Annals of the University of Craiova, Mathematics and Computer Science Series
Volume 45(2), 2018, Pages 258–262
ISSN: 1223-6934

A quenching result of weak solutions of semi-linear parabolic equations

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Abstract. We prove a global existence of weak solutions of semi-linear parabolic equations with a singular absorption. In particular, we show that any weak solution vanishes after a finite time under some circumstances. Our results improve the one of Montenegro, [16]

2010 Mathematics Subject Classification. 35K55, 35K67, 35K65.
Key words and phrases. Global existence, quenching type of parabolic equations.

1. Introduction

In this paper, we study nonnegative solutions of the following equation:

$$\begin{cases} \partial_t u - \Delta u + u^{-\beta}\chi_{\{u>0\}} = \lambda f(u) & \text{in } \Omega \times (0,T), \\ u(x,t) = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $\beta \in (0,1)$, $\lambda > 0$, and $\chi_{\{u>0\}}$ is the characteristic function, i.e:

$$\chi_{\{u>0\}} = \begin{cases} 1, & \text{if } u > 0, \\ 0, & \text{if } u \leq 0. \end{cases}$$

It is clear that the absorption term $u^{-\beta}\chi_{\{u>0\}}$ becomes singular when u is near to 0, and we impose $u^{-\beta}\chi_{\{u>0\}} = 0$ whenever $u = 0$. Through this paper, f is assumed a locally Lipschitz nonnegative function, and $f(0) = 0$.

Problem (1) can be considered as a limit of mathematical models describing enzymatic kinetics or the Langmuir-Hinshelwood model of the heterogeneous chemical catalyst, and it has been studied by the authors in [17], [12], [15], [9], [6], [5], [7], [8], [10], [17] and references therein. These authors have considered the existence and uniqueness, and the qualitative behavior of solutions of equation (1). One of the most interesting behaviors of solutions of equation (1) is the quenching phenomenon that any solution of equation (1) vanishes after a finite time, even begin with a positive initial data. Let us remind some results involving the quenching phenomenon. When $f = 0$, Phillips [17] showed that any solution is extinct after a finite time. Moreover, Montenegro [16] considered the case where f is sublinear. The author proved that there is a real number $\lambda_0 > 0$ such that for any $\lambda \in (0, \lambda_0)$, there is $t_0 > 0$ such that

$$u(x,t_0) = 0, \quad \forall x \in \Omega.$$

Received May 29, 2018. Accepted December 5, 2018.

And he called this phenomenon complete quenching. In this paper, we show that the complete quenching result still holds for a general $f(u)$ provided either small initial data, or small λ. We emphasize that our approach is different from the one used by Montenegro [16]. Then our complete quenching results are as follows:

Theorem 1.1. *For a given $\lambda > 0$, then every solution of* (1) *quenches after a finite time if $\|u_0\|_\infty$ is small enough.*

Theorem 1.2. *For a given $u_0 \in L^\infty(\Omega)$, there is a real number $\lambda_0 > 0$ such that every solution of equation* (1) *quenches after a finite time if $\lambda \in (0, \lambda_0)$.*

Note that the nonlinear source $f(u)$ may cause the blowing-up of solutions of (1). Thus, in order to prove Theorem 1.1 and Theorem 1.2, it is necessary to show first the global existence of solutions. Then, we have the following theorem.

Theorem 1.3. *Let $u_0 \in L^\infty(\Omega)$, and $\lambda > 0$. Assume that there is an open bounded domain Ω_0, and a positive real number k_0 such that $\Omega \subset\subset \Omega_0$, and*

$$\begin{cases} u_0(x) \leq k_0 \Phi_{\Omega_0}(x), \quad \text{for a.e } x \in \Omega, \\ \\ k_0 \lambda_{\Omega_0} \Phi_{\Omega_0}(x) + k_0^{-\beta} \Phi_{\Omega_0}^{-\beta}(x) \geq \lambda f(k_0 \Phi_{\Omega_0}(x)), \quad \forall x \in \Omega. \end{cases} \tag{2}$$

Then, any solution u of equation (1) *exists globally and satisfies*

$$u(x,t) \leq k_0 \Phi_{\Omega_0}(x), \quad \text{in } \Omega \times (0, \infty), \tag{3}$$

where λ_{Ω_0} (resp. Φ_{Ω_0}) is the first eigenvalue (resp. the first eigenfunction) of the following problem:

$$\begin{cases} -\Delta \Phi_{\Omega_0} = \lambda_{\Omega_0} \Phi_{\Omega_0} & \text{in } \Omega_0, \\ \Phi_{\Omega_0}(x) = 0, & \text{on } \partial\Omega_0. \end{cases} \tag{4}$$

Remark 1.1. For a given $\lambda > 0$; if u_0 is small enough then (2) holds, and conversely. Thus, we obtain the global existence of solutions if either u_0 or λ is small enough.

Before giving the proof of the above Theorem, let us define a weak solution of (1).

Definition 1.1. Let $0 \leq u_0 \in L^\infty(\Omega)$. A nonnegative function $u(x,t)$ is called a weak solution of equation (1) if $u^{-\beta}\chi_{\{u>0\}} \in L^1(\Omega \times (0,T))$, and $u \in L^2(0,T; W_0^{1,2}(\Omega)) \cap L^\infty(\Omega \times (0,T)) \cap \mathcal{C}([0,T); L^1(\Omega))$ satisfies equation (1) in the sense of distributions $\mathcal{D}'(\Omega \times (0,T))$, i.e:

$$\int_0^T \int_\Omega \left(-u\phi_t + \nabla u.\nabla \phi + u^{-\beta}\chi_{\{u>0\}}\phi - f(u,x,t)\phi\right) dxdt = 0, \quad \forall \phi \in \mathcal{C}_c^\infty(\Omega \times (0,T)).$$

Next, we recall a local existence of solution of (1) proved by Dao et al. [4].

Theorem 1.4. *Let $0 \leq u_0 \in L^\infty(\Omega)$. Then, there exists a finite time $T = T(u_0) > 0$ such that equation* (1) *has a maximal weak solution u in $\Omega \times (0,T)$. Moreover, there is a positive constant $C = C(f, \|u_0\|_\infty)$ such that*

$$|\nabla u(x,\tau)|^2 \leq C u^{1-\beta}\left(\tau^{-1} + 1\right), \quad \text{for a.e } (x,\tau) \in \Omega \times (0,T). \tag{5}$$

Besides, if $\nabla(u_0^{\frac{1}{\gamma}}) \in L^\infty(\Omega)$, with $\gamma = \frac{2}{1+\beta}$, then there is a positive constant $C = C(f, u_0)$ such that

$$|\nabla u(x,\tau)|^2 \leq C u^{1-\beta}(x,\tau), \quad \text{for a.e } (x,\tau) \in \Omega \times (0,T). \tag{6}$$

2. Global existence and complete quenching phenomenon of solutions

In this part, we study the global existence and the complete quenching phenomenon of solution through proving Theorem 1.1, 1.2, 1.3. Since u is the maximal solution, it is then sufficient to consider u.

Proof of Theorem 1.3: Let us first recall an approximating equation of (1):

$$(P_\varepsilon)\begin{cases} \partial_t u_\varepsilon - \Delta u_\varepsilon + g_\varepsilon(u_\varepsilon) = \lambda f(u_\varepsilon) & \text{in } \Omega\times(0,\infty),\\ u_\varepsilon = 0 & \text{on } \partial\Omega\times(0,\infty),\\ u_\varepsilon(0)=u_0 & \text{on } \Omega,\end{cases}$$

where $g_\varepsilon(s) = s^{-\beta}\psi_\varepsilon(s)$, with ψ_ε is a cut-off function with its value 0 in a neighborhood of $s=0$ to avoid the singularity of $s^{-\beta}$ at $s=0$, see [4], [5].
Now, let u be the maximal solution of equation (1) in $\Omega\times(0,T)$. To prove that u exists globally, we show that u is bounded by a constant not depending on t.
We first remind that $\inf\limits_{x\in\Omega}\{\Phi_{\Omega_0}(x)\} > 0$, for every $\Omega \subset\subset \Omega_0$.
Thus, for any $\varepsilon \in \left(0, \frac{1}{2}\inf\limits_{x\in\Omega}\{k_0\Phi_{\Omega_0}(x)\}\right)$, we have $g_\varepsilon(k_0\Phi_{\Omega_0}) = k_0^{-\beta}\Phi_{\Omega_0}^{-\beta}$.
Put

$$\mathcal{L}_\varepsilon(v) := v_t - \Delta v + g_\varepsilon(v) - \lambda f(v).$$

Then,

$$\mathcal{L}_\varepsilon(k_0\Phi_{\Omega_0}) = k_0\lambda_1(\Omega_0)\Phi_{\Omega_0} + k_0^{-\beta}\Phi_{\Omega_0}^{-\beta} - \lambda f(k_0\Phi_{\Omega_0}). \tag{7}$$

By (2), we observe that $\mathcal{L}_\varepsilon(k_0\Phi_{\Omega_0}) \ge 0$, in $\Omega\times(0,T)$. Therefore, $k_0\Phi_{\Omega_0}$ is a super-solution of equation (P_ε). The strong comparison theorem yields

$$u_\varepsilon(x,t) \le k_0\Phi_{\Omega_0}(x), \quad \forall (x,t)\in\Omega\times(0,T).$$

Since u_ε is bounded on $\Omega\times(0,T)$, the standard argument allows us to extend the existence of u_ε on $\Omega\times(0,T+\delta_0)$, $\Omega\times(0,T+2\delta_0)$,..., for some $\delta_0>0$, thereby proves the global existence of solution u_ε. Furthermore, we note that u_ε converges decreasingly to u above when $\varepsilon\to 0$, see [4]. Thus, u must exist globally, and the conclusion (3) follows immediately. □

Next, fixed $\lambda > 0$ we show that the maximal solution u must vanish identically after a finite time if $\|u_0\|_\infty$ is small enough.

Proof of Theorem 1.1. Since $\|u_0\|_\infty$ is small enough, we can choose a real number $k_0 > 0$ small as well, and an open bounded domain Ω_0 containing Ω, such that (2) holds. Thanks to Theorem 1.3, the maximal solution u exists globally, and it is bounded by $\alpha = k_0\sup\limits_{x\in\Omega}\{\Phi_{\Omega_0}(x)\}$. Note that α is as small as $\|u_0\|_\infty$.

Using the test function u to equation (1) gives us

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega u^2(t)dx + \int_\Omega\left(|\nabla u(t)|^2 + u^{1-\beta}(t)\right)dx =& \lambda\int_\Omega f(u)u\ dx\\ \le& \lambda\alpha^\beta \max_{0\le u\le\alpha}\{f(u)\}\int_\Omega u^{1-\beta}(t)dx.\end{aligned}$$

Or

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2(t)dx + \int_\Omega |\nabla u(t)|^2 dx + (1-c)\int_\Omega u^{1-\beta}(t)dx \leq 0, \tag{8}$$

where $c = \lambda\alpha^\beta \max_{0\leq u\leq\alpha}\{f(u)\}$ tends to 0 as $\alpha \to 0$. Thus, $(1-c_\alpha) > c_0 > 0$, provided that $\|u_0\|_\infty$ is small enough.
It follows from (8) that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2(t)dx + c_0\int_\Omega \left(|\nabla u(t)|^2 dx + u^{1-\beta}(t)\right)dx \leq 0. \tag{9}$$

Now, using Garliardo-Nirenberg's inequality yields

$$\|u(t)\|_{L^2(\Omega)} \leq c\|\nabla u(t)\|^\theta_{L^2(\Omega)}\|u(t)\|^{1-\theta}_{L^1(\Omega)} = c\left(\int_\Omega |\nabla u(t)|^2 dx\right)^{\frac{\theta}{2}}\left(\int_\Omega u(t)dx\right)^{1-\theta},$$

with $\theta = \frac{N+1}{N+2}$. Thus,

$$\|u(t)\|_{L^2(\Omega)} \leq c\left(\int_\Omega \left(|\nabla u(t)|^2 + u(t)\right)dx\right)^{\frac{\theta}{2}+1-\theta}$$

$$\leq c\left(\int_\Omega \left(|\nabla u(t)|^2 + \alpha^\beta u^{1-\beta}(t)\right)dx\right)^{1-\theta/2} \leq c_1\left(\int_\Omega \left(|\nabla u(t)|^2 + u^{1-\beta}(t)\right)dx\right)^{1-\theta/2},$$

with $c_1 = c_1(\beta,\alpha,N) > 0$.
Then, we obtain

$$\left(\int_\Omega u^2(t)dx\right)^{\frac{N+2}{N+3}} \leq c_2\int_\Omega \left(|\nabla u(t)|^2 + u^{1-\beta}(t)\right)dx, \tag{10}$$

with $c_2 = c_2(\beta,\alpha,N) > 0$.
By (9) and (10), there is a positive constant $c_3 = c_3(\beta,\alpha,N) > 0$ such that

$$y'(t) + c_3 y^{\frac{N+2}{N+3}}(t) \leq 0, \quad \text{for } t > 0, \tag{11}$$

with $y(t) = \int_\Omega |u(x,t)|^2 dx$.

If we can show that there exists a time $t_0 \in [0,\infty)$ such that $y(t_0) = 0$, it follows then from (11) that $y(t) = 0$, for any $t \geq t_0$, thereby proves Theorem 1.1.
Indeed, we assume a contradiction that $y(t) > 0$, for any $t > 0$. Solving the ordinary differential inequality (11) yields

$$y^{\frac{1}{N+3}}(t) + c_4 t \leq \|u_0\|^{\frac{2}{N+3}}_{L^2(\Omega)}, \quad c_4 = \frac{c_3}{N+3}, \quad \forall t > 0, \tag{12}$$

which leads to a contradiction as t is sufficiently large. Thus, u must quench after a finite time. □

Similarly, we obtain the complete quenching result for Theorem 1.2.

Remark 2.1. It follows from (12) that the extinction time of u, denoted by

$$T^\star \leq \frac{\|u_0\|^{\frac{2}{N+3}}_{L^2(\Omega)}}{c_4}.$$

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