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Dynamics of superior fractals via Jungck SP orbit with s-convexity

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Abstract. The purpose of this paper is to generate new fractals for complex-valued polynomials via Jungck SP orbit with s-convexity. In this paper, we obtain a new escape algorithm for quadratic, cubic and higher degree complex valued polynomials to generate fractals. Also, we provide an algorithm as well as source programs to generate fractals. We have shown that beautiful graphics can be generated by using new escape algorithm. Our results are the generalization of corresponding results which is obtained by us [13] via SP orbit with s-convexity and Kang et al. [15] via Modified Jungck three step orbit.

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1. Introduction

Generally, people believe that the geometry of nature is based on the simple figures like lines, circles, polygons, spheres, quadratic surfaces and so on. But there are so many examples in nature which show that the geometry does not depend on simple figures. Fractal geometry provides a general framework to study such type of figures. First of all, in 1982, Benoit Mandelbrot introduced the theory of fractals in his book [1] which consists different fractal shapes existing in the nature. Later on, many other mathematicians like George Cantor, Giuseppe Peano, David Hilbert, Helge von Koch, Waclaw Sierpinski etc. gave their contribution in the field of fractals. Fractal is a rough or fragmented geometric shape that can be subdivided into congruent parts, each of which is a reduced size copy of the whole.

Historically, different generalizations of fractals have been made. In 1919, French mathematician Gaston Julia [4] derived the Julia set when he was studying Cayley's problem which is related to the behavior of Newton's method in complex plane. After this, Mandelbrot extended the work of Julia and introduced Mandelbrot set in his first book [1]. He studied fractals in Complex plane. Further, some other functions such as rational [8], trigonometric and exponential [2] etc. were used in the generation of fractals. Mandelbrot and Julia sets were also extended from the complex numbers to quaternions [24], bicomplex numbers [23], tricomplex numbers [9] etc. Rani and Kumar [6, 7] introduced the superior iterate and generated superior Julia and Mandelbrot sets for quadratic and cubic polynomials. In 2009, D. Rochon [3] studied generalized Mandelbrot sets in bicomplex plane. Later on, the work of Rochon was extended by Wang et al. [19–22] and they carried further analysis of generalized Julia

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and Mandelbrot sets. Further, some researchers obtained some fixed point results in the generation of Julia and Mandelbrot sets with s-convexity (see [5, 16, 17]). Recently, S. Kumari et al. [13] obtained some fixed point results in the generation of new fractals using SP orbit with s-convexity.

In 2011, Chugh and Kumar [10] introduced Jungck-SP iterative scheme and with the help of examples, they proved that Jungck-SP iterative scheme converges faster than that of Jungck-Noor, Jungck-Ishikawa and Jungck-Mann iterative schemes. In 2015, by using Jungck Mann and Jungck Ishikawa orbits, Kang et al. [14] established the new escape criterion to generate Julia and Mandelbrot sets. In this sequel, they presented the generalization of Julia sets and Mandelbrot sets for complex-valued polynomials using Jungck three-step orbit [15]. Recently, authors [12] used SP orbit to study the chaotic behavior of logistic map.

In this paper, we derive escape criterions to generate new fractals as a beautiful graphics for quadratic, cubic and n^{th} degree polynomials via Jungck-SP orbit with s-convexity by using the software Mathematica 11.0.

2. Preliminaries

Definition 2.1. (Orbit) [11]. Let $x_0 \in R$, then the orbit of x_0 under the mapping F is defined as the sequence of points

$$x_0, x_1 = F(x_0), x_2 = F^2(x_0), ..., x_n = F^n(x_0), ...$$

Definition 2.2. (Julia Set) [4]. The Julia set of a function g is the boundary of the set of points $z \in \mathbb{C}$ that escape to infinity under repeated iteration by $g(z)$, i.e. the Julia set of a function g is defined as

$$J(g) = \partial\{z \in \mathbb{C} : g^n(z) \to \infty \ as \ n \to \infty\},$$

where $\mathbb{C}$ is the set of complex numbers, $g^n(z)$ is n^{th} iterate of function g.

Definition 2.3. (Mandelbrot Set) [1]. The Mandelbrot set M consists of all parameters for which the Julia set of g is connected, i.e.,

$$M = \{c \in \mathbb{C} : J(g) \ is \ connected\}.$$

The set M contains a lot of information about the structure of Julia set.
The Mandelbrot set M for the Quadratic function $P_c(z) = z^2 + c$ is defined as the collection of all $c \in \mathbb{C}$ for which the orbit of the point 0 is bounded, i.e.

$$M = \{c \in \mathbb{C} : \{P_c^n(0)\}; \ n = 0, 1, 2, ... \ is \ bounded\}.$$

We take the initial point 0 as 0 is the only critical point of $P_c(z)$ (see [11], p. 249).

Definition 2.4. Let (X, d) be a metric space and $I = [0, 1]$. A mapping $\omega\colon X^2 \times I \to X$ on X is said to be convex structure on X, if for any $(x, y, \alpha) \in X^2 \times I$ and $u \in X$, the following inequality holds:

$$d(\omega(x, y, \alpha), u) \leq \alpha^s d(x, u) + (1 - \alpha)^s d(y, u).$$

The metric space (X, d) with convex structure ω is said to be convex metric space [18]. Moreover, a nonempty subset C of X is said to be convex if $\omega(x, y, \alpha) \in C$ for all $(x, y, \alpha) \in C^2 \times I$.

Definition 2.5. Let (X, d) be a metric space and $I = [0, 1]$ and $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ are real sequences in $[0, 1]$ with $a_n + b_n + c_n = 1$. A mapping $\omega\colon X^3 \times I^3 \to X$ is said to be convex structure on X, if for any $(x, y, z, a_n, b_n, c_n) \in X^3 \times I^3$ and $u \in X$, the following inequality holds:

$$d(\omega(x, y, z, a_n, b_n, c_n), u) \leq (a_n)^s d(x, u) + (b_n)^s d(y, u) + (c_n)^s d(z, u).$$

Moreover, a nonempty subset C of X is said to be convex if $\omega(x, y, z, a_n, b_n, c_n) \in C$ for all $(x, y, z, a_n, b_n, c_n) \in C^3 \times I^3$.

Definition 2.6. Let X be a subset of real complex number and $T\colon X \to X$. Consider a sequence $\{z_n\}$ of iterates for initial point $z_0 \in X$ such that

$$\begin{aligned} Sz_{n+1} &= (1-\alpha_n)^s Su_n + \alpha_n^s Tu_n, \\ Su_n &= (1-\beta_n)^s Sv_n + \beta_n^s Tv_n, \\ Sv_n &= (1-\gamma_n)^s Sz_n + \gamma_n^s Tz_n, \end{aligned}$$

where $\alpha_n, \beta_n, \gamma_n$ are sequences of positive numbers in $[0, 1]$. Then the above sequence $\{z_n\}$ of iterates is said to be Jungck SP orbit with s-convexity having six tuples $(T, z_0, \alpha_n, \beta_n, \gamma_n, s)$.

3. Main results

The escape criterion has an renowned place in the generation of fractals. We prove following escape criterions for quadratic, cubic and higher degree polynomials in Jungck SP orbit with s-convexity. Throughout this paper, we assume that $z_0 = z \in \mathbb{C}$ and $\alpha_n = \alpha, \beta_n = \beta, \gamma_n = \gamma$, then Jungck SP iteration scheme with s-convexity can be written in the following manner :

$$\begin{aligned} Sz_{n+1} &= (1-\alpha)^s Su_n + \alpha^s P_c(u_n), \\ Su_n &= (1-\beta)^s Sv_n + \beta^s P_c(v_n), \\ Sv_n &= (1-\gamma)^s Sz_n + \gamma^s P_c(z_n), \end{aligned}$$

where $P_c(z_n)$ is a quadratic, cubic or higher degree complex polynomial and $0 < \alpha, \beta, \gamma, s \leq 1$.

3.1. Escape criterion for quadratic polynomials.

Let $Q(z) = z^2 - az + c$ be a quadratic complex polynomial. We choose $P_c(z) = z^2 + c$ and $Sz = az$, where a and c are complex numbers.

Theorem 3.1. *Suppose $|z| \geq |c| > 2(1 + |a|)/s\alpha$, $|z| \geq |c| > 2(1 + |a|)/s\beta$ and $|z| \geq |c| > 2(1 + |a|)/s\gamma$, where $0 < \alpha, \beta, \gamma, s \leq 1$ and c is a complex number. Define*

$$\begin{aligned} Sz_1 &= (1-\alpha)^s Su + \alpha^s P_c(u) \\ Sz_2 &= (1-\alpha)^s Su_1 + \alpha^s P_c(u_1) \\ &\cdots \\ &\cdots \\ &\cdots \\ Sz_n &= (1-\alpha)^s Su_{n-1} + \alpha^s P_c(u_{n-1}) \end{aligned}$$

where $P_c(z)$ is a quadratic polynomial in terms of α and $n = 1, 2, 3, ...,$ then $|z_n| \to \infty$ as $n \to \infty$.

Proof. Consider

$$\begin{aligned}|Sv| &= |(1-\gamma)^s Sz + \gamma^s P_c(z)|, \qquad for\ P_c(z) = z^2 + c\\ |Sv| &= |(1-\gamma)^s Sz + \gamma^s (z^2 + c)|\\ &= |(1-\gamma)^s Sz + (1-(1-\gamma))^s (z^2 + c)|.\end{aligned}$$

By binomial expansion up to linear terms of γ and $(1-\gamma)$, we obtain

$$\begin{array}{rcll} |Sv| &=& |(1-s\gamma)Sz + (1-s(1-\gamma))(z^2+c)| & \\ |av| &=& |(1-s\gamma)az + (1-s(1-\gamma))(z^2+c)| & \\ &=& |(1-s\gamma)az + (1-s+s\gamma)(z^2+c)| & \\ &\geq& |(1-s\gamma)az + s\gamma(z^2+c)|, & (\because 1-s+s\gamma \geq s\gamma)\\ &\geq& |s\gamma z^2 + (1-s\gamma)az| - |s\gamma c| & \\ &\geq& |s\gamma z^2 + (1-s\gamma)az| - |s\gamma z|, & (\because |z| \geq |c|)\\ &\geq& |s\gamma z^2| - |(1-s\gamma)az| - |s\gamma z| & \\ &\geq& |s\gamma z^2| - |az| + |s\gamma az| - |s\gamma z| & \\ &\geq& |s\gamma z^2| - |az| - s\gamma|z| & (\because |a| \geq 0)\\ &\geq& |s\gamma z^2| - |az| - |z| & (\because s\gamma < 1)\\ &=& |s\gamma z^2| - |z|(|a|+1) & \\ &=& |z|\{s\gamma|z| - (|a|+1)\}. & \end{array}$$

Thus,

$$\begin{array}{rcl} |a||v| &\geq& |z|\{s\gamma|z| - (|a|+1)\}\\ |v| &\geq& |z|(1+1/|a|)\{s\gamma|z|/(|a|+1) - 1\}\\ &\geq& |z|\{s\gamma|z|/(|a|+1) - 1\},\\ i.e.,\ |v| &\geq& |z|\{s\gamma|z|/(|a|+1) - 1\}. \end{array} \tag{1}$$

Also,

$$\begin{array}{rcl} |Su| &=& |(1-\beta)^s Sv + \beta^s P_c(v)|, \qquad for\ P_c(v) = v^2 + c\\ |Su| &=& |(1-\beta)^s Sv + \beta^s (v^2 + c)|\\ &=& |(1-\beta)^s Sv + (1-(1-\beta))^s (v^2 + c)|. \end{array}$$

By binomial expansion up to linear terms of β and $(1-\beta)$, we obtain

$$\begin{array}{rcll} |Su| &=& |(1-s\beta)Sv + (1-s(1-\beta))(v^2+c)| & \\ |au| &=& |(1-s\beta)av + (1-s(1-\beta))(v^2+c)| & \\ &=& |(1-s\beta)av + (1-s+s\beta)(v^2+c)| & \\ &\geq& |(1-s\beta)av + s\beta(v^2+c)|, & (\because 1-s+s\beta \geq s\beta)\\ &\geq& |(1-s\beta)a\{|z|(s\gamma|z|/(|a|+1)-1)\} + s\beta[\{|z|(s\gamma|z|/(|a|+1)-1)\}^2 + c]| & \end{array}$$

Since $|z| > (2(1+|a|)/s\gamma)$, we have $(s\gamma|z|/(|a|+1)-1) > 1$. This gives

$$\begin{aligned}
|au| &\geq |(1-s\beta)a|z| + s\beta(|z|^2+c)| \\
&\geq |s\beta z^2 + (1-s\beta)az| - |s\beta c| \\
&\geq |s\beta z^2 + (1-s\beta)az| - |s\beta z|, && (\because |z| \geq |c|) \\
&\geq |s\beta z^2| - |(1-s\beta)az| - |s\beta z| \\
&\geq |s\beta z^2| - |az| + |s\beta az| - |s\beta z| \\
&\geq |s\beta z^2| - |az| - s\beta|z| && (\because |a| \geq 0) \\
&\geq |s\beta z^2| - |az| - |z| && (\because s\beta < 1) \\
&= |s\beta z^2| - |z|(|a|+1) \\
&= |z|\{s\beta|z| - (|a|+1)\}.
\end{aligned}$$

Thus,

$$\begin{aligned}
|a||u| &\geq |z|\{s\beta|z| - (|a|+1)\} \\
|u| &\geq |z|(1+1/|a|)\{s\beta|z|/(|a|+1) - 1\} \\
&\geq |z|\{s\beta|z|/(|a|+1) - 1\}, \\
\textit{i.e.},\ |u| &\geq |z|\{s\beta|z|/(|a|+1) - 1\}. \qquad (2)
\end{aligned}$$

Now, for $Sz_n = (1-\alpha)^s Su_{n-1} + \alpha^s P_c(u_{n-1})$, we have

$$\begin{aligned}
|Sz_1| &= |(1-\alpha)^s Su + \alpha^s P_c(u)| \\
&= |(1-\alpha)^s Su + \alpha^s(u^2+c)| \\
&= |(1-\alpha)^s Su + (1-(1-\alpha))^s(u^2+c)|.
\end{aligned}$$

By binomial expansion up to linear terms of α and $(1-\alpha)$, we obtain

$$\begin{aligned}
|Sz_1| &= |(1-s\alpha)Su + (1-s(1-\alpha))(u^2+c)| \\
|az_1| &= |(1-s\alpha)au + (1-s(1-\alpha))(u^2+c)| \\
&= |(1-s\alpha)au + (1-s+s\alpha)(u^2+c)| \\
&\geq |(1-s\alpha)au + s\alpha(u^2+c)|, && (\because 1-s+s\alpha \geq s\alpha) \\
&\geq |(1-s\alpha)a\{|z|(s\beta|z|/(|a|+1)-1)\} + s\alpha[\{|z|(s\beta|z|/(|a|+1)-1)\}^2 + c]|.
\end{aligned}$$

Since $|z| > (2(1+|a|)/s\beta)$, we have $(s\beta|z|/(|a|+1)-1) > 1$. This gives

$$\begin{aligned}
|az_1| &\geq |(1-s\alpha)az + s\alpha(z^2+c)| \\
&\geq |s\alpha z^2 + (1-s\alpha)az| - |s\alpha z|, && (\because |z| \geq |c|) \\
&\geq |s\alpha z^2| - |(1-s\alpha)az| - |s\alpha z| \\
&\geq |s\alpha z^2| - |az| + |s\alpha az| - |s\alpha z| \\
&\geq |s\alpha z^2| - |az| - s\alpha|z|, && (\because |a| \geq 0) \\
&\geq |s\alpha z^2| - |az| - |z|, && (\because s\alpha < 1) \\
&= |s\alpha z^2| - |z|(|a|+1) \\
&= |z|\{s\alpha|z| - (|a|+1)\}.
\end{aligned}$$

Thus,

$$\begin{aligned} |a||z_1| &\geq |z|\{s\alpha|z| - (|a|+1)\} \\ |z_1| &\geq |z|(1+1/|a|)\{s\alpha|z|/(|a|+1) - 1\} \\ &\geq |z|\{s\alpha|z|/(|a|+1) - 1\}, \\ i.e., |z_1| &\geq |z|\{s\alpha|z|/(|a|+1) - 1\}. \end{aligned}$$

Since $|z| \geq |c| > (2(1+|a|)/s\alpha)$, $|z| \geq |c| > (2(1+|a|)/s\beta)$ and $|z| \geq |c| > (2(1+|a|)/s\gamma)$ exist. Therefore, we have $s\alpha|z|/(1+|a|) - 1 > 1$. Hence, there exists a $\lambda > 0$ such that $s\alpha|z|/(1+|a|) - 1 > \lambda + 1 > 1$. Consequently, we have

$$|z_1| > (1+\lambda)|z|.$$

Particularly, $|z_n| > |z|$. So, repeating this process n times we have,

$$|z_n| > (1+\lambda)^n|z|.$$

Thus, the orbit of z tends to infinity as n tends to infinity. Hence the result. □

From the above theorem, we obtain the following corollaries:

Corollary 3.2. *Let $|c| > 2(1+|a|)/s\alpha$, $|c| > 2(1+|a|)/s\beta$ and $|c| > 2(1+|a|)/s\gamma$, then the orbit of Jungck $SP(Q, 0, \alpha, \beta, \gamma, s)$ with s-convexity escapes to infinity.*

In the above theorem, the escape criterion proved gives us a little more information. In the proof, we used the only fact that $|z| \geq |c|$ and $|c| > 2(1+|a|)/s\alpha$, $|c| > 2(1+|a|)/s\beta$ and $|c| > 2(1+|a|)/s\gamma$. Thus, we have the following corollary as a refinement of the escape criterion:

Corollary 3.3. *(Escape Criterion). Suppose $|z| > \max\{|c|, 2(1+|a|)/s\alpha, 2(1+|a|)/s\beta, 2(1+|a|)/s\gamma\}$, then $|z_n| > (1+\lambda)^n|z|$ and $|z_n| \to \infty$ as $n \to \infty$.*

We notice that we may apply Corollary 3.3 to $|z_k|$ for some $k \geq 0$ to have the following result:

Corollary 3.4. *Suppose $|z_k| > \max\{|c|, 2(1+|a|)/s\alpha, 2(1+|a|)/s\beta, 2(1+|a|)/s\gamma\}$ for some $k \geq 0$ then $|z_{k+1}| > (1+\lambda)|z_k|$ and $|z_k| \to \infty$ as $k \to \infty$.*

Using this corollary, we obtain an algorithm to generate connected Julia sets of quadratic complex polynomials $Q_c(z)$ for any number $c \in \mathbb{C}$. If for some n, the orbit of z, i.e. $\{z_n\}$ lies outside the circle of radius $\max\{|c|, 2(1+|a|)/s\alpha, 2(1+|a|)/s\beta, 2(1+|a|)/s\gamma\}$, then the orbit escapes to infinity, which means that z does not lie in the connected Julia set. If $\{z_n\}$ does not exceed this bound, then by definition, z lies in the connected Julia set and collection of such points is known as Mandelbrot set.

3.2. Escape criterion for cubic polynomials. Now, we prove the following theorem for a cubic complex polynomial $Q_{a,c}(z) = z^3 - az + c$, we take $P_c(z) = z^3 + c$ and $Sz = az$, where a, c are complex numbers, as this polynomial is equivalent to all other cubic polynomials.

Theorem 3.5. *Suppose that $|z| \geq |c| > (2(1+|a|)/s\alpha)^{1/2}$, $|z| \geq |c| > (2(1+|a|)/s\beta)^{1/2}$ and $|z| \geq |c| > (2(1+|a|)/s\gamma)^{1/2}$, where $0 < \alpha, \beta, \gamma, s \leq 1$, and a, c are complex numbers. Define*

$$Sz_1 = (1-\alpha)^s Su + \alpha^s P_c(u)$$

$$Sz_2 = (1-\alpha)^s Su_1 + \alpha^s P_c(u_1)$$
$$\cdots$$
$$\cdots$$
$$\cdots$$
$$Sz_n = (1-\alpha)^s Su_{n-1} + \alpha^s P_c(u_{n-1}), n = 1, 2, 3, \dots ,$$

where $P_c(u)$ is a cubic polynomial in terms of α, then $z_n \to \infty$ as $n \to \infty$.

Proof. Let us consider,

$$\begin{aligned}
|Sv| &= |(1-\gamma)^s Sz + \gamma^s P_c(z)|, \quad for\ P_c(z) = z^3 + c \\
|Sv| &= |(1-\gamma)^s Sz + \gamma^s (z^3 + c)| \\
&= |(1-\gamma)^s Sz + (1-(1-\gamma))^s (z^3 + c)|.
\end{aligned}$$

By binomial expansion up to linear terms of γ and $(1-\gamma)$, we obtain

$$\begin{aligned}
|Sv| &= |(1-s\gamma)Sz + (1-s(1-\gamma))(z^3+c)| \\
|av| &= |(1-s\gamma)az + (1-s(1-\gamma))(z^3+c)| \\
&= |(1-s\gamma)az + (1-s+s\gamma)(z^3+c)| \\
&\geq |(1-s\gamma)az + s\gamma(z^3+c)|, && (\because 1-s+s\gamma \geq s\gamma) \\
&\geq |s\gamma z^3 + (1-s\gamma)az| - |s\gamma c| \\
&\geq |s\gamma z^3 + (1-s\gamma)az| - |s\gamma z|, && (\because |z| \geq |c|) \\
&\geq |s\gamma z^3| - |(1-s\gamma)az| - |s\gamma z| \\
&\geq |s\gamma z^3| - |az| + |s\gamma az| - |s\gamma z| \\
&\geq |s\gamma z^3| - |az| - s\gamma|z|, && (\because |a| \geq 0) \\
&\geq |s\gamma z^3| - |az| - |z|, && (\because s\gamma < 1) \\
&= |s\gamma z^3| - |z|(|a|+1) \\
&= |z|\{s\gamma|z|^2 - (|a|+1)\}.
\end{aligned}$$

Thus,

$$\begin{aligned}
|a||v| &\geq |z|\{s\gamma|z|^2 - (|a|+1)\} \\
|v| &\geq |z|(1+1/|a|)\{s\gamma|z|^2/(|a|+1) - 1\} \\
&\geq |z|\{s\gamma|z^2|/(|a|+1) - 1\}, \\
i.e.,\ |v| &\geq |z|\{s\gamma|z|^2/(|a|+1) - 1\}. \qquad (3)
\end{aligned}$$

Also,

$$\begin{aligned}
|Su| &= |(1-\beta)^s Sv + \beta^s P_c(v)|, \quad for P_c(v) = v^3 + c \\
|Su| &= |(1-\beta)^s Sv + \beta^s (v^3 + c)| \\
&= |(1-\beta)^s Sv + (1-(1-\beta))^s (v^3 + c)|.
\end{aligned}$$

By binomial expansion up to linear terms of β and $(1-\beta)$, we obtain

$$\begin{aligned}
|Su| &= |(1-s\beta)Sv+(1-s(1-\beta))(v^3+c)| \\
|au| &= |(1-s\beta)av+(1-s(1-\beta))(v^3+c)| \\
&= |(1-s\beta)av+(1-s+s\beta)(v^3+c)| \\
&\geq |(1-s\beta)av+s\beta(v^3+c)|, \qquad (\because 1-s+s\beta \geq s\beta) \\
&\geq |(1-s\beta)a\{|z|(s\gamma|z|^2/(|a|+1)-1)\}+s\beta[\{|z|(s\gamma|z|^2/(|a|+1)-1)\}^3+c]|.
\end{aligned}$$

Since $|z|>(2(1+|a|)/s\gamma)^{1/2}$, we have $(s\gamma|z^2|/(|a|+1)-1)>1$. This gives

$$\begin{aligned}
|au| &\geq |(1-s\beta)a|z|+s\beta(|z|^3+c)| \\
&\geq |s\beta z^3+(1-s\beta)az|-|s\beta c| \\
&\geq |s\beta z^3+(1-s\beta)az|-|s\beta z|, \qquad (\because |z|\geq|c|) \\
&\geq |s\beta z^3|-|(1-s\beta)az|-|s\beta z| \\
&\geq |s\beta z^3|-|az|+|s\beta az|-|s\beta z| \\
&\geq |s\beta z^3|-|az|-s\beta|z|, \qquad (\because |a|\geq 0) \\
&\geq |s\beta z^3|-|az|-|z|, \qquad (\because s\beta<1) \\
&= |s\beta z^3|-|z|(|a|+1) \\
&= |z|\{s\beta|z|^2-(|a|+1)\}.
\end{aligned}$$

Thus,

$$\begin{aligned}
|a||u| &\geq |z|\{s\beta|z|^2-(|a|+1)\} \\
|u| &\geq |z|(1+1/|a|)\{s\beta|z|^2/(|a|+1)-1\} \\
&\geq |z|\{s\beta|z|^2/(|a|+1)-1\}, \\
i.e.,\ |u| &\geq |z|\{s\beta|z|^2/(|a|+1)-1\}. \qquad (4)
\end{aligned}$$

Now, for $Sz_n=(1-\alpha)^sSu_{n-1}+\alpha^sP_c(u_{n-1})$, we have

$$\begin{aligned}
|Sz_1| &= |(1-\alpha)^sSu+\alpha^sP_c(u)| \\
&= |(1-\alpha)^sSu+\alpha^s(u^3+c)| \\
&= |(1-\alpha)^sSu+(1-(1-\alpha))^s(u^3+c)|.
\end{aligned}$$

By binomial expansion up to linear terms of α and $(1-\alpha)$, we obtain

$$\begin{aligned}
|Sz_1| &= |(1-s\alpha)Su+(1-s(1-\alpha))(u^3+c)| \\
|az_1| &= |(1-s\alpha)au+(1-s(1-\alpha))(u^3+c)| \\
&= |(1-s\alpha)au+(1-s+s\alpha)(u^3+c)| \\
&\geq |(1-s\alpha)au+s\alpha(u^3+c)|, \qquad (\because 1-s+s\alpha\geq s\alpha) \\
&\geq |(1-s\alpha)a\{|z|(s\beta|z|^2/(|a|+1)-1)\}+s\alpha[\{|z|(s\beta|z|^2/(|a|+1)-1)\}^3+c]|.
\end{aligned}$$

Since $|z| > (2(1+|a|)/s\beta)^{1/2}$, we have $(s\beta|z^2|/(|a|+1)-1) > 1$. This gives

$$\begin{aligned} |az_1| &\geq |(1-s\alpha)az + s\alpha(z^3+c)| \\ &\geq |s\alpha z^3 + (1-s\alpha)az| - |s\alpha z|, && (\because |z| \geq |c|) \\ &\geq |s\alpha z^3| - |(1-s\alpha)az| - |s\alpha z| \\ &\geq |s\alpha z^3| - |az| + |s\alpha az| - |s\alpha z| \\ &\geq |s\alpha z^3| - |az| - s\alpha|z|, && (\because |a| \geq 0) \\ &\geq |s\alpha z^3| - |az| - |z|, && (\because s\alpha < 1) \\ &= |s\alpha z^3| - |z|(|a|+1) \\ &= |z|\{s\alpha|z|^2 - (|a|+1)\}. \end{aligned}$$

Thus,

$$\begin{aligned} |a||z_1| &\geq |z|\{s\alpha|z|^2 - (|a|+1)\} \\ |z_1| &\geq |z|(1+1/|a|)\{s\alpha|z|^2/(|a|+1) - 1\} \\ &\geq |z|\{s\alpha|z|^2/(|a|+1) - 1\}, \\ i.e.,\ |z_1| &\geq |z|\{s\alpha|z|^2/(|a|+1) - 1\}. \end{aligned}$$

Since $|z| \geq |c| > (2(1+|a|)/s\alpha)^{1/2}$, $|z| \geq |c| > (2(1+|a|)/s\beta)^{1/2}$ and $|z| \geq |c| > (2(1+|a|)/s\gamma)^{1/2}$ exist. Therefore, we have $s\alpha|z|^2/(1+|a|) - 1 > 1$. Hence, there exists a $\lambda > 0$ such that $s\alpha|z|^2/(1+|a|) - 1 > \lambda + 1 > 1$. Consequently, we have

$$|z_1| > (1+\lambda)|z|.$$

Particularly, $|z_n| > |z|$. So, using the same argument n times, we have

$$|z_n| > (1+\lambda)^n|z|.$$

Thus, the orbit of z tends to infinity as n tends to infinity. Hence the result. □

The following corollaries are the consequences of above result:

Corollary 3.6. *(Escape Criterion) Let $Q_{a,c}(z) = z^3 - az + c$, where a, c are complex numbers. If $|z| > \max\{|c|, (|a|+2/s\alpha)^{1/2}, (|a|+2/s\beta)^{1/2}, (|a|+2/s\gamma)^{1/2}\}$, then $|z_n| \to \infty$ as $n \to \infty$.*
This gives the escape criterion for a cubic polynomial.

Corollary 3.7. *For some $k \geq 0$, let us assume $|z_k| > \max\{|c|, (2(1+|a|)/s\alpha)^{1/2}, (2(1+|a|)/s\beta)^{1/2}, (2(1+|a|)/s\gamma)^{1/2}\}$. Then $|z_{k+1}| > \lambda|z_k|$ and $|z_k| \to \infty$ as $k \to \infty$.*

We see that Corollary 3.7 gives the algorithm to generate fractals for cubic polynomial $Q_{a,c}(z)$.

3.3. A general escape criterion. Now, we find out a general escape criterion for higher degree polynomials of the form $G_c(z) = z^n - az + c$, we choose $P_c(z) = z^n + c$ and $Sz = az$, where a and c are complex numbers and $n = 1, 2, 3, \ldots$.

Theorem 3.8. *For a general function $P_c(z) = z^n + c$; $n = 1, 2, 3, \ldots$, where $0 < \alpha, \beta, \gamma, s \leq 1$, and c is a complex number. Define*

$$Sz_1 = (1-\alpha)^s Su + \alpha^s P_c(u)$$
$$Sz_2 = (1-\alpha)^s Su_1 + \alpha^s P_c(u_1)$$
$$\cdots$$

$$\ldots$$

$$\ldots$$

$$Sz_n = (1-\alpha)^s Su_{n-1} + \alpha^s P_c(u_{n-1}), n = 2, 3, \ldots .$$

Then, the general escape criterion is $\max\{|c|, (2(1+|a|)/s\alpha)^{1/(n-1)}, (2(1+|a|)/s\beta)^{1/(n-1)}, (2(1+|a|)/s\gamma)^{1/(n-1)}\}$.

Proof. We shall prove the theorem by using the method of induction. For $n = 1$, we have $P_c(z) = z + c$ and this implies

$$|z| > \max\{|c|, 0, 0, 0\}.$$

For $n = 2$, we have $P_c(z) = z^2 + c$, so by Theorem 3.1, the escape criterion is

$$|z| > \max\{|c|, 2(1+|a|)/s\alpha, 2(1+|a|)/s\beta, 2(1+|a|)/s\gamma\}.$$

Similarly, for n=3, we get $P_c(z) = z^3 + c$. Then, the escape criterion from the Theorem 3.5 is given by

$$|z| > \max\{|c|, (2(1+|a|)/s\alpha)^{1/2}, (2(1+|a|)/s\beta)^{1/2}, (2(1+|a|)/s\gamma)^{1/2}\}.$$

Hence, the theorem is true for $n = 1, 2, 3$. Now, suppose that theorem is true for any n. We shall prove that the result holds for $n+1$. Let us suppose that $P_c(z) = z^{n+1} + c$ and $|z| \geq |c| > (2(1+|a|)/s\alpha)^{1/n}$, $|z| \geq |c| > (2(1+|a|)/s\beta)^{1/n}$ and $|z| \geq |c| > (2(1+|a|)/s\gamma)^{1/n}$.

Then, consider

$$\begin{aligned}
|Sv| &= |(1-\gamma)^s Sz + \gamma^s P_c(z)|, \quad for\ P_c(z) = z^{n+1} + c \\
|Sv| &= |(1-\gamma)^s Sz + \gamma^s (z^{n+1} + c)| \\
&= |(1-\gamma)^s Sz + (1-(1-\gamma))^s (z^{n+1} + c)|.
\end{aligned}$$

By binomial expansion up to linear terms of γ and $(1-\gamma)$, we obtain

$$\begin{aligned}
|Sv| &= |(1-s\gamma)Sz + (1-s(1-\gamma))(z^{n+1} + c)| \\
|av| &= |(1-s\gamma)az + (1-s(1-\gamma))(z^{n+1} + c)| \\
&= |(1-s\gamma)az + (1-s+s\gamma)(z^{n+1} + c)| \\
&\geq |(1-s\gamma)az + s\gamma(z^{n+1} + c)|, && (\because 1-s+s\gamma \geq s\gamma) \\
&\geq |s\gamma z^{n+1} + (1-s\gamma)az| - |s\gamma c| \\
&\geq |s\gamma z^{n+1} + (1-s\gamma)az| - |s\gamma z|, && (\because |z| \geq |c|) \\
&\geq |s\gamma z^{n+1}| - |(1-s\gamma)az| - |s\gamma z| \\
&\geq |s\gamma z^{n+1}| - |az| + |s\gamma az| - |s\gamma z| \\
&\geq |s\gamma z^{n+1}| - |az| - s\gamma|z|, && (\because |a| \geq 0) \\
&\geq |s\gamma z^{n+1}| - |az| - |z|, && (\because s\gamma < 1) \\
&= |s\gamma z^{n+1}| - |z|(|a|+1) \\
&= |z|\{s\gamma|z|^n - (|a|+1)\}.
\end{aligned}$$

Thus,

$$
\begin{aligned}
|a||v| &\geq |z|\{s\gamma|z|^n - (|a|+1)\} \\
|v| &\geq |z|(1+1/|a|)\{s\gamma|z|^n/(|a|+1) - 1\} \\
&\geq |z|\{s\gamma|z^n|/(|a|+1) - 1\}, \\
i.e.,\ |v| &\geq |z|\{s\gamma|z|^n/(|a|+1) - 1\}.
\end{aligned} \tag{5}
$$

Also,

$$
\begin{aligned}
|Su| &= |(1-\beta)^s Sv + \beta^s P_c(v)|, \qquad for\ P_c(v) = v^{n+1} + c \\
|Su| &= |(1-\beta)^s Sv + \beta^s (v^{n+1} + c)| \\
&= |(1-\beta)^s Sv + (1-(1-\beta))^s (v^{n+1} + c)|.
\end{aligned}
$$

By binomial expansion up to linear terms of β and $(1-\beta)$, we obtain

$$
\begin{aligned}
|Su| &= |(1-s\beta)Sv + (1-s(1-\beta))(v^{n+1}+c)| \\
|au| &= |(1-s\beta)av + (1-s(1-\beta))(v^{n+1}+c)| \\
&= |(1-s\beta)av + (1-s+s\beta)(v^{n+1}+c)| \\
&\geq |(1-s\beta)av + s\beta(v^{n+1}+c)|, \qquad (\because 1-s+s\beta \geq s\beta) \\
&\geq |(1-s\beta)a\{|z|(s\gamma|z|^n/(|a|+1)-1)\} + s\beta[\{|z|(s\gamma|z|^n/(|a|+1)-1)\}^{n+1} + c]|.
\end{aligned}
$$

Since $|z| > (2(1+|a|)/s\gamma)^{1/n}$, we have $(s\gamma|z^n|/(|a|+1)-1) > 1$. This gives

$$
\begin{aligned}
|au| &\geq |(1-s\beta)a|z| + s\beta(|z|^{n+1}+c)| \\
&\geq |s\beta z^{n+1} + (1-s\beta)az| - |s\beta c| \\
&\geq |s\beta z^{n+1} + (1-s\beta)az| - |s\beta z|, \qquad (\because |z| \geq |c|) \\
&\geq |s\beta z^{n+1}| - |(1-s\beta)az| - |s\beta z| \\
&\geq |s\beta z^{n+1}| - |az| + |s\beta az| - |s\beta z| \\
&\geq |s\beta z^{n+1}| - |az| - s\beta|z|, \qquad (\because |a| \geq 0) \\
&\geq |s\beta z^{n+1}| - |az| - |z|, \qquad (\because s\beta < 1) \\
&= |s\beta z^{n+1}| - |z|(|a|+1) \\
&= |z|\{s\beta|z|^n - (|a|+1)\}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
|a||u| &\geq |z|\{s\beta|z|^n - (|a|+1)\} \\
|u| &\geq |z|(1+1/|a|)\{s\beta|z|^n/(|a|+1) - 1\} \\
&\geq |z|\{s\beta|z|^n/(|a|+1) - 1\}, \\
i.e.,\ |u| &\geq |z|\{s\beta|z|^n/(|a|+1) - 1\}.
\end{aligned} \tag{6}
$$

Now, for $Sz_n = (1-\alpha)^s Su_{n-1} + \alpha^s P_c(u_{n-1})$, we have

$$
\begin{aligned}
|Sz_1| &= |(1-\alpha)^s Su + \alpha^s P_c(u)| \\
&= |(1-\alpha)^s Su + \alpha^s (u^{n+1}+c)| \\
&= |(1-\alpha)^s Su + (1-(1-\alpha))^s (u^{n+1}+c)|
\end{aligned}
$$

By binomial expansion up to linear terms of α and $(1-\alpha)$, we obtain

$$\begin{aligned}
|Sz_1| &= |(1-s\alpha)Su + (1-s(1-\alpha))(u^{n+1}+c)| \\
|az_1| &= |(1-s\alpha)au + (1-s(1-\alpha))(u^{n+1}+c)| \\
&= |(1-s\alpha)au + (1-s+s\alpha)(u^{n+1}+c)| \\
&\geq |(1-s\alpha)au + s\alpha(u^{n+1}+c)|, \qquad (\because 1-s+s\alpha \geq s\alpha) \\
&\geq |(1-s\alpha)a\{|z|(s\beta|z|^n/(|a|+1)-1)\} + s\alpha[\{|z|(s\beta|z|^n/(|a|+1)-1)\}^{n+1}+c]|.
\end{aligned}$$

Since $|z| > (2(1+|a|)/s\beta)^{1/n}$, we have $(s\beta|z^n|/(|a|+1)-1) > 1$. This gives

$$\begin{aligned}
|az_1| &\geq |(1-s\alpha)az + s\alpha(z^{n+1}c)| \\
&\geq |s\alpha z^{n+1} + (1-s\alpha)az| - |s\alpha z|, \qquad (\because |z| \geq |c|) \\
&\geq |s\alpha z^{n+1}| - |(1-s\alpha)az| - |s\alpha z| \\
&\geq |s\alpha z^{n+1}| - |az| + |s\alpha az| - |s\alpha z| \\
&\geq |s\alpha z^{n+1}| - |az| - s\alpha|z|, \qquad (\because |a| \geq 0) \\
&\geq |s\alpha z^{n+1}| - |az| - |z|, \qquad (\because s\alpha < 1) \\
&= |s\alpha z^{n+1}| - |z|(|a|+1) \\
&= |z|\{s\alpha|z|^n - (|a|+1)\}.
\end{aligned}$$

Thus,

$$\begin{aligned}
|a||z_1| &\geq |z|\{s\alpha|z|^n - (|a|+1)\} \\
|z_1| &\geq |z|(1+1/|a|)\{s\alpha|z|^n/(|a|+1)-1\} \\
&\geq |z|\{s\alpha|z|^n/(|a|+1)-1\}, \\
i.e.,\ |z_1| &\geq |z|\{s\alpha|z|^n/(|a|+1)-1\}.
\end{aligned}$$

Since $|z| \geq |c| > (2(1+|a|)/s\alpha)^{1/n}$, $|z| \geq |c| > (2(1+|a|)/s\beta)^{1/n}$ and $|z| \geq |c| > (2(1+|a|)/s\gamma)^{1/n}$ exist. Therefore, we have $s\alpha|z^n|/(1+|a|)-1 > 1$. Hence there exists a $\lambda > 0$ such that $s\alpha|z^n|/(1+|a|)-1 > \lambda+1 > 1$. Consequently, we have

$$|z_1| > (1+\lambda)|z|.$$

So, applying the same argument n times, we have

$$|z_n| > (1+\lambda)^n|z|.$$

Thus, the orbit of z tends to infinity as n tends to infinity. Hence the result. □

Corollary 3.9. *Assume that* $|c| > (2(1+|a|)/\alpha)^{1/n-1}, |c| > (2(1+|a|)/\beta)^{1/n-1}$ *and* $|c| > (2(1+|a|)/\gamma)^{1/n-1}$ *exists. Then, the orbit* $SP(P_c, 0, \alpha, \beta, \gamma, s)$ *tends to infinity.*

Corollary 3.10. *(Escape Criterion) Let us suppose that for some* $k \geq 0$, $|z_k| > \max\{|c|, (2(1+|a|)/s\alpha)^{1/k-1}, (2(1+|a|)/s\beta)^{1/k-1}, (2(1+|a|)/s\gamma)^{1/k-1}\}$, *then* $|z_k| > \lambda|z_{k-1}|$ *and* $|z_k| \to \infty$ *as* $k \to \infty$.

Using Corollary 3.10, we obtain an algorithm to generate fractals for the functions of the type $G_c(z) = z^n - az + c$; $n = 2, 3, \ldots$.

4. Algorithm for generating Fractals

With the help of general escape criterion, derived in Theorem 3.8, we use the following algorithm to generate all fractals (Mandelbrot sets and Julia sets):

(1) **Setup :**
Choose a complex number $c = l + m\iota$.
Initialize values to variables $\alpha,\ \beta,\ \gamma,\ a,\ s$.
Take $z_0 = x + y\iota$ as first iteration.

(2) **Iterate :**

$$Sz_{n+1} = (1-\alpha)^s Su_n + \alpha^s G_c(u_n),$$
$$Su_n = (1-\beta)^s Sv_n + \beta^s G_c(v_n),$$
$$Sv_n = (1-\gamma)^s Sz_n + \gamma^s G_c(z_n),$$

where $G_c(z_n) = z^n - az + c,\ n = 2, 3, ...$ and $Sz = az$.

(3) **Stop :**
$|z_n| >$ escape radius

$$= \max\left\{|c|, (2(1+|a|)/s\alpha)^{1/n-1}, (2(1+|a|)/s\beta)^{1/n-1}, (2(1+|a|)/s\gamma)^{1/n-1}\right\}.$$

(4) **Count :** number of iterations to escape.

(5) **Color :** point depends on number of iterations required to escape.

Note: To generate Mandelbrot set, we take $z_0 = 0$ as our first iteration while in case of Julia set z_0 is considered non-zero, i.e., $z_0 \neq 0$.

With the help of this algorithm, we make a program in Mathematica 11.0 to generate fractals. To make source program, firstly we use Block construct. Then, we input the values of parameters $\alpha, \beta, \gamma, a, s, l, m$ ended with semicolons. Iterate the given n^{th} degree polynomial $Q_c(z)$ by implementing the general escape criterion derived in Theorem 3.8 to generate fractals of n^{th} degree polynomials ($n = 2, 3, 4, ...$) using while loop. In the last, to plot required fractals, we apply DensityPlot which includes no. of iterations, range of x-axis and y-axis, no. of points and color function.

4.1. Source program to generate Mandelbrot sets using above algorithm.

Here, we provide a source program to obtain Quadratic Mandelbrot set shown in Fig.1.

$iter[x,y, \lim] := Block[\{c, z, ct, \alpha, \beta, \gamma, a, s\},$
$c = x + yI;$
$z = c;$
$\alpha = 0.7;$
$\beta = 0.2;$
$\gamma = 0.4;$
$a = 3;$
$s = 0.1;$
$ct = 0;$

$While[\Big(Abs[z] < Max\left\{Abs[c], \frac{2*(1+Abs[a])}{s*\alpha}, \frac{2*(1+Abs[a])}{s*\beta}, \frac{2*(1+Abs[a])}{s*\gamma}\right\}\Big) \&\&(ct <= \lim), ++ ct;$
$G_c(z) = z^2 - az + c;$
$Su_n = (1-\alpha)^s z + \alpha^s G_c(z);$
$G_c(q) = q^2 - aq + c;$
$Sv_n = (1-\beta)^s q + \beta^s G_c(q);$
$G_c(p) = p^2 - ap + c;$
$Sw_n = (1-\gamma)^s p + \alpha^s G_c(p);$
$return[ct];$
]
$DensityPlot[-iter[x, y, no.of iterations], \{x, xmin, xmax\}, \{y, ymin, ymax\},$
$PlotPoints \to 200, Mesh \to False, ColorFunction \to (Hue[1 - \#]\&)]$

4.2. Source program to generate Julia sets. We obtain Quadratic Julia set, shown in Fig.16 by running the following program in the Mathematica 11.0.

$iter[x,y, \lim] := Block[\{c, z, ct, \alpha, \beta, \gamma, a, s, l, m\},$
$z = x + yI;$
$c = l + mI;$
$\alpha = 0.7;$
$\beta = 0.2;$
$\gamma = 0.4;$
$a = 3;$
$s = 0.1;$
$l = 0.5;$
$m = -0.5;$
$ct = 0;$
$While[\Big(Abs[z] < Max\left\{Abs[c], \frac{2*(1+Abs[a])}{s*\alpha}, \frac{2*(1+Abs[a])}{s*\beta}, \frac{2*(1+Abs[a])}{s*\gamma}\right\}\Big) \&\&(ct <= \lim), ++ ct;$
$G_c(z) = z^2 - az + c;$
$Su_n = (1-\alpha)^s z + \alpha^s G_c(z);$
$G_c(q) = q^2 - aq + c;$
$Sv_n = (1-\beta)^s q + \beta^s G_c(q);$
$G_c(p) = p^2 - ap + c;$
$Sw_n = (1-\gamma)^s p + \alpha^s G_c(p);$
$return[ct];$
]
$DensityPlot[-iter[x, y, no.of iterations], \{x, xmin, xmax\}, \{y, ymin, ymax\},$
$PlotPoints \to 200, Mesh \to False, ColorFunction \to (Hue[1 - \#]\&)]$

Note: We construct all fractals using the same source program only by changing the values of parameters as described in the caption of each figure.

5. Generation of Mandelbrot sets in Jungck SP orbit with s-convexity

Using source program given in Section 4.1, we generate following Mandelbrot sets via Jungck SP orbit with s-convexity by running the program in Mathematica 11.0.

5.1. Mandelbrot sets for quadratic polynomial.

- We generate Quadratic Mandelbrot sets (shown in Figs.1 and 2) by taking polynomial $Q_c(z) = z^2 - 3z + c$ and $P_c(z) = z^2 + c$. We choose the parameters $\alpha = 0.7, \beta = 0.2, \gamma = 0.4$ and vary the value of convex parameter s.
- For the polynomial $Q_c(z) = z^2 - 4z + c$ and $P_c(z) = z^2 + c$. Take the parameters $\alpha = \beta = \gamma = 0.5$ and for different values of s, corresponding Quadratic Mandelbrot sets are presented by Figs. 3 and 4.
- Now for $Q_c(z) = z^2 - z + c$ and $P_c(z) = z^2 + c$. The Quadratic Mandelbrot sets for parameters $\alpha = \beta = \gamma = 0.5$ and for different values of s are shown in Figs. 5 and 6.

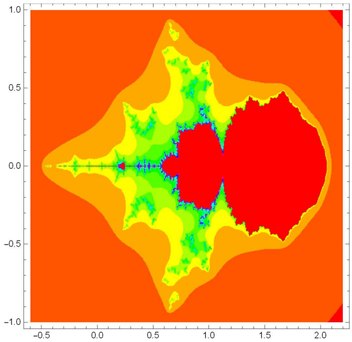

Fig. 1. Mandelbrot set for $\alpha = 0.7, \beta = 0.2, \gamma = 0.4, a = 3, s = 0.1$

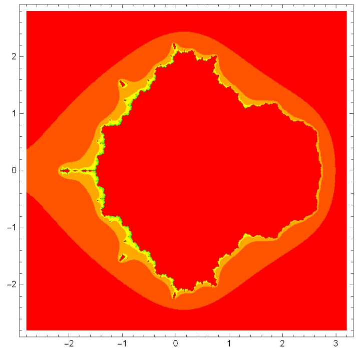

Fig. 2. Mandelbrot set for $\alpha = 0.7, \beta = 0.2, \gamma = 0.4, a = 3, s = 0.8$

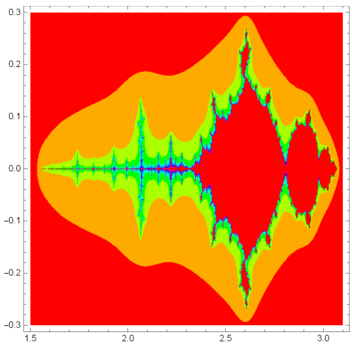

Fig. 3. Mandelbrot set for $\alpha = \beta = \gamma = 0.5, a = 4, s = 0.1$

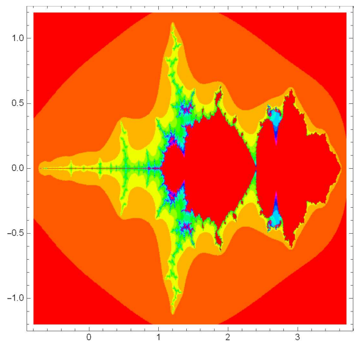

Fig. 4. Mandelbrot set for $\alpha = \beta = \gamma = 0.5, a = 4, s = 0.7$

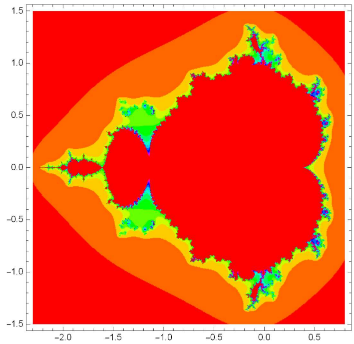

Fig. 5. Mandelbrot set for $\alpha = \beta = \gamma = 0.5, a = 1, s = 0.3$

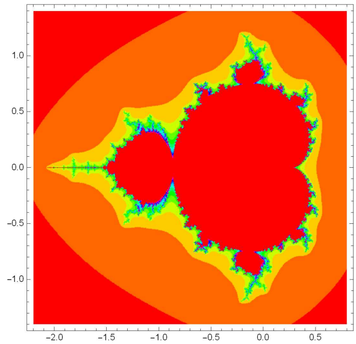

Fig. 6. Mandelbrot set for $\alpha = \beta = \gamma = 0.5, a = 1, s = 0.1$

5.2. Mandelbrot sets for higher degree polynomials.

- We suppose $Q_c(z) = z^3 - 3z + c$ and $P_c(z) = z^3 + c$. The cubic Mandelbrot sets for parameters $\alpha = 0.1, \beta = 0.9, \gamma = 0.5$ and taking the values of s in ascending order are presented by Figs. 7, 8 and 9.
- The Mandelbrot sets for fifth degree polynomial $Q_c(z) = z^5 - 3z + c$, $P_c(z) = z^5 + c$ and for parameters $\alpha = 0.1, \beta = 0.9, \gamma = 0.5$ are shown in Figs. 10 and 11. For the same values of parameters, Mandelbrot set for tenth degree polynomial $Q_c(z) = z^{10} - 3z + c$, $P_c(z) = z^{10} + c$ is presented by Figure 12.
- We generate Mandelbrot sets for higher values of n (see Figs. 13, 14 and 15) by choosing parameters $\alpha = 0.9, \beta = 0.1, \gamma = 0.5$ and $s = 0.7$.

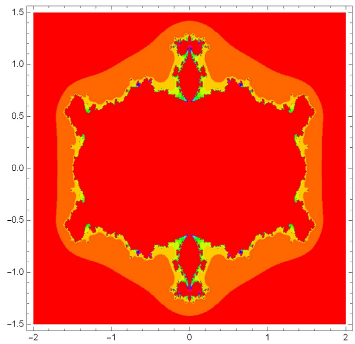

Fig. 7. Mandelbrot set for $\alpha = 0.1, \beta = 0.9, \gamma = 0.5, s = 0.1, a = 3, n = 3$

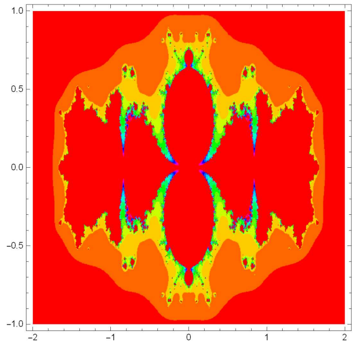

Fig. 8. Mandelbrot set for $\alpha = 0.1, \beta = 0.9, \gamma = 0.5, s = 0.7, a = 3, n = 3$

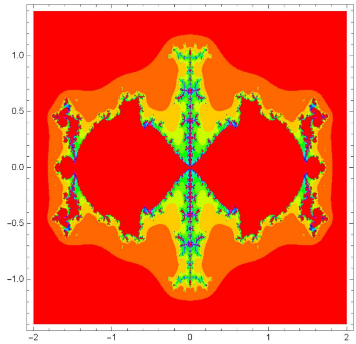

Fig. 9. Mandelbrot set for $\alpha = 0.1, \beta = 0.9, \gamma = 0.5, s = 1, a = 3, n = 3$

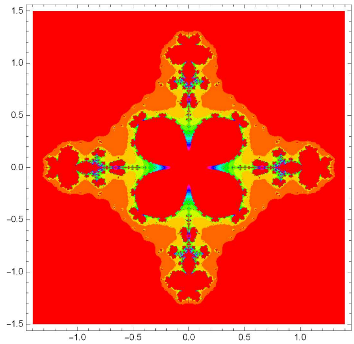

Fig. 10. Mandelbrot set for $\alpha = 0.1, \beta = 0.9, \gamma = 0.5, s = 0.7, a = 3$ and $n = 5$

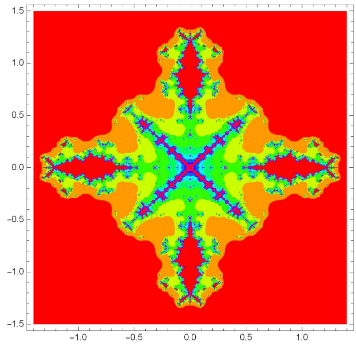

Fig. 11. Mandelbrot set for $\alpha = 0.1, \beta = 0.9, \gamma = 0.5, s = 1, a = 3$ and $n = 5$

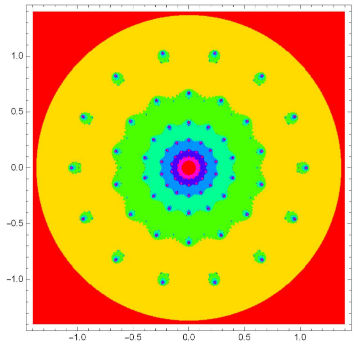

Fig. 12. Mandelbrot set for $\alpha = 0.1, \beta = 0.9, \gamma = 0.5, s = 1, a = 3$ and $n = 15$

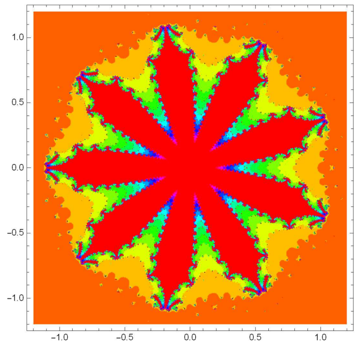

Fig. 13. Mandelbrot set for $\alpha = 0.9, \beta = 0.1, \gamma = 0.5, s = 0.7, a = 3$ and $n = 10$

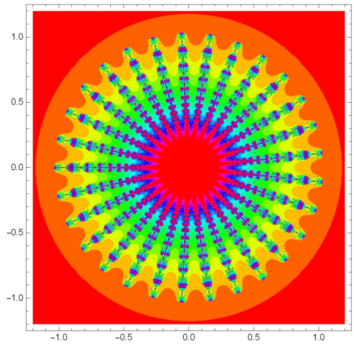

Fig. 14. Mandelbrot set for $\alpha = 0.9, \beta = 0.1, \gamma = 0.5, s = 0.7, a = 3$ and $n = 30$

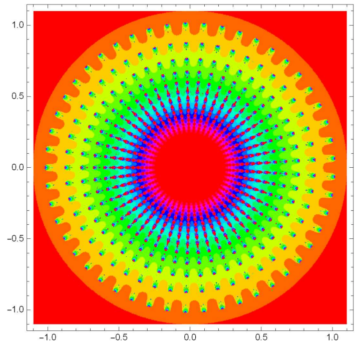

Fig. 15. Mandelbrot set for $\alpha = 0.9, \beta = 0.1, \gamma = 0.5, s = 0.7, a = 3$ and $n = 50$

6. Generation of Julia sets in Jungck SP orbit with s-convexity

By running the source program given in Section 4.2, we generate following Julia sets via Jungck SP orbit with s-convexity in Mathematica 11.0.

6.1. Julia sets for quadratic polynomial $Q_c(z) = z^2 - az + c$ and $P_c(z) = z^2 + c$.

- We generate Quadratic Julia sets for the parameters $\alpha = 0.3, \beta = 0.1, \gamma = 0.6, c = 0.5 - 0.5\iota$ and $a = 1, s = 0.2$ (see Fig. 16) and for $a = 3, s = 0.8$ (Fig. 17).
- By choosing $\alpha = 0.2, \beta = 0.1, \gamma = 0.9, c = 0.5 - 0.2\iota$ and $a = 3$, Quadratic Julia sets for different values of s are shown in Figs. 18 and 19.
- Now, for $\alpha = 0.1, \beta = 0.4, \gamma = 0.9, c = 2.5 - 0.9\iota$ and $a = (1 + \sqrt{6})$. Quadratic Julia sets for $s = 0.2$ and $s = 0.5$ are shown in Figs. 20 and 21 respectively. A totally disconnected Julia set is presented by Fig. 22 for $\alpha = 0.3, \beta = 0.9, \gamma = 0.4, c = 2.5 - 0.9\iota$ and $a = (1 + \sqrt{6})$.

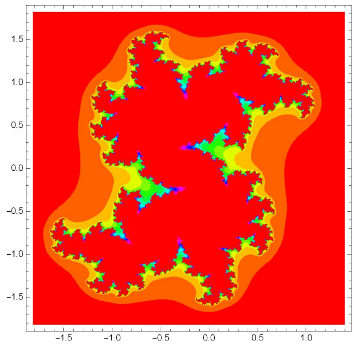

Fig. 16. Quadratic Julia set for $\alpha = 0.3, \beta = 0.1, \gamma = 0.6, c = 0.5 - 0.5\iota, a = 1, s = 0.2$

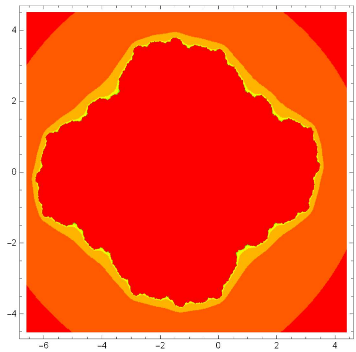

Fig. 17. Quadratic Julia set for $\alpha = 0.3, \beta = 0.1, \gamma = 0.6, c = 0.5 - 0.5\iota, a = 3, s = 0.8$

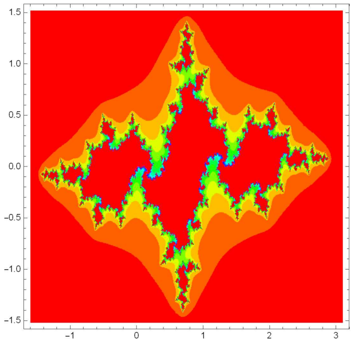

Fig. 18. Quadratic Julia set for $\alpha = 0.2, \beta = 0.1, \gamma = 0.9, c = 0.5 - 0.2\iota, a = 3, s = 0.2$

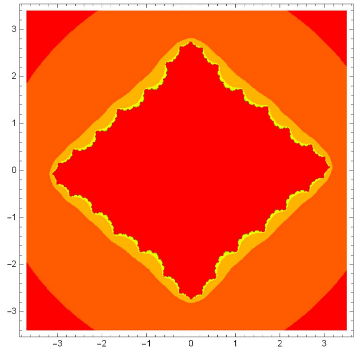

Fig. 19. Quadratic Julia set for $\alpha = 0.2, \beta = 0.1, \gamma = 0.9, c = 0.5 - 0.2\iota, a = 3, s = 0.5$

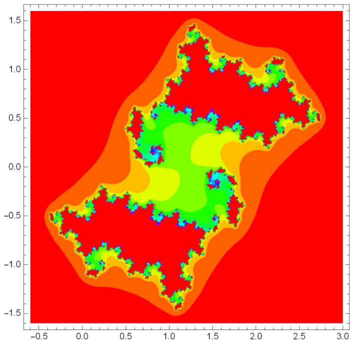

Fig. 20. Quadratic Julia set for $\alpha = 0.1, \beta = 0.4, \gamma = 0.9, c = 2.5 - 0.9\iota, a = (1 + \sqrt{6}), s = 0.2$

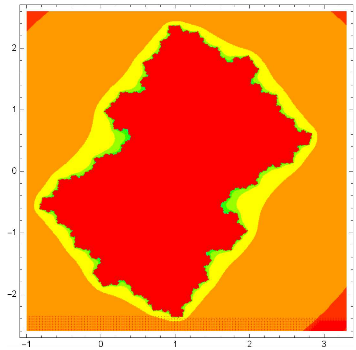

Fig. 21. Quadratic Julia set for $\alpha = 0.1, \beta = 0.4, \gamma = 0.9, c = 2.5 - 0.9\iota, a = (1 + \sqrt{6}), s = 0.5$

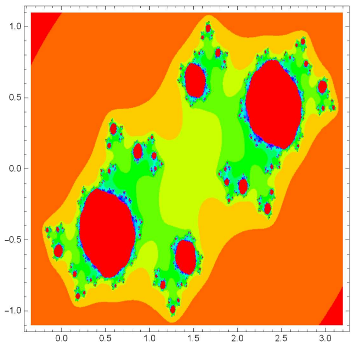

Fig. 22. Quadratic Julia set for $\alpha = 0.3, \beta = 0.9, \gamma = 0.4, c = 2.5 - 0.9\iota, a = (1 + \sqrt{6}), s = 0.3$

6.2. Julia sets for higher degree polynomials.

- The cubic Julia sets for polynomial $Q_c(z) = z^3 - az + c$, where $P_c(z) = z^3 + c$ and for parameters $\alpha = 0.1, \beta = 0.4, \gamma = 0.1, c = -0.5 + 0.09\iota$, $a = (1 + \sqrt{6})$ are presented by Figs. 23 and 24 for $s = 0.2$ and $s = 0.5$ respectively.

- Taking the same polynomials as above, the cubic Julia set for $\alpha = 0.1, \beta = 0.4, \gamma = 0.9, c = -0.5 + 0.09\iota$ and $a = (1 + \sqrt{6})$ is shown in Fig. 25. Further, we generate cubic Julia sets for $\alpha = 0.3, \beta = 0.1, \gamma = 0.5, c = 0.2 - 0.2\iota$, $a = 3$ by using different values of parameter s (see Figs. 26 and 27).
- We generate fifth order Julia sets (see Figs. 28 and 29) by considering $\alpha = 0.5, \beta = 0.5, \gamma = 0.5, c = 0.1 - 0.1\iota$, $a = 2$ for different values of s. A Julia set of fifteenth order is presented for $\alpha = 0.7, \beta = 0.7, \gamma = 0.7, c = 0.2 - 0.2\iota$, $a = 2$ by Fig. 30.

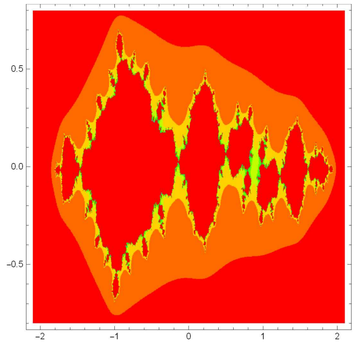

Fig. 23. Cubic Julia set for $\alpha = 0.1, \beta = 0.4, \gamma = 0.1, c = -0.5 + 0.09\iota$, $a = (1 + \sqrt{6}), s = 0.2$

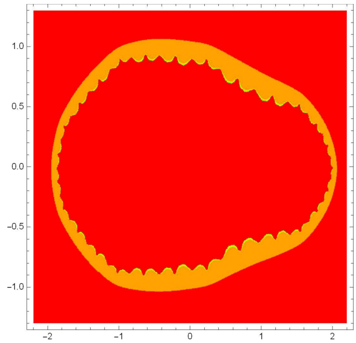

Fig. 24. Cubic Julia set for $\alpha = 0.1, \beta = 0.4, \gamma = 0.1, c = -0.5 + 0.09\iota$, $a = (1 + \sqrt{6}), s = 0.5$

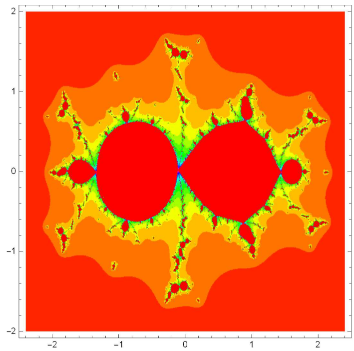

Fig. 25. Cubic Julia set for $\alpha = 0.1, \beta = 0.4, \gamma = 0.9, c = -0.5 + 0.09\iota, a = (1 + \sqrt{6}), s = 1$

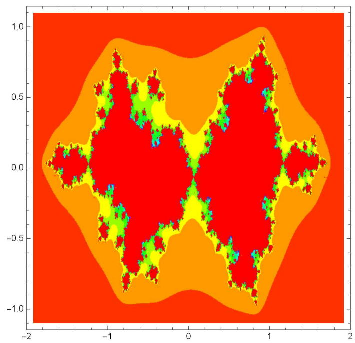

Fig. 26. Cubic Julia set for $\alpha = 0.3, \beta = 0.1, \gamma = 0.5, c = 0.2 - 0.2\iota, a = 3, s = 0.2$

7. Conclusion

In this experimental study, we generate fractals in Jungck SP orbit with s-convexity and derive following results:

(1) We use maximum number of 30 iterations to generate these fractals.

(2) It is observed that the structure of Mandelbrot sets and Julia sets varies with the variation of convex parameter s.

(3) As the value of parameter s comes closer to 1, the number of attached tiny bulbs increases and main body of fractals becomes fattier.
(4) Some fractals generated by us may be useful for graphic designers (see Figs. 11-15).
(5) Our algorithm and source programs enable one to generate more beautiful fractals.

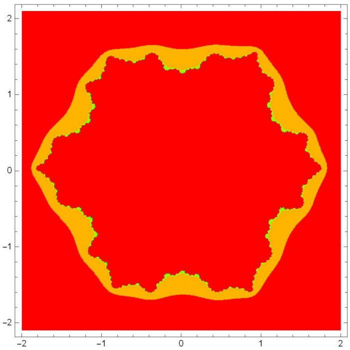

Fig. 27. Cubic Julia set for $\alpha = 0.3, \beta = 0.1, \gamma = 0.5, c = 0.2 - 0.2\iota, a = 3, s = 0.5$

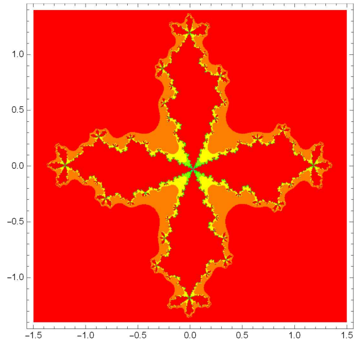

Fig. 28. Julia set for $\alpha = 0.5, \beta = 0.5, \gamma = 0.5, c = 0.1 - 0.1\iota, a = 2, s = 0.6$ and $n = 5$

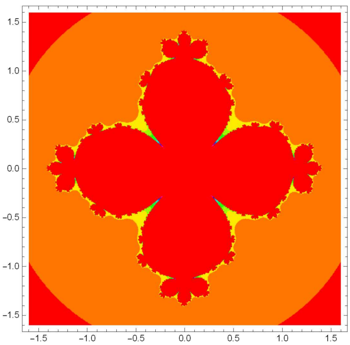

Fig. 29. Julia set for $\alpha = 0.5, \beta = 0.5, \gamma = 0.5, c = 0.1 - 0.1\iota, a = 2, s = 1$ and $n = 5$

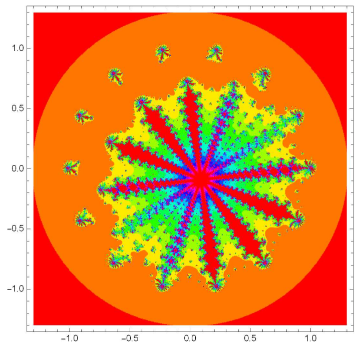

Fig. 30. Julia set for $\alpha = 0.7, \beta = 0.7, \gamma = 0.7, c = 0.2 - 0.2\iota, a = 2, s = 0.2$ and $n = 15$

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