p^+ < \infty.$$

We accept the following hypothesis for the functions a, A, V and f:
(**V1**) The potential sequence $V(k)$ is such that $V(k) \geq V_0 > 0$ for all $k \in \mathbb{Z}$, and $V(k) \to +\infty$ as $|k| \to \infty$.

(**a0**) $A(k, 0) = 0$, for all $k \in \mathbb{Z}$.

(**a1**)$(a(k,t) - a(k,s))(t - s) \geq 0$ for all $t, s \in \mathbb{R}$ and $k \in \mathbb{Z}$.

(**a2**) The following inequality holds

$$|a(k,t)| \leq c_1(a_0(k) + |t|^{p(k)-1})$$

for all $t \in \mathbb{R}$ and $k \in \mathbb{Z}$, where c_1 is a positive real number and $a_0 \in l^{p'(k)}(\mathbb{Z})$ is a nonnegative function.

(**a3**) The following inequality holds

$$a(k,t)t \geq c_2|t|^{p(k)}$$

for all $t \in \mathbb{R}$ and $k \in \mathbb{Z}$, where c_2 is a positive real number.

(**f1**) $f(k,t) : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ is a continuous function in t; there are functions $\theta, \beta : \mathbb{Z} \to (0, \infty)$ with $\theta \in l^{p'(k)}(\mathbb{Z})$ and $\beta \in l^{r(k)}(\mathbb{Z})$, $r(k) = \frac{p(k)}{p(k)-q(k)}$, and there exists a real number $\lambda > 0$ such that

$$\limsup_{|t|\to+\infty} \frac{|f(k,t)|}{\theta(k) + \beta(k)|t|^{q(k)-1}} \leq \lambda \text{ uniformly for } k \in \mathbb{Z}.$$

(f2)

$$\lim_{t\to 0}\frac{|f(k,t)|}{|t|^{p(k)-1}}=0 \text{ uniformly for } k\in\mathbb{Z}.$$

Definition 2.1. A function $u\in E$ is a homoclinic solution for problem (1)-(2) if

$$\sum_{k\in\mathbb{Z}}a(k-1,\triangle u(k-1))\triangle v(k-1)+\sum_{k\in\mathbb{Z}}V(k)|u|^{p(k)-2}uv-\sum_{k\in\mathbb{Z}}f(k,u)v=0 \qquad (10)$$

for all $v\in E$, and $\lim_{|k|\to\infty}u(k)=0$.

Let us define the functionals $T,S:E\to\mathbb{R}$, as follows:

$$T(u)=\sum_{k\in\mathbb{Z}}A(k-1,\triangle u(k-1))+\sum_{k\in\mathbb{Z}}\frac{V(k)}{p(k)}|u|^{p(k)}$$

and

$$S(u)=\sum_{n\in\mathbb{Z}}F(k,u),$$

where $F(k,t)=\int_0^t f(k,s)ds$.

Proposition 2.8. *Under assumptions (V1), (a2) and (f2) the functionals T and S are well defined on E and are of the class C^1 with the derivatives $T',S':E\to E^*$ given by*

$$\langle T'(u),v\rangle=\sum_{k\in\mathbb{Z}}a(k-1,\triangle u(k-1))\triangle v(k-1)+\sum_{k\in\mathbb{Z}}V(k)|u|^{p(k)-2}uv, \qquad (11)$$
$$\langle S'(u),v\rangle=\sum_{k\in\mathbb{Z}}f(k,u)v,$$

for all $u,v\in E$, respectively, where $\langle\cdot,\cdot\rangle=\langle\cdot,\cdot\rangle_{E^*\times E}$ stands for the duality pairing.

Proof. By (f2) there exists $\delta>0$ such that

$$|f(k,t)|\le|t|^{p(k)-1}, \text{ for all } k\in\mathbb{Z} \text{ and } |t|\le\delta. \qquad (12)$$

Therefore, $|F(k,t)|\le\frac{1}{p(k)}|t|^{p(k)}$ for all $k\in\mathbb{Z}$ and $|t|\le\delta$. Moreover, for any $u\in E$ there exist $N\in\mathbb{N}$ such that $|u(k)|\le\delta$ for all $|k|>N$. Using this information, we have

$$|S(u)| \le \sum_{k\in\mathbb{Z}}|F(k,u)|\le\sum_{|k|\le N}|F(k,u)|+\frac{1}{p^-}\sum_{|k|>N}|u|^{p(k)}<\infty. \qquad (13)$$

On the other hand, from the definition of A, we obtain

$$A(k,\xi)=\int_0^1\frac{d}{dt}A(k,t\xi)dt=\int_0^1 a(k,t\xi)\xi dt.$$

Then, using (a2) and above information we have

$$|A(k,\xi)|\le\int_0^1|a(k,\xi)||\xi|dt\le c_1(a_0(k)|\xi|+|\xi|^{p(k)}). \qquad (14)$$

Therefore, using the inequality

$$\begin{aligned}|\triangle u(k-1)|^{p(k-1)} &= |u(k)-u(k-1)|^{p(k-1)}\\ &\le 2^{p^+-1}(|u(k)|^{p(k-1)}+|u(k-1)|^{p(k-1)}),\forall|k|\le N\end{aligned}$$

or

$$\begin{aligned}\sum_{k\in\mathbb{Z}}|\triangle u(k-1)|^{p(k-1)} &\leq 2^{p^+-1}\left(\sum_{k\in\mathbb{Z}}|u(k)|^{p(k)}+\sum_{k\in\mathbb{Z}}|u(k)|^{p(k)}\right) \\ &\leq 2^{p^+}\sum_{k\in\mathbb{Z}}|u(k)|^{p(k)}\leq 2^{p^+}\|u\|^{\gamma}_{p(k)}, \quad \gamma\in[p^-,p^+]\end{aligned} \tag{15}$$

along with the Young inequality and the embedding $E\hookrightarrow l^{p(k)}(\mathbb{Z})$, it leads to

$$\begin{aligned}|T(u)| &\leq \sum_{k\in\mathbb{Z}}|A(k-1,\triangle u(k-1))|+\sum_{k\in\mathbb{Z}}\frac{V(k)}{p(k)}|u|^{p(k)} \\ &\leq \sum_{k\in\mathbb{Z}}c_1(a_0(k-1)|\triangle u(k-1)|+|\triangle u(k-1)|^{p(k-1)})+\|u\|^{\gamma}_{E} \\ &\leq c_1\sum_{k\in\mathbb{Z}}\frac{p(k-1)-1}{p(k-1)}|a_0|^{\frac{p(k-1)}{p(k-1)-1}}+c_1\sum_{k\in\mathbb{Z}}\frac{1}{p(k-1)}|\triangle u(k-1)|^{p(k-1)} \\ &+ \sum_{k\in\mathbb{Z}}|\triangle u(k-1)|^{p(k-1)}+\|u\|^{\gamma}_{E} \\ &\leq c_2\|a_0\|_{p'(k)}+(1+c_3 2^{p^+})\|u\|^{\gamma}_{E}<\infty.\end{aligned}$$

Therefore, T is well defined on E.
For fixed $u,v\in E$ and $t\in(0,1)$, using (14)-(15) and the Young inequality several times, we have

$$\begin{aligned}\sum_{k\in\mathbb{Z}}|A(k-1,\triangle u(k-1)+t\triangle v(k-1))| &\leq c_1\sum_{k\in\mathbb{Z}}a_0(k-1)|\triangle u(k-1)+t\triangle v(k-1))| \\ &\quad +c_1\sum_{k\in\mathbb{Z}}|\triangle u(k-1)+t\triangle v(k-1))|^{p(k-1)} \\ &\leq 2^{p^+}C(\|u\|^{\gamma}_{E},\|v\|^{\gamma}_{E})<\infty\end{aligned}$$

and

$$\begin{aligned}\sum_{k\in\mathbb{Z}}|A(k-1,\triangle u(k-1)| &\leq c_1\sum_{k\in\mathbb{Z}}(a_0(k-1)|\triangle u(k-1)|+|\triangle u(k-1)|^{p(k-1)}) \\ &\leq c_2 2^{p^+}\|u\|^{\gamma}_{E}<\infty.\end{aligned}$$

Therefore,

$$\begin{aligned}&\lim_{t\to 0^+}\sum_{k\in\mathbb{Z}}\frac{A(k-1,\triangle u(k-1)+t\triangle v(k-1))-A(k-1,\triangle u(k-1))}{t} \\ &\quad =\sum_{k\in\mathbb{Z}}\lim_{t\to 0^+}\frac{A(k-1,\triangle u(k-1)+t\triangle v(k-1))-A(k-1,\triangle u(k-1))}{t} \\ &\quad =\sum_{k\in\mathbb{Z}}a(k-1,\triangle u(k-1))\triangle v(k-1).\end{aligned}$$

By an analogous way, we can obtain

$$\lim_{t\to 0^+}\sum_{k\in\mathbb{Z}}\frac{V(k)}{p(k)}\frac{|u(k)+t\triangle v(k))|^{p(k)}-|u(k)|^{p(k)}}{t}$$
$$=\sum_{k\in\mathbb{Z}}\lim_{t\to 0^+}\frac{V(k)}{p(k)}\frac{|u(k)+t\triangle v(k))|^{p(k)}-|u(k)|^{p(k)}}{t}=\sum_{k\in\mathbb{Z}}V(k)|u|^{p(k)-2}uv.$$

Thus,

$$\lim_{t\to 0^+}\sum_{k\in\mathbb{Z}}\frac{T(u+tv)-T(u)}{t}=\langle T'(u),v\rangle$$
$$=\sum_{k\in\mathbb{Z}}a(k-1,\triangle u(k-1))\triangle v(k-1)+\sum_{k\in\mathbb{Z}}V(k)|u|^{p(k)-2}uv. \tag{16}$$

Since the right-hand side of the above expression, as a function of v, is a continuous linear functional on E, (16) is the Gateaux differential of T.

Next, we proceed with S. Let us choose fixed functions $u,v\in E$ such that (12) holds and $\max\{|u(k)|,|v(k)|\}\le\frac{\delta}{2}$ for all $k\in\mathbb{Z}$, $|k|>N$. Let t be a parameter with $0<t<1$. Then, given any $\varepsilon>0$, it holds

$$\sum_{|k|\le N}\left|\frac{F(k,u+tv)-F(k,u)}{t}-f(k,u)v\right|\le\frac{\varepsilon}{4p^-p^+}.$$

On the other hand, by the mean value theorem, there exists $\sigma\in(0,1)$ such that

$$\frac{F(k,u+tv)-F(k,u)}{t}=f(k,u+t\sigma v)v,\quad \forall|k|>N.$$

Let us define a function $w\in l^{p(k)}(\mathbb{Z})$ such that $w(k)=0$ for all $|k|\le N$ and $w(k)=u(k)+t\sigma v(k)$ for all $|k|>N$. Then $|w(k)|\le\delta$ for all $k\in\mathbb{Z}$. Moreover, given any $\varepsilon>0$, there exists $N\in\mathbb{N}$ such that

$$\sum_{|k|>N}|w|^{p(k)}<\frac{\varepsilon}{4(p^+)^2},\ \sum_{|k|>N}|u|^{p(k)}<\frac{\varepsilon}{4(p^+)^2},\ \sum_{|k|>N}|v|^{p(k)}<\frac{\varepsilon}{4(p^+)^2}. \tag{17}$$

Therefore, by (f2) and (12), the Young inequality, it follows

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}}\left|\frac{F(k,u+tv)-F(k,u)}{t}-f(k,u)v\right|\\
&\le\frac{\varepsilon}{4p^-p^+}+\sum_{|k|>N}|f(k,u+t\sigma v)v|+\sum_{|k|>N}|f(k,u)v|\\
&\le\frac{\varepsilon}{4p^-p^+}+\sum_{|k|>N}|w|^{p(k)-1}|v|+\sum_{|k|>N}|u|^{p(k)-1}|v|\\
&\le\frac{\varepsilon}{4p^-p^+}+\frac{(p^+-1)}{p^-}\left(\sum_{|k|>N}(|w|^{p(k)}+|u|^{p(k)})\right)+\frac{1}{p^-}\sum_{|k|>N}|v|^{p(k)}\\
&\le\frac{\varepsilon}{4p^-p^+}+\frac{p^+}{p^-}\left(\frac{\varepsilon}{4(p^+)^2}+\frac{\varepsilon}{4(p^+)^2}\right)+\frac{1}{p^-}\frac{\varepsilon}{4(p^+)^2}\\
&\le\frac{\varepsilon}{4p^-p^+}+\frac{2\varepsilon}{4p^-p^+}+\frac{\varepsilon}{4p^-p^+}<\varepsilon.
\end{aligned}$$

Hence, we have

$$\begin{aligned}\lim_{t\to 0^+}\sum_{k\in\mathbb{Z}}\frac{F(k,u+tv)-F(k,u)}{t} &= \sum_{k\in\mathbb{Z}}\lim_{t\to 0^+}\frac{F(k,u+tv)-F(k,u)}{t} \\ &= \langle S'(u),v\rangle=\sum_{k\in\mathbb{Z}}f(k,u)v\end{aligned} \tag{18}$$

and hence (18) is the Gateaux differential of S.
In conclusion, the operator $I : E \to E^*$ defined by

$$\langle I(u),v\rangle=\langle T'(u),v\rangle-\langle S'(u),v\rangle,\quad \text{for all } u,v\in E \tag{19}$$

is Gateaux differentiable on E.
Now, we proceed for the continuity of $I : E \to E^*$. To this end, we assume, for a sequence $(u_n) \subset E$, that $u_n \to u \in E$, and show that, for all $v \in E$ with $\|v\|_E \leq 1$, and given any $\varepsilon > 0$, whenever $n > N_0 \in \mathbb{N}$, it holds

$$\begin{aligned}|\langle I(u_n)-I(u),v\rangle| &\leq \sum_{k\in\mathbb{Z}}|a(k-1,\triangle u_n(k-1))-a(k-1,\triangle u(k-1))||\triangle v(k-1)| \\ &\quad+\sum_{k\in\mathbb{Z}}V(k)||u_n|^{p(k)-2}u_n-|u|^{p(k)-2}u||v|+\sum_{k\in\mathbb{Z}}|f(k,u_n)-f(k,u)||v| \\ &< \varepsilon.\end{aligned} \tag{20}$$

Since $u_n \to u \in E$, by Proposition (2.7) and (15), given any $\varepsilon > 0$, there is a $N \in \mathbb{N}$ such that

$$\sum_{|k|>N}|\triangle u_n(k-1)|^{p(k)}<\frac{\varepsilon}{7(p^+)^2}\ \forall n\in\mathbb{N},\ \sum_{|k|>N}|\triangle u(k-1)|^{p(k)}<\frac{\varepsilon}{7(p^+)^2}$$

and

$$\sum_{|k|>N}|\triangle v(k-1)|^{p(k)}<\frac{\varepsilon}{7(p^+)^2},\ \sum_{|k|>N}|a_0(k-1)|^{p'(k)}<\frac{\varepsilon}{7(p^+)^2}.$$

Moreover, by the continuity of the finite sum when $n > \max\{N, N_0\}$, it satisfies

$$\sum_{|k|\leq N}|a(k-1,\triangle u_n(k-1))-a(k-1,\triangle u(k-1))||\triangle v(k-1)|<\frac{\varepsilon}{7p^-p^+}.$$

Therefore, using (a2), the Young inequality and the above information lead to

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}}|a(k-1,\triangle u_n(k-1))-a(k-1,\triangle u(k-1))||\triangle v(k-1)| \leq \frac{\varepsilon}{7p^-p^+}\\
&\quad + 2c_1\sum_{|k|>N} a_0(k-1)|\triangle v(k-1)| + c_1\sum_{|k|>N}|\triangle v(k-1)||\triangle u_n(k-1)|^{p(k-1)-1}\\
&\quad + c_1\sum_{|k|>N}|\triangle v(k-1)||\triangle u(k-1)|^{p(k-1)-1}\\
&\leq \frac{\varepsilon}{7p^-p^+} + \frac{(p^+-1)}{p^-}\Big(\sum_{|k|>N}(|\triangle u_n(k-1)|^{p(k)} + |\triangle u(k-1)|^{p(k)}\\
&\quad + |a_0(k-1)|^{p'(k)})\Big) + \frac{3}{p^-}\sum_{|k|>N}|\triangle v(k-1)|^{p(k)}\\
&\leq \frac{\varepsilon}{7p^-p^+} + \frac{3\varepsilon}{7p^-p^+} + \frac{3\varepsilon}{7p^-(p^+)^2} \leq \frac{7\varepsilon}{7p^-p^+} < \frac{\varepsilon}{3}. \qquad (21)
\end{aligned}$$

Now, we mention the following inequality given in [8]: for $1 < m < \infty$ there exist constants $C_m > 0$ such that

$$||\xi|^{m-2}\xi - |\zeta|^{m-2}\zeta| \leq C_m|\xi-\zeta|(|\xi|+|\zeta|)^{m-2}, \ \ \forall \xi,\zeta\in\mathbb{R}.$$

Since $u_n \to u \in E$, (u_n) is bounded and $u_n(k) \to u(k)$ as $n\to\infty$, for all $k\in\mathbb{Z}$. Then considering the inequality given above, it leads

$$\sum_{k\in\mathbb{Z}}V(k)||u_n|^{p(k)-2}u_n - |u|^{p(k)-2}u||v| \leq C_p\sum_{k\in\mathbb{Z}}V(k)|u_n-u|(|u_n|+|u|)^{p(k)-2}|v| \searrow 0$$

and hence

$$\sum_{k\in\mathbb{Z}}V(k)||u_n|^{p(k)-2}u_n - |u|^{p(k)-2}u||v| < \frac{\varepsilon}{3}, \quad \forall n > N_0. \qquad (22)$$

On the other hand, since $u_n \to u \in E$, by the compact embedding $E \hookrightarrow l^{p(k)}(\mathbb{Z})$, there exists $\omega \in l^{p(k)}(\mathbb{Z})$ such that $|u_n(k)| \leq \omega(k)$ for all $k\in\mathbb{Z}$ and for all $n\in\mathbb{N}$. Moreover, since $u_n, u\in E$ there exist $\delta>0$, $N\in\mathbb{N}$ such that $|u_n(k)|\leq\delta$ for all $n\in\mathbb{N}$, $|k|>N$ and $|u(k)|\leq\delta$ for all $|k|>N$. Additionally, by (f2) we have $|f(k,t)|\leq|t|^{p(k)-1}$ for all $k\in\mathbb{Z}$ and $|t|\leq\delta$. Therefore, considering this information along with (17) and the Young inequality, we have

$$\begin{aligned}
|\langle S'(u_n)-S'(u),v\rangle| &\leq \sum_{|k|\leq N}|f(k,u_n)-f(k,u)|\,|v| + \sum_{|k|>N}|f(k,u_n)-f(k,u)|\,|v|\\
&\leq \frac{\varepsilon}{4p^-p^+} + \sum_{|k|>N}|u_n|^{p(k)-1}|v| + \sum_{|k|>N}|u|^{p(k)-1}|v|\\
&\leq \frac{\varepsilon}{4p^-p^+} + \frac{(p^+-1)}{p^-}\left(\sum_{|k|>N}(|\omega|^{p(k)}+|u|^{p(k)})\right) + \frac{1}{p^-}\sum_{|k|>N}|v|^{p(k)}\\
&\leq \frac{\varepsilon}{4p^-p^+} + \frac{p^+}{p^-}\left(\frac{\varepsilon}{4(p^+)^2}+\frac{\varepsilon}{4(p^+)^2}\right) + \frac{1}{p^-}\frac{\varepsilon}{4(p^+)^2}\\
&\leq \frac{4\varepsilon}{4p^-p^+} < \frac{\varepsilon}{3}, \quad \forall n > \max\{N, N_0\}, \qquad (23)
\end{aligned}$$

since by the continuity of the finite sum $\sum_{|k|\leq N} |f(k,u_n)-f(k,u)|\,|v| < \frac{\varepsilon}{4p^-p^+}$ for all $n > \max\{N, N_0\}$.
Overall, combining (21), (22) and (23), we obtain (20), that is, $I : E \to E^*$ is continuous. □

3. The auxiliary results

Proposition 3.1.
(i) *T' is a strictly monotone operator;*
(ii) *T' is of type (S_+), that is, if $u_n \rightharpoonup u$ in E and* $\limsup_{n\to\infty}\langle T'(u_n)-T'(u), u_n - u\rangle \leq 0$ *then $u_n \to u$ in E;*
(iii) *T' is a homeomorphism;*
(iv) *S' is compact.*

Proof. (i) We make use of the well-known inequality (see [34])

$$\langle |x|^{r-2}x - |y|^{r-2}y, x-y\rangle \geq C_r|x-y|^r \text{ if } r\geq 2,\ \forall x,y\in\mathbb{R}. \tag{24}$$

Thus, for all $u, v \in E$ such that $u \neq v$, along with (V1),(a1), we obtain

$$\begin{aligned}
&\langle T'(u)-T'(v), u-v\rangle\\
\geq\ &\sum_{k\in\mathbb{Z}} \left(a(k-1,\triangle u(k-1)) - a(k,\triangle v(k-1))\right)\left(\triangle u(k-1)-\triangle v(k-1)\right)\\
+\ &\sum_{k\in\mathbb{Z}} V(k)\left(|u|^{p(k)-2}u - |v|^{p(k)-2}v\right)(u-v) > 0.
\end{aligned}$$

Therefore, T' is strictly monotone.
(ii) For a sequence $(u_n)\subset E$ assume that $u_n \rightharpoonup u_0 \in E$. Then, (u_n) is bounded in E, and hence, there exists a constant $M>0$ such that $\rho_E(u_n)\leq M$ for all n, and

$$u_n(k)\to u_0(k) \text{ for all } k\in\mathbb{Z} \text{ as } n\to+\infty.$$

Also assume that

$$\limsup_{n\to\infty}\langle T'(u_n)-T'(u_0), u_n-u_0\rangle \leq 0. \tag{25}$$

Then, considering these information along with strict monotonicity of T', we have

$$\limsup_{n\to\infty}\langle T'(u_n)-T'(u_0), u_n-u_0\rangle = \lim_{n\to\infty}\langle T'(u_n)-T'(u_0), u_n-u_0\rangle = 0. \tag{26}$$

Therefore, using (3.1) and (a1), we have

$$\begin{aligned}
0 \leq C_p\sum_{k\in\mathbb{Z}} V(k)|u_n-u_0|^{p(k)} &\leq \sum_{k\in\mathbb{Z}} V(k)\left(|u_n|^{p(k)-2}u_n - |u_0|^{p(k)-2}u_0\right)(u_n-u_0)\\
&\leq \langle T'(u_n)-T'(u_0), u_n-u_0\rangle.
\end{aligned}$$

Taking limit and considering (26) leads us to the inequality

$$0\leq C_p\lim_{n\to\infty}\sum_{k\in\mathbb{Z}} V(k)|u_n-u_0|^{p(k)} \leq \lim_{n\to\infty}\langle T'(u_n)-T'(u_0), u_n-u_0\rangle = 0,$$

which means, by Proposition 2.1, $u_n\to u_0$ in E.
(iii) Since T' is continuous due to Proposition 2.8, it is enough to show that T' has a continuous inverse $(T')^{-1} : E^*\to E$. First, we show that T' is coercive. Without

loss of generality, we may assume that $\|u\|_E > 1$. Then, by (a3) and Proposition 2.1, we have

$$\langle T'(u), u\rangle = \sum_{k\in\mathbb{Z}} a(k-1, \triangle u(k-1))\triangle u(k-1) + \sum_{k\in\mathbb{Z}} V(k)|u|^{p(k)} \geq c\|u\|_E^{p^-}$$

or

$$\frac{\langle T'(u), u\rangle}{\|u\|_E} \geq c\|u\|_E^{p^- -1} \tag{27}$$

which means that T' is coercive. Moreover, we know from (i) that T' is strictly monotone, which ensures that T' is an injection. By Minty-Browder theorem (see [36]), we obtain that T' is a surjection. As a consequence, T' has an inverse mapping $(T')^{-1} : E^* \to E$. We now show that $(T')^{-1}$ is continuous. To this end, let $(u_n^*), u^* \in E^*$ with $u_n^* \to u^*$, and let $(T')^{-1}(u_n^*) = u_n, (T')^{-1}(u^*) = u$. Then, $T'(u_n) = u_n^*$ and $T'(u) = u^*$ which means, by the coercivity of T', that (u_n) is bounded in E. Therefore, there exist $\hat{u} \in E$ and a subsequence, not relabelled, $(u_n) \subset E$ such that $u_n \rightharpoonup \hat{u}$ in E. Since the weak limit is unique, we must have $u_n \rightharpoonup u$ in E. Additionally, considering that $u_n^* \to u^*$ in E^*, we have

$$\lim_{n\to\infty} \langle u_n^* - u^*, u_n - u\rangle = \lim_{n\to\infty} \langle T'(u_n) - T'(u), u_n - u\rangle = 0.$$

Since T' is of type (S_+), we have $u_n \to u$ in E, that is, $(T')^{-1} : E^* \to E$ is continuous. Overall, T' is a homeomorphism.

(iv) The compactness of S' follows by Proposition 2.8. Indeed, if we let $u_n \rightharpoonup u$ in E and apply the same arguments as we did to obtain (23), we have

$$\lim_{n\to\infty} \sup \|S'(u_n) - S'(u)\|_{E^*} = \lim_{n\to\infty} \sup_{\|v\|_E\leq 1} |\langle S'(u_n) - S'(u), v\rangle| = 0.$$

Therefore, $S' : E \to E^*$ is strongly continuous, that is, S' is compact. □

4. The existence result

First, we would like to remark that, due to (10) and (19), to show that $u \in E$ is a solution to problem (1)-(2) for all $v \in E$ is equivalent to show that

$$T'u = S'u \text{ in } E^*. \tag{28}$$

The following theorem establishes a topological existence result for problem (1)-(2).

Theorem 4.1. *Suppose that (V1), (a1)-(a3) and (f1)-(f2) hold. Additionally, assume that the following conditions are fulfilled:*
(i) *T' is a homeomorphism;*
(ii) *S' is compact;*
(iii) *The mapping $T' - S'$ is coercive.*
Then operator equation (28) has a nontrivial solution in E, which in turn becomes a homoclinic solution to problem (1)-(2).

Proof. Due to Proposition 3.1, we skip (i),(ii) and proceed with (iii). As in Proposition 3.1 (iii), we can obtain that

$$\|T'u\|_{E^*} \geq c_1\|u\|_E^{p^- -1}, \text{ for all } u \in E, \|u\|_E > 1, c_1 > 0. \tag{29}$$

By (f1), there is $M > 0$ such that $|f(k,t)| \leq \theta(k)+\beta(k)|t|^{q(k)-1}$ for all $k \in \mathbb{Z}$, $|t| > M$. Using the Hölder inequality and the embeddings, we have

$$\begin{aligned}
|\langle S'(u), v\rangle| &\leq \sum_{k\in\mathbb{Z}} \theta(k)|v| + \sum_{k\in\mathbb{Z}} \beta(k)|u|^{q(k)-1}|v| \\
&\leq \|\theta\|_{p'(k)}\|v\|_{p(k)} + \|\beta\|_{r(k)}\||u|^{q(k)-1}\|_{\frac{p(k)}{q(k)-1}}\|v\|_{p(k)} \\
&\leq (c_2\|u\|_E^{\alpha-1} + c_3)\|v\|_E
\end{aligned}$$

and hence

$$\|S'u\|_{E^*} \leq c_2\|u\|_E^{\alpha-1} + c_3, \ \text{ for all } \ u \in E, \ \|u\|_E > 1, \ \alpha \in [q^-, q^+], \ c_2, c_3 > 0. \tag{30}$$

Then, by (29) and (30), it leads

$$\|(T' - S')u\|_{E^*} \geq \|T'u\|_{E^*} - \|S'u\|_{E^*} \geq c_1\|u\|_E^{p^- -1} - c_2\|u\|_E^{q^+ -1} - c_3 \tag{31}$$

and thus, we obtain coercivity, that is, $\|(T'-S')u\|_{E^*} \to \infty$ as $\|u\|_E \to \infty$. Therefore, there exists a constant $R_0 > 1$ such that

$$\|(T' - S')u\|_{E^*} > 1 \ \text{ for all } \ u \in E, \ \|u\|_E \geq R_0. \tag{32}$$

Since T' is a homeomorphism of E onto E^*, (28) can be equivalently written as

$$u = (T')^{-1}(S'u). \tag{33}$$

Moreover, since compactness is a topological property we can define a compact operator $\mathcal{K}$ by $\mathcal{K} := (T')^{-1}(S') : E \to E$. Now, we will seek a solution to the operator equation

$$u = \mathcal{K}u \tag{34}$$

that is, we search a fixed point for the operator $\mathcal{K}$. Let us define the set

$$\mathcal{F} = \{u \in E : \ u = \tau(T')^{-1}(S'u) \ \text{ for some } \ \tau \in [0,1]\}.$$

To this end, for an $u \in \mathcal{F}\backslash\{0\}$, we have

$$\|T'(\frac{u}{\tau})\|_{E^*} = \|S'u\|_{E^*}. \tag{35}$$

Then, considering (35) along with (29) and (30), it follows

$$\frac{c_1}{\tau^{p^- -1}}\|u\|_E^{\gamma-1} \leq c_2\|u\|_E^{\alpha-1} + c_3 \tag{36}$$

that is, $\mathcal{F}$ is bounded in E, where the exponents $\gamma \in [p^-, p^+]$ and $\alpha \in [q^-, q^+]$ are determined according to the $\|u\|_E$ while its value varies in the intervals $(0,1)$ or $[1,\infty)$. Therefore, there is some constant $R_1 \geq R_0$ such that the inclusion $\mathcal{F} \subseteq B_{R_1}(0)$ holds. Thus, we can write

$$\mathcal{K} : \overline{B_{R_1}(0)} \to E$$

where $\mathcal{K}$ is still compact. On the other hand, by (32) we have $u - \mathcal{K}u \neq 0$ for any $u \in \partial B_{R_1}(0)$. Otherwise, we would have $u = \mathcal{K}u = (T')^{-1}(S'u)$ for any $u \in \partial B_{R_1}(0)$; however, this would lead us to $(T' - S')u = 0$. This can not happen since by (32) we must have $\|(T' - S')u\|_{E^*} > 0$ for any $u \in \partial B_{R_1}(0)$. Therefore, we can associate the Leray-Schauder degree of mapping, a $\mathbb{Z}$-valued function $d_{LS}(I - \mathcal{K}, B_{R_1}(0), 0)$, to $\mathcal{K}$. Next, let us define the mapping

$$H(u,t) = u - t\mathcal{K}u \ \text{ for } \ u \in \overline{B_{R_1}(0)} \ \text{ and } \ t \in [0,1]. \tag{37}$$

Apparently $H(u,t)$ is a continuous mapping on $\overline{B_{R_1}(0)}\times[0,1]$ such that $H(u,t)\neq 0$ for all $u\in\partial B_{R_1}(0)$ and $t\in[0,1]$. Assume by contradiction that there exist $\tilde{u}\in\partial B_{R_1}(0)$ and $\tilde{t}\in[0,1]$ such that

$$\tilde{u}-\tilde{t}\mathcal{K}\tilde{u}=0. \tag{38}$$

Then

$$0=\|\tilde{u}-\tilde{t}\mathcal{K}\tilde{u}\|_E\geq\|\tilde{u}\|_E-\tilde{t}\|\mathcal{K}\tilde{u}\|_E\geq(1-\tilde{t})R_1\geq 0 \tag{39}$$

since $\|\tilde{u}\|_E=R_1$, and hence, $\|\mathcal{K}\tilde{u}\|_E\leq R_1$. Therefore, it must be $\tilde{t}=1$. This result contradicts the fact $u-\mathcal{K}u\neq 0$ for any $u\in\partial B_{R_1}(0)$. Overall,

$$H(u,t)\neq 0 \text{ for } u\in\partial B_{R_1}(0) \text{ and } t\in[0,1]. \tag{40}$$

Therefore, $H(\cdot,t)$ is a homotopy of the mappings $I=H(\cdot,0)$ and $I-\mathcal{K}=H(\cdot,1)$. Taking into account the homotopy invariance and normalization properties of degree, we obtain that

$$d_{LS}(I-\mathcal{K},B_{R_1}(0),0)=d_{LS}(I,B_{R_1}(0),0)=1 \tag{41}$$

which means that $\mathcal{K}$ has a fixed point located in $B_{R_1}(0)$. In conclusion, there exists a function $u\in E$ such that $T'u=S'u$ in E^*, that is, u is a nontrivial homoclinic solution for problem (1)-(2). □

References

[1] C.O. Alves, Existence of standing waves solution for a nonlinear Schrödinger equation in $\mathbb{R}^N$, *Journal of Elliptic and Parabolic Equations* **1** (2015), no. 2, 231–241.

[2] C.O. Alves, G.M. Figueiredo, M.F. Furtado, Multiple solutions for a nonlinear Schrödinger equation with magnetic fields, *Communications in Partial Differential Equations* **36** (2011), 1565–1586.

[3] A. Ambrosetti, P.H. Rabinowitz, Dual variational methods in critical point theory, *J. Funct. Anal.* **14** (1973), 349–381.

[4] M. Avci, Existence results for anisotropic discrete boundary value problems, *Elec. J. Dif. Equa.* **148** (2016), 1–11.

[5] M. Avci, A. Pankov, Nontrivial solutions of discrete nonlinear equations with variable exponent, *J. Math. Anal. Appl.* **431** (2015), no. 1, 22–33.

[6] L. Boccardo, P. Drábek, D. Giachetti, M. Kućera, Generalization of Fredholm alternative for nonlinear differential operators, *Nonlinear Anal. Theory Methods Appl.* **10** (1986), 1083–1103.

[7] H. Brezis, *Analyse Functionelle*, Mason, Paris, 1983.

[8] M. Chipot, *Elliptic Equations: An Introductory Course*, Birkhäuser Verlag AG, Basel, 2009.

[9] F.E. Browder, Nonlinear elliptic boundary value problems and the generalized topological degree, *Bull. Amer. Math. Soc.* **76** (1970), no. 5, 999–1005.

[10] R. F. Brown, *A Topological Introduction to Nonlinear Analysis, Third Edition*, Birkhäuser, Basel, 2014.

[11] D.V. Cruz-Uribe, A. Fiorenza, *Variable Lebesgue Spaces: Foundations and Harmonic Analysis*, Springer, Basel, 2013.

[12] L. Diening, P. Harjuletho, P. Hästö, M. Růž ička, *Lebesgue and Sobolev spaces with variable exponents*, Springer-Verlag, Berlin, 2011.

[13] S. Heidarkhani, G.A. Afrouzi, S. Moradi, G. Caristi, Existence of multiple solutions for a perturbed discrete anisotropic equation, *Journal of Difference Equations and Applications* **23** (2017), no. 9, 1491–1507.

[14] G. Dinca, P. Jebelean, J. Mawhin, Variational and topological methods for Dirichlet problems with p-Laplacian, *Portugal. Math.* **53** (2001), no. 3, 339–377.

[15] P. Drábek, J. Milota, *Methods of Nonlinear Analysis: Applications to Differential Equations, Second Edition*, Birkhäuser, Basel, 2013.

[16] S. Flash, C.R. Willis, Discrete breathers, *Phys. Repts.* **295** (1998) 181–264.

[17] A. Guiro, B. Kone, S. Ouaro, Weak homoclinic solutions of anisotropic difference equations with variable exponents, *Adv. Difference Equa.* **154** (2012), 1–12.
[18] D. Hennig, G.P. Tsironis, Wave transmission in non-linear lattices, *Phys. Repts.* **309** (1999), 333–432.
[19] A. Iannizzotto, S.A. Tersian, Multiple homoclinic solutions for the discrete p-Laplacian via critical point theory, *J. Math. Anal. Appl.* **403** (2013), 173–182.
[20] A. Iannizzotto, V. Rădulescu, Positive homoclinic solutions for the discrete p-Laplacian with a coercive weight function, *Differential Integr. Equat.* **27** (2014), 35–44.
[21] P.G. Kevrekidis, *The Discrete Nonlinear Schrödinger Equation*, Springer, Berlin, 2009.
[22] P.G. Kevrekidis, D.J. Frantzeskakis, R. Garretero-González, *Emergent Nonlinear Phenomena in Bose-Einstein Condensates*, Springer, Berlin, 2008.
[23] J. Leray, J.L. Lions, Quelques résultats de viśk sur les problémes elliptiques nonlinéaires par les méthodes de Minty-Browder, *Bull. Soc. Math. France* **93** (1965), 97–107.
[24] P. Matei, On the Fréchet differentiability of Luxemburg norm in the sequence spaces $l^{p(n)}$ with variable exponents, *Romanian J. Math. Comp. Sci.* **4** (2014), no. 1, 167–179.
[25] J. Mawhin, Leray-Schauder degree: A half century of extensions and applications, *Topological Methods in Nonlinear Analysis* **14** (1999), 195–228.
[26] M. Mihăilescu, V. Rădulescu, S. Tersian, Homoclinic solutions of difference equations with variable exponents, *Topol. Meth. Nonlin. Anal.* **38** (2011), 277–289.
[27] D.N. Mills, *Nonlinear Optics: Basic Concepts*, Springer, Berlin, 1998.
[28] G.J. Minty, On a monotonicity method for the solution of nonlinear equations in Banach spaces, *Proc. Nat. Acad. Sci. U.S.A.* **50** (1963), 1038–1041.
[29] M.K. Moghadam, M. Avci, Existence results to a nonlinear $p(k)$-Laplacian difference equation, *J. Difference Equa. App.* **23** (2017), no. 10, 1652–1669.
[30] G.M. Bisci, D. Repovs, On sequences of solutions for discrete anisotropic equations, *Expo. Math.* **32** (2014), 284–295.
[31] A. Pankov, Gap solitons in periodic discrete nonlinear Schrödinger equations, *Nonlinearity* **19** (2006), 27–40.
[32] A. Pankov, Gap solitons in periodic discrete nonlinear Schrödinger equations II: A generalized Nehari manifold approach, *Discr. Contin. Dyn. Syst.* **19** (2007), no. 2, 419–430.
[33] M. Růžička, *Electrorheological Fluids: Modeling and Mathematical Theory*, Springer, Berlin, 2000.
[34] J. Simon, Régularité de la solution d'une équation nonlinéaire dans $\mathbb{R}^N$, *Lecture Notes in Math.* **665** (1978), 205–227.
[35] Z. Yucedag, Existence of solutions for anisotropic discrete boundary value problems of Kirchhoff type, *Int. J. Dif. Equa. Appl.* **13** (2014), no. 1, 1–15.
[36] E. Zeidler, *Nonlinear Functional Analysis and its Applications II/B: Nonlinear Monotone Operators*, Springer-Verlag, New York, 1990.
[37] G. Zhang, A. Pankov, Standing waves of the discrete nonlinear Schrödinger equations with growing potentials, *Commun. Math. Anal.* **5** (2008), no. 2, 38–49.
[38] G. Zhang, A. Pankov, Standing wave solutions of the discrete non-linear Schrödinger equations with unbounded potentials II, *Applicable Anal.* **89** (2010), no. 9, 1541–1557.

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