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Numerical simulations in the study of frictionless viscoelastic contact problems

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Abstract. The frictionless contact problem between a viscoelastic body and a deformable foundation is considered. The process is assumed to be quasistatic and the contact is modelled with normal compliance. A fully discrete numerical scheme is presented for the model and implemented in a computer code. It is based on the finite element method and the backward's Euler scheme. Numerical simulations in 1D and 2D test problems are shown.

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1. Introduction

Phenomena of contact between deformable bodies abound in industry and everyday life. Contact of braking pads with wheels, tires with roads, pistons with skirts are just a few simple examples. Because of the importance of contact processes in structural and mechanical systems, a considerable effort has been made in its modelling and numerical simulations. The literature concerning this topic is extensive, see for instance the references in [4].

The quasistatic frictionless contact problem between a viscoelastic body and a deformable obstacle was considered in [8]. There, the contact was modeled with normal compliance, that is, the penetration of the body into the foundation is allowed but penalized; an existence and uniqueness result was obtained by using arguments of variational inequalities and fixed point. The numerical analysis of this model was provided in [3]; there, semi-discrete and fully discrete schemes were considered and error estimates were derived.

In this paper we deal with the frictionless version of the contact problem studied in [8]. The analysis of this problem, including the existence of a unique weak solution and error estimates for the approximation schemes, is a straightforward consequence of the results obtained in the references above. For this reason we only recall it and focus on the numerical simulations of the model, which represent the main goal of this work.

The paper is structured as follows. In Section 2 we present the mechanical problem as well as its variational formulation. In Section 3 we describe a fully discrete approximation scheme for the model, based on the finite element method and the backward's Euler scheme. Our main interest lies in Section 4 where we present numerical simulations in the study of one- and two-dimensional test problems.

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2. Mechanical problem and variational formulation

The physical setting is the following. A viscoelastic body occupies a regular domain $\Omega \subset \mathbb{R}^d$ $(d = 1, 2, 3)$ with the boundary Γ partitioned into three disjoint measurable parts Γ_1, Γ_2 and Γ_3 such that meas $(\Gamma_1) > 0$. We are interested in the evolution process of the mechanical state of the body in the time interval $[0, T]$ with $T > 0$. The body is clamped on Γ_1 and so the displacement field vanishes there. Surface tractions of density $\boldsymbol{f}_2$ act on Γ_2 and volume forces of density $\boldsymbol{f}_0$ act in Ω. We assume that the forces and tractions change slowly in time so that the acceleration of the system is negligible. The body may come in frictionless contact with an obstacle, called the foundation. There is a gap g between the potential contact surface Γ_3 and the foundation, measured along the direction of the unit outward normal vector $\boldsymbol{\nu}$.

Under these conditions, the classical formulation of the mechanical problem is the following:

Problem P**.** *Find a displacement field* $\boldsymbol{u} : \Omega \times [0, T] \to \mathbb{R}^d$ *and a stress field* $\boldsymbol{\sigma} : \Omega \times [0, T] \to S_d$ *such that*

$$\boldsymbol{\sigma} = \mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) + \mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}) \quad \text{in} \quad \Omega \times (0, T), \tag{1}$$

$$\mathrm{Div}\, \boldsymbol{\sigma} + \boldsymbol{f}_0 = \boldsymbol{0} \quad \text{in} \quad \Omega \times (0, T), \tag{2}$$

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on} \quad \Gamma_1 \times (0, T), \tag{3}$$

$$\boldsymbol{\sigma\nu} = \boldsymbol{f}_2 \quad \text{on} \quad \Gamma_2 \times (0, T), \tag{4}$$

$$-\sigma_\nu = p_\nu(u_\nu - g), \quad \boldsymbol{\sigma}_\tau = \boldsymbol{0} \quad \text{on} \quad \Gamma_3 \times (0, T), \tag{5}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0 \quad \text{in} \quad \Omega. \tag{6}$$

Here S_d represents the space of second-order symmetric tensors on $\mathbb{R}^d$. The relation (1) is the viscoelastic constitutive law in which $\mathcal{A}$ and $\mathcal{B}$ are given nonlinear operators, called the *viscosity* and *elasticity* operators, respectively. As usual, $\boldsymbol{\varepsilon}(\boldsymbol{u})$ denotes the infinitesimal strain tensor and the dot above represents the derivative with respect to the time variable. Relation (2) represents the equilibrium equation in which Div denotes the divergence operator, and (3) and (4) are the displacement-traction boundary conditions. The function $\boldsymbol{u}_0$ in (6) denotes the initial displacement.

The first equality in (5) represents the *normal compliance* contact condition where u_ν is the normal displacement, σ_ν denotes the normal stress, and p_ν is a prescribed function. Here, $u_\nu - g$, when positive, represents the penetration of the surface asperities into those of the foundation. As an example of normal compliance function p_ν we may consider

$$p_\nu(r) = c_\nu r_+ \tag{7}$$

where c_ν is a positive constant and $r_+ = \max\{0, r\}$. Formally, Signorini's nonpenetration condition is obtained in the limit $c_\nu \to \infty$. The normal compliance contact condition was introduced in [7] and used in a large number of papers, see e.g. [2, 5, 6, 8] and the references therein. Finally, in (5), $\boldsymbol{\sigma}_\tau$ represents the tangential stress which is assumed to be zero, since frictionless contact is being considered.

To study Problem P we introduce the function spaces

$$V = \{\boldsymbol{v} = (v_i) \in [H^1(\Omega)]^d \,;\, \boldsymbol{v} = \boldsymbol{0} \quad \text{on} \quad \Gamma_1\},$$
$$Q = \{\boldsymbol{\tau} = (\tau_{ij})_{i,j=1}^d \in [L^2(\Omega)]^{d\times d} \,;\, \tau_{ij} = \tau_{ji},\ 1 \le i, j \le d\}.$$

These are real Hilbert spaces endowed with their canonical inner products denoted $(\cdot,\cdot)_V$ and $(\cdot,\cdot)_Q$, respectively. We denote by $\boldsymbol{f}(t)$ the element of V given by

$$(\boldsymbol{f}(t),\boldsymbol{v})_V = \int_\Omega \boldsymbol{f}_0(t)\cdot\boldsymbol{v}\,dx + \int_{\Gamma_2} \boldsymbol{f}_2(t)\cdot\boldsymbol{v}\,da$$

for all $\boldsymbol{v}\in V$ and $t\in[0,T]$, and let $j : V\times V \to \mathbb{R}$ be the functional defined by

$$j(\boldsymbol{v},\boldsymbol{w}) = \int_{\Gamma_3} p_\nu(v_\nu - g)w_\nu\,da \quad \forall\,\boldsymbol{v},\boldsymbol{w}\in V.$$

Here and below the index ν denotes the normal component of vectors in V. Using Green's formula we obtain the following variational formulation of Problem P in terms of the displacement field:

Problem P_V**.** *Find a displacement field* $\boldsymbol{u} : \Omega\times[0,T]\to V$ *such that* $\boldsymbol{u}(0) = \boldsymbol{u}_0$ *and*

$$\begin{aligned}(\mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)),\boldsymbol{\varepsilon}(\boldsymbol{v}))_Q + (\mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)),\boldsymbol{\varepsilon}(\boldsymbol{v}))_Q + j(\boldsymbol{u}(t),\boldsymbol{v})&\\ = (\boldsymbol{f}(t),\boldsymbol{v})_V \quad \forall\,\boldsymbol{v}\in V,\, t\in[0,T].&\end{aligned}$$

Problem P_V represents a special case of the mathematical model treated in [8]. Therefore, keeping in mind the results in [8], under appropriate assumptions on the data, we can state the existence and the uniqueness of a solution for Problem P_V.

3. Numerical approximation

In this section we consider a fully discrete approximation scheme for Problem P_V. Let us denote by $V^h\subset V$ an arbitrary finite dimensional subspace of V, where $h>0$ is a discretization parameter. Assume Ω is a polyhedral domain, and $\mathcal{T}^h$ is a finite element triangulation compatible with the boundary partition $\Gamma = \Gamma_1\cup\Gamma_2\cup\Gamma_3$. In our numerical simulations, we choose V^h to be the finite element space of continuous piecewise affine functions:

$$V^h = \{\boldsymbol{v}^h\in[\mathcal{C}(\bar\Omega)]^d\,;\,\boldsymbol{v}^h|_T\in[P_1(T)]^d \quad \forall T\in\mathcal{T}^h,\quad \boldsymbol{v}^h=\boldsymbol{0}\quad\text{on}\quad\Gamma_1\}. \tag{8}$$

For the time discretization we use a general partition of the time interval $[0,T] : 0 = t_0 < t_1 < \ldots < t_N = T$. Denote the step size $k_n = t_n - t_{n-1}$ for $n = 1,\ldots,N$ and let $k = \max_n k_n$ be the maximal step size. For a continuous function $\boldsymbol{w}(t)$, we use the notation $\boldsymbol{w}_n = \boldsymbol{w}(t_n)$. For a sequence $\{\boldsymbol{w}_n\}_{n=0}^N$ we denote $\Delta\boldsymbol{w}_n = \boldsymbol{w}_n - \boldsymbol{w}_{n-1}$ for the difference and $\delta\boldsymbol{w}_n = \Delta\boldsymbol{w}_n/k_n$ for the corresponding divided difference. In the following, c denotes a generic positive constants independent of the discretization parameters h and k.

Let $\boldsymbol{u}_0^h\in V^h$ be chosen to approximate the initial value $\boldsymbol{u}_0$. A fully discrete approximation scheme for Problem P_V is the following:

Problem P_V^{hk}**.** *Find a displacement field* $\boldsymbol{u}^{hk} = \{\boldsymbol{u}_n^{hk}\}_{n=0}^N\subset V^h$ *such that* $\boldsymbol{u}_0^{hk} = \boldsymbol{u}_0^h$*, and for* $n = 1,\ldots,N$*,*

$$(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_n^{hk}),\boldsymbol{\varepsilon}(\boldsymbol{w}^h))_Q + (\mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}),\boldsymbol{\varepsilon}(\boldsymbol{w}^h))_Q + j(\boldsymbol{u}_{n-1}^{hk},\boldsymbol{w}^h) = (\boldsymbol{f}_n,\boldsymbol{w}^h)_V \quad \forall\,\boldsymbol{w}^h\in V^h,$$

where $\boldsymbol{v}_n^{hk} = \delta\boldsymbol{u}_n^{hk} = (\boldsymbol{u}_n^{hk} - \boldsymbol{u}_{n-1}^{hk})/k_n$ *is the discrete velocity field.*

By using arguments similar to those used in [3, 4], we deduce that Problem P_V^{hk} has a unique solution $\boldsymbol{u}^{hk}\subset V^h$. Proceeding now as in [1] and assuming that $\ddot{\boldsymbol{u}}\in$

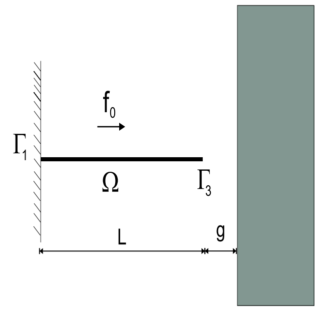

FIGURE 1. Contact of a rod with a deformable foundation.

$L^\infty(0,T;V)$, we can show the following error estimate:

$$\max_{1\le n\le N}\{|\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}|_V+\quad|\dot{\boldsymbol{u}}_n-\delta\boldsymbol{u}_n^{hk}|_V\}\le c\Big(k|\ddot{\boldsymbol{u}}|_{L^\infty(0,T:V)}+|\boldsymbol{u}_0-\boldsymbol{u}_0^h|_V$$
$$+\max_{1\le n\le N}|\dot{\boldsymbol{u}}_n-\boldsymbol{w}^h|_V\Big)\qquad\forall\boldsymbol{w}^h\in V^h.$$

This estimate is the basis for error estimation. For example, if we assume V^h is defined by (8) and $\dot{\boldsymbol{u}}\in L^\infty(0,T;[H^2(\Omega)]^d)$, we obtain a linear convergence with respect to the parameters k and h.

4. Numerical simulations

To show the performance of the numerical method described in the previous section, we have done a number of numerical experiments. In this section we describe numerical results for solving Problem P in one and two dimensions.

4.1. One dimensional example. We consider a cantilever viscoelastic rod $\Omega=(0,L)$ which is fixed at its left end $x=0$ and is subjected to the action of a body force of density $f_0(x,t)$ in the x-direction (see Figure 1). An initial gap g is assumed between its right end $x=L$ and an obstacle with normal compliance. Contact occurs at certain time $t_c>0$ and then compression starts to spread in from the contacting end. This problem corresponds to Problem P with $\Omega=(0,L)$, $\Gamma_1=\{0\}$, $\Gamma_2=\emptyset$, $\Gamma_3=\{L\}$. We use a linearly viscoelastic constitutive law, i.e.

$$\sigma=a\varepsilon(\dot{u})+b\varepsilon(u).$$

Here $\varepsilon(u)=\dfrac{\partial u}{\partial x}$, while a, b are material constants, independent of x and t, such that $a>0$. We choose (7) as normal compliance contact function.

A complete description of this problem is the following:

Problem $1D$. *Find a displacement field* $u : [0, L] \times [0, T] \to \mathbb{R}$ *and a stress field* $\sigma : [0, L] \times [0, T] \to \mathbb{R}$ *such that*

$$\sigma(x,t) = a\frac{\partial^2 u(x,t)}{\partial x \partial t} + b\frac{\partial u(x,t)}{\partial x} \qquad \text{in} \quad (0,L)\times(0,T),$$

$$\frac{\partial \sigma(x,t)}{\partial x} + f_0(x,t) = 0 \qquad \text{in} \quad (0,L)\times(0,T),$$

$$u(0,t) = 0 \qquad \text{for} \ \ t \in (0,T),$$

$$-\sigma(L,t) = c_\nu \max\{0, u(L,t) - g\} \qquad \text{for } t \in (0,T),$$

$$u(x,0) = u_0(x) \qquad \text{in} \ \ (0,L).$$

For computation we used the following data:

$$L = 1\,m, \quad T = 0.1\,sec, \quad a = 100\,N sec/m, \quad b = 1\,N/m,$$
$$f_0(x,t) = 10\,N/m \quad \forall x \in (0,1), \ t \in [0, 0.1],$$
$$c_\nu = 1\,N/m, \quad g = 0.25\,m, \quad u_0(x) = 0\,m \quad \forall x \in (0,1).$$

The exact solution of Problem $1D$ with the previous data can be obtained through an elementary but tedious calculation. It is given below.

- For $0 \le t \le t_c = \ln\left(\frac{20}{19}\right)$ there is no contact and

$$\begin{cases} u(x,t) = 5(1 - e^{-t})(2x - x^2), \\ \\ \sigma(x,t) = 10 - 10x. \end{cases}$$

- For $t_c = \ln\left(\frac{20}{19}\right) \le t \le 0.1$ there is contact and

$$\begin{cases} u(x,t) = -\dfrac{50}{19}\,e^{-2t}x + 5x(x-1)e^{-t} + \dfrac{61}{8}x - 5x^2, \\ \\ \sigma(x,t) = \dfrac{50}{19}\,e^{-2t} + \dfrac{61}{8} - 10x. \end{cases}$$

We have implemented the numerical method described in Section 3 on a standard workstation. We used continuous piecewise linear functions for the space V^h with parameter $h = 0.01$. The time step is uniform, and $k = 0.01$.

In Figure 2 the displacement fields at times $t = 0.03$, 0.06, 0.1 and the corresponding errors are shown.

The evolution in time of the displacement at nodes $x = 0.25$, 0.5, 1 for the approximate solutions and the corresponding error are plotted in Figure 3. In Figure 4, the evolution of the discrete stresses on the contact node $x = 1$ and their difference with the exact solution are drawn. We remark that before the contact occurs, the point $x = 1$ is stress free and, after the contact occurs, the reaction of the obstacle is towards the rod and increases when the penetration increases. Linear convergence of the method is clearly observed.

4.2. Two-dimensional example. As a two-dimensional example of Problem P, we consider the plane stress viscoelastic problem depicted in Figure 5. The domain $\Omega = (0,10) \times (0,2)$ is a cross-section of a three dimensional body subjected to the action of vertical body forces assumed to be linearly increasing in time. No tractions act on the part $\Gamma_2 = [0,10] \times \{2\} \cup (2,8) \times \{0\}$ of the boundary. Both ends of

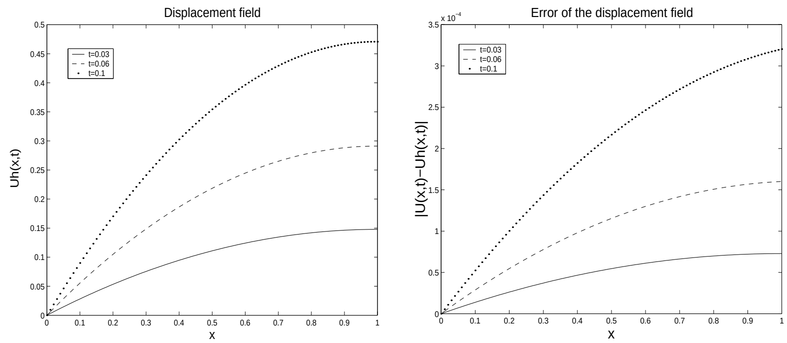

FIGURE 2. Problem $1D$: Displacement fields and corresponding error at several times.

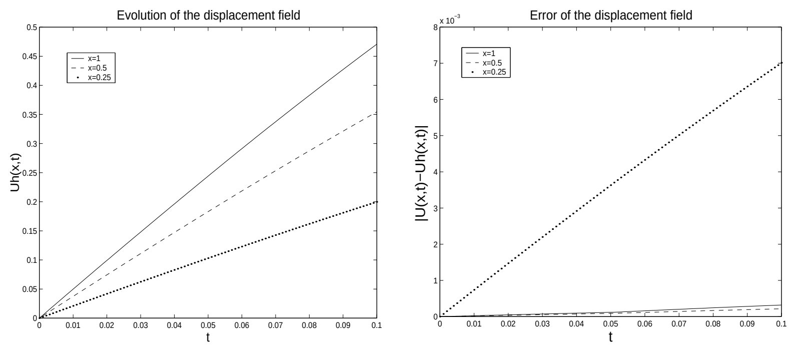

FIGURE 3. Problem $1D$: Evolution of the displacements of nodes $x = 0.25$, 0.5, 1 and the corrsponding errors.

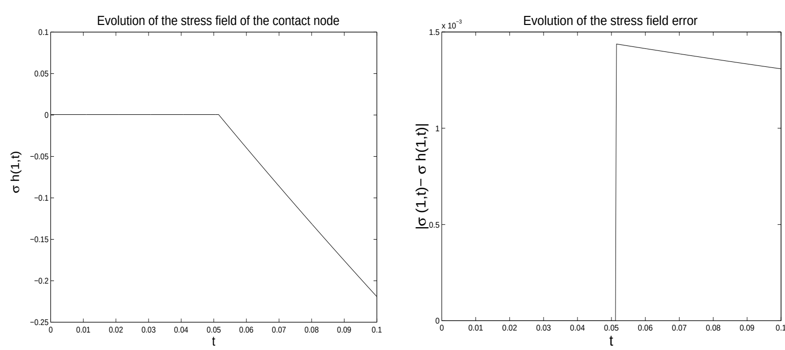

FIGURE 4. Problem $T1D$: Evolution of the stress field in $x = 1$ and the corresponding error.

the body are supposed to be clamped, i.e. $\Gamma_1 = \{0, 10\} \times (0, 2)$. We assume the body is in frictionless contact with an elastic foundation with normal compliance on $\Gamma_3 = [0, 2] \times \{0\} \cup [8, 10] \times \{0\}$.

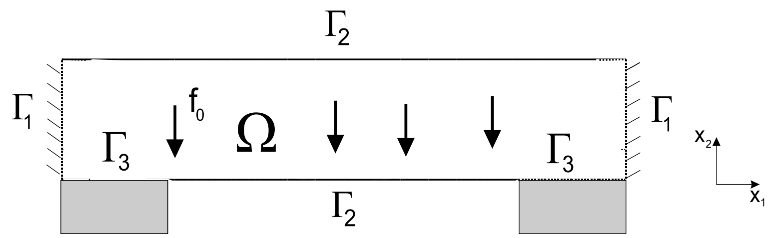

FIGURE 5. Test 2D: Contact of a 2D viscoelastic body.

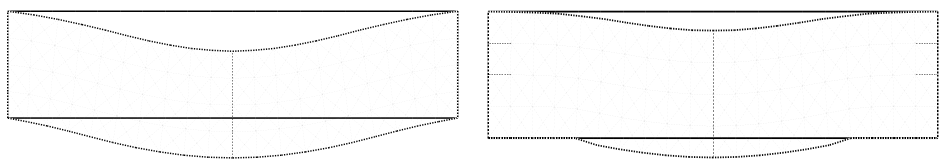

FIGURE 6. Problem $2D$: Initial boundary and deformed mesh at final time for $c_\nu = 10\, N/cm^2$ and $c_\nu = 10^8\, N/cm^2$.

The elasticity tensor $\mathcal{B}$ satisfies

$$(\mathcal{B}\boldsymbol{\tau})_{\alpha\beta} = \frac{E\kappa}{1-\kappa^2}(\tau_{11}+\tau_{22})\delta_{\alpha\beta} + \frac{E}{1+\kappa}\tau_{\alpha\beta}, \qquad 1 \le \alpha, \beta \le 2,$$

where E and κ are Young's modulus and Poisson's ratio of the material respectively, and $\delta_{\alpha\beta}$ denotes the Kronecker symbol. The viscosity tensor $\mathcal{A}$ has a similar form, i.e.

$$(\mathcal{A}\boldsymbol{\tau})_{\alpha\beta} = \mu(\tau_{11}+\tau_{22})\delta_{\alpha\beta} + \eta\tau_{\alpha\beta}, \qquad 1 \le \alpha, \beta \le 2,$$

where μ and η are viscosity constants. Note that the von Mises norm for a plane stress field $\boldsymbol{\tau} = (\tau_{\alpha\beta})$ is given by

$$\|\boldsymbol{\tau}\| = \left(\tau_{11}^2 + \tau_{22}^2 - \tau_{11}\tau_{22} + 3\tau_{12}^2\right)^{\frac{1}{2}}.$$

For computation we used the following data:

$$T = 1\,\text{sec}, \quad \boldsymbol{f}_0(x_1, x_2, t) = (0, -t)\, N/m^3, \quad \boldsymbol{f}_2 = (0,0)\, N/m^2,$$
$$p_\nu(r) = c_\nu(r)_+, \quad g = 0\, m, \quad \boldsymbol{u}_0 = \mathbf{0}\, m,$$
$$E = 100N/m^2, \quad \kappa = 0.3, \quad \mu = 30N \cdot sec/m^2, \quad \eta = 20N \cdot sec/m^2.$$

As in the previous example, V^h is composed by continuous piecewise linear functions and a uniform partition was considered with $k = 0.01$.

In Figure 6 the initial boundary and deformed mesh at final time $t = 1sec.$ are shown for values $c_\nu = 10,\ 10^8 N/cm^2$ (Signorini's conditions). In Figure 7, the von Mises norm for stress field at final time $t = 1sec.$ is plotted in the deformed configuration for $c_\nu = 10^8\, N/cm^2$.

Finally, the evolution of the contact line displacement is shown in Figure 8 for several values of $\beta = 1/c_\nu$.

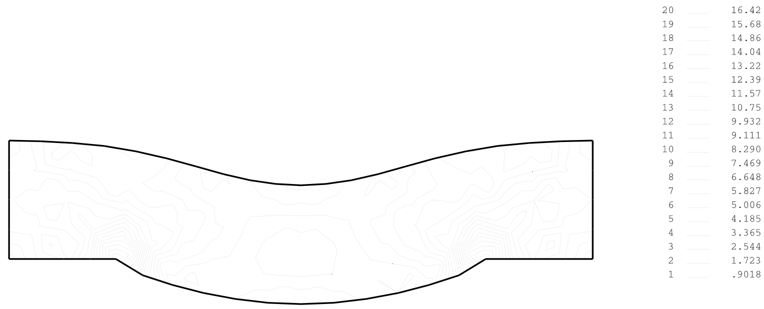

FIGURE 7. Test $2D$: Von-Mises stress norm at final time in the deformed configuration for $c_\nu = 10^8\, N/cm^2$.

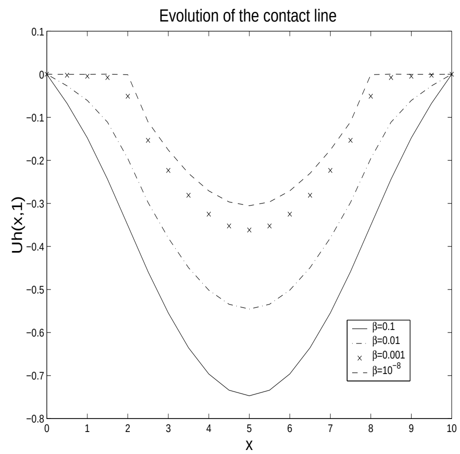

FIGURE 8. Test $2D$: Evolution of the contact line.

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