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The Chebyshev wavelet of the second kind for solving fractional delay differential equations

Hamid Safdari, Hamid Mesgarani, Mohamad Javidi, and Yones Esmaeelzade

Abstract. This article extends a numerical method for solving the fractional delay differential equations specified in terms of Caputo derivatives. The approach is based on the second kind Chebyshev wavelet. Our investigation is concentrated on convergence analysis and we prove a theorem for the error bound of the Chebyshev wavelets of the fractional differential in Caputo sense. Also, we discuss convergence analysis of the collocation method. In the end, some examples are presented to indicate the credibility and applicability of the numerical technique. The obtained results are compared with other numerical methods which our results are much more accurate than others.

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1. Introduction

The differential equations of the Fractional order are derived of the mathematical models in most disciplines such as mathematics, engineering, and science in applied branches as electrochemistry, biology, biophysics, control theory, signal and image processing of experimental data [1, 2, 3]. In the development of many applications, the researchers arrived the fractional delay differential equations (FDDE's) in some of the fields that these equations usually can only be solved by numerical methods because do not have analytic solutions. These numerical methods have been attracted by many researchers in recent years. For this reason, we refer the readers to the numerous works for solving these equations by with different ways as:

In 2011, Bhalekar and Daftardar approximated FDDE's by reaching the Adams-Bashforth-Moulton predictor-corrector methods (EABMPC method) [4]. Lately in [5], Wang solved the differential equation in delayed fractional order by the Adams-Bashforth-Moulton method with the linear interpolation method. Wang et al. extended a numerical scheme for nonlinear FDDE's on Grunwald-Letnikov definition in [6]. Also, in [7] by applying an adaptation of a fractional backward difference method, Morgado et al. solved a linear fractional differential equation with finite delay. In 2013, Moghadam and Miostaghim have established an innovative approach based on the finite difference for solving the fractional differential equations with delay in [8]. Also, Heris obtained the fractional backward differential formulas of the fifth-order and the fourth-order for FDDE's with periodic and anti-Periodic conditions in [9, 10].

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About the existence of the solution of FDDE's, we refer [11]. In [12], sufficient conditions are defined for the uniqueness of the solution. In [13], Eva presented some analytical and numerical methods for the stability investigation of linear fractional-order delay differential equations. Also, Morgado et al. investigated the analysis of numerical schemes for FDDE's in [7].

The main intention of this article is to introduce approximate of the second kind Chebyshev wavelet for solving the fractional differential equations (FDDE's) of the general form

$$\begin{aligned} &{}_0^c\mathbb{D}_t^\alpha y(t) = \mathfrak{F}(t, y(t), y(t-\zeta)) \quad t \in [0, \zeta], \\ &y(t) = \phi(t), \qquad t \in [-\zeta, 0], \qquad 0 < \alpha < 1, \end{aligned} \tag{1.1}$$

with $y(0) = -\phi(\zeta)$ anti-periodic condition or $y(0) = \phi(\zeta)$ periodic condition where ${}_0^c\mathbb{D}_t^\alpha$ is the Caputo derivative, ζ is a the delay constant and $\mathfrak{F}$ is a linear function as $\mathfrak{F}(t, y(t), y(t-\zeta)) = a(t)y(t) + b(t)y(t-\zeta) + c(t)$. The numerical methods for solving equation (1.1) headed to a system of algebraic equations which may be a large system and lead to numerous computational complexity and great storage needs. Hence, we use the spectral method based on the Chebyshev wavelet basis of the second kind because this wavelet is structurally sparse, this reduces the computational complexity of the resulting linear algebraic system. This property of the sparse of the coefficient matrix increases the convergence rate of the numerical approach.

The residue of this article is arranged as: Section 2 recommends preliminaries and notations of fractional derivatives. In Section 3, we describe the Chebyshev wavelet of the second kind and their properties. In Section 4, we summarize the collocation methods of the second kind Chebyshev wavelet for the numerical scheme of FDDE's. Convergence analysis and upper bound of the collocation methods based on the basis of the second kind Chebyshev wavelet is presented in Section 5. Finally, we are shown the accuracy of the approach by solving five numerical examples.

2. Preliminaries And Notations of Fractional Derivatives

In mathematics, the first appearance of the concept of a fractional derivative is instituted in a letter to Guillaume de I'Hopital by the famous mathematician Leibniz in 1965 [14]. Basically, the fractional computation was the field of mathematical analysis proposing at the study of integrals and derivatives of arbitrary orders. In this section, We recollect definitions and properties of fractional derivatives which will be used later [15, 16].

The Riemann-Liouville derivative of fractional order $\beta > 0$ has disutility for describing some natural phenomena. Thus, we recommend Caputo's derivative of fractional order $\beta > 0$ is determined as

$${}_0^C\mathbb{D}_t^\beta g(t) = \begin{cases} \frac{1}{\Gamma(n-\beta)} \int_0^t (t-\tau)^{n-\beta-1} g^{(n)}(\tau) d\tau, & n-1 < \beta < n \in \mathbb{N}, \\ \frac{d^n}{dt^n} g(t), & \beta = n, \end{cases} \tag{2.1}$$

This derivative is defined by the Italian mathematician Caputo in 1967 [17]. Now, the most used definition is due to the Riemann-Liouville fractional integral of order β, which is described as

$$\mathfrak{I}^\beta g(x) = \frac{1}{\Gamma(\beta)} \int_a^x (x-\tau)^{\beta-1} g(\tau) d\tau,$$

that $x > a$ and $a, x \in \mathbb{R}$. The important properties of integral operator are the identity operator i.e. $\mathfrak{I}^0 g(x) = g(x)$ and the linearity i.e. $\mathfrak{I}^\beta(\gamma f(x) + \eta g(x)) = \gamma \mathfrak{I}^\beta f(x) + \eta \mathfrak{I}^\beta g(x)$, $\beta \in \mathbb{R}^+, \gamma, \delta \in \mathbb{C}$.

Some properties of the Caputo fractional derivative ${}_0^C\mathbb{D}_t^\beta$, which are need here, we mention the following:

1. $\mathbb{D}^\beta(f(x)g(x)) = \sum_{i=0}^{\infty} \binom{\beta}{i} (\mathbb{D}^{\beta-i} f(x)) g^{(i)}(x) - \sum_{i=0}^{n-1} \frac{x^{i-\beta}}{\Gamma(i+1-\beta)} ((f(x)g(x))^{(i)}(0))$,
2. $\lim_{\beta \to n} \mathbb{D}^\beta g(t) = g^{(n)}(x)$, $\lim_{\beta \to n-1} \mathbb{D}^\beta g(t) = g^{(n-1)}(x) - g^{(n-1)}(0)$,
3. $\mathbb{D}^\beta \mathbb{D}^n g(x) = \mathbb{D}^{\beta+n} g(x) \neq \mathbb{D}^n \mathbb{D}^\beta g(x)$,
4. $\mathbb{D}^\beta(c) = 0$, is constant.

3. The second kind Chebyshev wavelets

We can create a class of bases for $L_2(\mathbb{R})$ that each basis is spaned of dilates and translates of a finite set of functions. We specify a space $\mathcal{V}_{\mathfrak{z}}^k$ of piecewise polynomial functions. $\mathcal{V}_{\mathfrak{z}}^k$ is set of f in the interval $[2^{-\mathfrak{m}}\mathfrak{n}, 2^{-\mathfrak{m}}(\mathfrak{n}+1)]$, f is a polynomial of degree less than $\mathfrak{z}$, $\mathfrak{n} = 0, 1, \ldots .2^{\mathfrak{m}} - 1$ and vanishes elsewhere. The dimension of the space $\mathcal{V}_{\mathfrak{z}}^k$ is $2^{\mathfrak{m}}\mathfrak{z}$ and we have

$$\mathcal{V}_0^k \subset \mathcal{V}_1^k \subset \ldots \subset \mathcal{V}_{\mathfrak{m}}^k \subset \ldots \ . \tag{3.1}$$

The space $\mathcal{W}_{\mathfrak{m}}^{\mathfrak{z}}$ is orthogonal complement of $\mathcal{V}_{\mathfrak{m}}^{\mathfrak{z}}$ in $\mathcal{V}_{\mathfrak{m}+1}^{\mathfrak{z}}$, as

$$\mathcal{V}_{\mathfrak{m}}^{\mathfrak{z}} \oplus \mathcal{W}_{\mathfrak{m}}^{\mathfrak{z}} = \mathcal{V}_{\mathfrak{m}+1}^{\mathfrak{z}}, \qquad \mathcal{W}_{\mathfrak{m}}^{\mathfrak{z}} \perp \mathcal{V}_{\mathfrak{m}}^{\mathfrak{z}}. \tag{3.2}$$

So we obtain the decomposition $\mathcal{V}_{\mathfrak{m}}^{\mathfrak{z}} = \mathcal{V}_0^{\mathfrak{z}} \oplus \mathcal{W}_0^{\mathfrak{z}} \oplus \mathcal{W}_1^{\mathfrak{z}} \oplus \ldots \oplus \mathcal{W}_{\mathfrak{m}-1}^{\mathfrak{z}}$.

Space $\mathcal{V}_{\mathfrak{m}}^{\mathfrak{z}}$ is spanned by the orthogonal functions of the Chebyshev polynomials of the second kind where space dimension $\mathcal{V}_{\mathfrak{m}}^{\mathfrak{z}}$ is $2^{\mathfrak{m}}\mathfrak{z}$. Wavelets constitute a class formulated from dilation and transformation of an individual function denominated the mother wavelet ψ. If $\mathfrak{a}$, $\mathfrak{b}$ be the dilation and translation parameter respectively, we can introduce the following set of continuous wavelets as

$$\psi_{\mathfrak{a},\mathfrak{b}}(t) = |\mathfrak{a}|^{-\frac{1}{2}} \psi\left(\frac{t-\mathfrak{b}}{\mathfrak{a}}\right), \qquad \mathfrak{a}, \mathfrak{b} \in \mathbb{R}, \mathfrak{a} \neq 0, \tag{3.3}$$

Now we confined the parameters $\mathfrak{a}$ and $\mathfrak{b}$ to discrete values as $\mathfrak{a} = \mathfrak{a}_0{}^{-\mathfrak{z}}$, $\mathfrak{b} = \mathfrak{n}\mathfrak{b}_0\mathfrak{a}_0{}^{-\mathfrak{z}}, \mathfrak{a}_0 > 1, \mathfrak{b}_0 > 0$, we could introduce the following family of discrete wavelets

$$\psi_{\mathfrak{z},\mathfrak{n}}(t) = |\mathfrak{a}_0|^{\frac{\mathfrak{z}}{2}} \psi\left(\mathfrak{a}_0{}^{\mathfrak{z}} t - \mathfrak{n}\mathfrak{b}_0\right), \qquad \mathfrak{z}, \mathfrak{n} \in \mathbb{Z}, \tag{3.4}$$

where $\psi_{\mathfrak{z},\mathfrak{n}}$ design a wavelet basis for $L_2(\mathbb{R})$. When $\mathfrak{a}_0 = 2$ and $\mathfrak{b}_0 = 1$, then $\psi_{\mathfrak{z},\mathfrak{n}}(t)$ forms an orthonormal basis [18, 19, 20]. The second kind Chebyshev wavelet $\psi_{\mathfrak{n},\mathfrak{m}} = \psi(\mathfrak{z}, \mathfrak{n}, \mathfrak{m}, t)$ entail four arguments, $\mathfrak{n} = 1, 2, \ldots, 2^{\mathfrak{z}-1}$, $\mathfrak{z}$ is presumed any nonnegative integer, $\mathfrak{m}$ is the degree of Chebyshev polynomial of second kind and t is the normalized time. They are described on the interval $[0, 1)$ as

$$\psi_{\mathfrak{n},\mathfrak{m}}(t) = \begin{cases} 2^{\frac{\mathfrak{z}}{2}} \mathfrak{U}_{\mathfrak{m}}(2^{\mathfrak{z}} t - 2\mathfrak{n} + 1), & \frac{\mathfrak{n}-1}{2^{\mathfrak{z}-1}} \leq t < \frac{\mathfrak{n}}{2^{\mathfrak{z}-1}}, \\ 0, & \text{otherwise}, \end{cases} \tag{3.5}$$

where

$$\mathfrak{U}_{\mathfrak{m}}(t) = \sqrt{\frac{2}{\pi}} U_m(t), \qquad \mathfrak{m} = 0, 1, 2, ..., M-1. \tag{3.6}$$

Here $U_m(t)$ are the second kind Chebyshev polynomials of degree m which are orthonormal with respect to the weight function $\mathfrak{W}(t) = \sqrt{1-t^2}$ on the interval $[-1, 1]$. These bases are orthonormal respect to the weight function $\mathfrak{W}_{\mathfrak{n},\mathfrak{z}}(t) = \mathfrak{W}(2^{\mathfrak{z}}t - 2\mathfrak{n}+1)$ on the interval $[0, 1]$ as

$$\int_0^1 \boldsymbol{\psi}_{\mathfrak{n},\mathfrak{m}}(t)\boldsymbol{\psi}_{\mathfrak{n},\mathfrak{m}'}(t)\mathfrak{W}_{\mathfrak{n},\mathfrak{z}}(t)dt = \delta_{m,m'}.$$

We can be expanded any function $f(t) \in L_2(\mathbb{R})$ determined on $[0, 1]$ into Chebyshev wavelet basis as

$$f(t) = \sum_{\mathfrak{n}=1}^{\infty}\sum_{\mathfrak{m}=0}^{\infty} \mathfrak{c}_{\mathfrak{n}\mathfrak{m}}\psi_{\mathfrak{n},\mathfrak{m}}(t) = f_{\mathfrak{n}\mathfrak{m}}(t), \tag{3.7}$$

The series (3.7) is denominated a wavelet series portrayal of $f(t)$ and the wavelet coefficients $\mathfrak{c}_{\mathfrak{n}\mathfrak{m}}$ are given by $\mathfrak{c}_{\mathfrak{n}\mathfrak{m}} = \langle f(t), \boldsymbol{\psi}_{\mathfrak{n},\mathfrak{m}}(t)\rangle_{\mathfrak{W}_{\mathfrak{n},\mathfrak{z}}(t)} = \int_0^1 f(t)\boldsymbol{\psi}_{\mathfrak{n},\mathfrak{m}}(t)\mathfrak{W}_{\mathfrak{n},\mathfrak{z}}(t)dt$. If the infinite series (3.7) is truncated, then we can be written as

$$f(t) \approx \mathfrak{P}_{\mathfrak{J}}(f(t)) = \sum_{\mathfrak{n}=1}^{2^{\mathfrak{z}-1}}\sum_{\mathfrak{m}=0}^{M-1} \mathfrak{c}_{\mathfrak{n}\mathfrak{m}}\boldsymbol{\psi}_{\mathfrak{n},\mathfrak{m}}(t) = \tilde{f}_{\mathfrak{n}\mathfrak{m}}(t) = \mathfrak{C}^T\boldsymbol{\Psi}(t), \tag{3.8}$$

where $\mathfrak{C}$ and $\boldsymbol{\Psi}$ are two vectors given by

$$\begin{aligned}
\mathfrak{C} =&[\mathfrak{c}_{10}, \mathfrak{c}_{11}, ..., \mathfrak{c}_{1M-1}, \mathfrak{c}_{20}, \mathfrak{c}_{21}, ..., \mathfrak{c}_{2M-1}, ..., \mathfrak{c}_{2^{\mathfrak{z}-1}0}, ..., \mathfrak{c}_{2^{\mathfrak{z}-1}M-1}]^T,\\
\boldsymbol{\Psi} =&[\psi_{10}, \psi_{11}, ..., \psi_{1M-1}, \psi_{20}, \psi_{21}, ..., \psi_{2M-1}, ..., \psi_{2^{\mathfrak{z}-1}0}, ..., \psi_{2^{\mathfrak{z}-1}M-1}]^T.
\end{aligned}$$

4. Chebyshev Wavelet Collocation Scheme for Linear FDDE's

In this section, we describe the collocation method for solving equation (1.1) with based on the Chebyshev wavelet of the second kind. So we use the Caputo fractional derivative definition of arbitrary order $0 < \beta < 1$ for mentioned equation as

$$\begin{aligned}
&\frac{1}{\Gamma(1-\beta)}\int_0^t (t-\tau)^{-\beta}y'(\tau)d\tau = \mathfrak{F}(t, y(t), y(t-\zeta)), \quad t \in [0, \zeta],\\
&y(t) = \phi(t), \qquad t \in [-\zeta, 0],\\
&y(0) = -\phi(\zeta), \;\; \textit{or } y(0) = \phi(\zeta),
\end{aligned} \tag{4.1}$$

where $\mathfrak{F}$ is a linear function. The approximation of the function $y'(\tau)$ may be written as a linear combination of Chebyshev wavelet basis as (3.8) for the vector space $L_2(\mathbb{R})$. So, the first sentence of (4.1) takes the following form:

$$\frac{1}{\Gamma(1-\beta)}\int_0^t \left[(t-\tau)^{-\beta}\sum_{\mathfrak{n}=1}^{2^{\mathfrak{z}-1}}\sum_{\mathfrak{m}=0}^{M-1}\mathfrak{c}_{\mathfrak{n}\mathfrak{m}}\boldsymbol{\psi}_{\mathfrak{n},\mathfrak{m}}(\tau)\right]d\tau - a(t)y(t) = \mathfrak{F}(t, y(t-\zeta)), \tag{4.2}$$

Now, we employ the integral of the series expand (4.2) to approximate the unknown function $y'(t)$ and then we apply the delay condition $y(t-\zeta) = \phi(t)$. The equation

(4.2) can be written as

$$\sum_{\mathfrak{n}=1}^{2^{\mathfrak{z}-1}} \sum_{\mathfrak{m}=0}^{M-1} \mathfrak{c}_{\mathfrak{n}\mathfrak{m}} \left[\frac{1}{\Gamma(1-\beta)} \int_0^t (t-\tau)^{-\beta} \psi_{\mathfrak{n},\mathfrak{m}}(\tau) d\tau - a(t) \left(\int \psi_{\mathfrak{n},\mathfrak{m}}(t) dt + c \right) \right] = \mathfrak{F}(t, \phi(t)).$$

The value constant c is obtained from the periodic or anti-periodic condition which $c = y(0) - \int \psi_{\mathfrak{n},\mathfrak{m}}(t)dt \Big|_{t=0}$. So, we rewrite the above equation for the collocation nodes $t_i = \frac{i-1}{M \times 2^{\mathfrak{z}-1}}$, $1 \leq i \leq M \times 2^{\mathfrak{z}-1}$ as

$$\sum_{\mathfrak{n}=1}^{2^{\mathfrak{z}-1}} \sum_{\mathfrak{m}=0}^{M-1} \mathfrak{c}_{\mathfrak{n}\mathfrak{m}} \left[\frac{1}{\Gamma(1-\beta)} \int_0^{t_i} (t_i-\tau)^{-\beta} \psi_{\mathfrak{n},\mathfrak{m}}(\tau) d\tau - a(t_i) \left(\int \psi_{\mathfrak{n},\mathfrak{m}}(t) dt + c \right) \right] \Big|_{t=t_i} = \mathcal{F}(t_i), \tag{4.3}$$

where $\mathcal{F}(t_i) = \mathfrak{F}(t_i, \phi(t_i))$, for all $1 \leq i \leq M \times 2^{\mathfrak{z}-1}$.

In this method, a linear algebraic system is acquired, which when solved gives us the unknown coefficients $\{\mathfrak{c}_{\mathfrak{n}\mathfrak{m}}\}, 1 \leq \mathfrak{n} \leq 2^{\mathfrak{z}-1}, 0 \leq \mathfrak{m} \leq M-1$. Because the above integral has the answer analytically in system (4.3), we did not use numerical methods for approximating the integral solution. If this integral was not solved numerically, we would use Legendre-Gauss-Lobatto method, which is precise for the $2m-1$ degree.

5. Outline of the error convergence

In this segment, we investigate some theorems for estimating the approximate function of the wavelet basis. Let us express the following theorems which will be used later in the proof of the main theorems.

Theorem 5.1. [21] *Suppose $f(t)$ defined on $[0,1)$ with bounded second derivative can be expanded as an infinite sum of Chebyshev wavelets and the series converges uniformly to $f(t)$, that is*

$$f(t) = \sum_{\mathfrak{n}=1}^{\infty} \sum_{\mathfrak{m}=0}^{\infty} \mathfrak{c}_{\mathfrak{n}\mathfrak{m}} \psi_{\mathfrak{n},\mathfrak{m}}(t), \tag{5.1}$$

where $\mathfrak{c}_{\mathfrak{n}\mathfrak{m}} = \langle f(t), \psi_{\mathfrak{n},\mathfrak{m}}(t) \rangle_{\mathfrak{W}_{\mathfrak{n},\mathfrak{z}}(t)}$.

Theorem 5.2. [22] *Assume f be a function pertaining to $\mathbb{C}^a \cap L_2(\mathbb{R}), a > 0$, and Φ and Ψ be a scaling function and a mother wavelet. Also, assume $\Phi, \Psi \in \mathbb{C}^a$ for some $0 < a < r$ provide the following decay condition*

$$|\Phi(t)| \leq c(1+|t|)^{-1-\varepsilon}, \qquad |\Psi(t)| \leq c(1+|t|)^{-1-\varepsilon}, \qquad c, \varepsilon > 0, \tag{5.2}$$

Then $\mathfrak{P}_{\mathfrak{J}}(f(t))$ (3.8) converges to $f(t)$ in the infinity norm. Moreover, $||\mathfrak{P}_{\mathfrak{J}}(f(t)) - f(t)||_{\infty} \leq c2^{-\mathfrak{J}a}$ for some constant c depending only on f .

Now, using the approximate function (3.7), the residual $\mathfrak{R}_{\mathfrak{n}\mathfrak{m}}(t)$ can be expressed as

$$\mathfrak{R}_{\mathfrak{n}\mathfrak{m}} f(t) = {}_0^C\mathbb{D}_t^{\beta} f(t) - \mathfrak{P}_{\mathfrak{J}}({}_0^C\mathbb{D}_t^{\beta} f(t)) = \sum_{\mathfrak{n}=2^{\mathfrak{J}}}^{+\infty} \sum_{\mathfrak{m}=M}^{+\infty} \mathfrak{c}_{\mathfrak{n}\mathfrak{m}} \psi_{\mathfrak{n},\mathfrak{m}}(t). \tag{5.3}$$

In order to illustrate the error estimate, we have given the following theorem.

Theorem 5.3. *Assume that the operator 3.8, $\mathfrak{P}_{\mathfrak{J}}({}_0^C\mathbb{D}_t^\beta f(t))$, obtained by using the second kind Chebyshev wavelets and ${}_0^C\mathbb{D}_t^\beta f(t)$ is bounded by the second derivative, then we have the following relation:*

$$\lim_{\mathfrak{J}\to\infty} |\mathfrak{R}_{\mathfrak{nm}} f(t)| \to 0. \tag{5.4}$$

Proof. We have $\int_0^1 \boldsymbol{\psi}_{\mathfrak{n},\mathfrak{m}}(t)\boldsymbol{\psi}_{\mathfrak{n},\mathfrak{m}'}(t)\mathfrak{W}_{\mathfrak{n},\mathfrak{z}}(t)dt = \delta_{m,m'}$ because the Chebyshev wavelet sequence is orthonormal, where $\mathfrak{W}_{\mathfrak{n},\mathfrak{z}}$ is the weight function and $\delta_{m,m'}$ is the Kronecker delta, then

$$\begin{aligned}
\mathfrak{c}_{\mathfrak{nm}} &= \int_0^1 \mathfrak{W}_{\mathfrak{n},\mathfrak{z}}(t)\boldsymbol{\psi}_{\mathfrak{n},\mathfrak{m}}(t){}_0^C\mathbb{D}_t^\beta f(t)dt \\
&= \int_{\frac{\mathfrak{n}-1}{2^{\mathfrak{z}-1}}}^{\frac{\mathfrak{n}}{2^{\mathfrak{z}-1}}} 2^{\frac{\mathfrak{z}}{2}}\sqrt{\frac{2}{\pi}}\mathfrak{W}_{\mathfrak{n},\mathfrak{z}}(t)U_{\mathfrak{m}}(2^{\mathfrak{z}}t-2\mathfrak{n}+1){}_0^C\mathbb{D}_t^\beta f(t)dt.
\end{aligned}$$

Now, suppose $x = 2^{\mathfrak{z}}t - 2\mathfrak{n} + 1$, then

$$\mathfrak{c}_{\mathfrak{nm}} = \int_{-1}^1 2^{\frac{\mathfrak{z}}{2}}\sqrt{\frac{2}{\pi}}\mathfrak{W}(x)U_{\mathfrak{m}}(x){}_0^C\mathbb{D}_t^\beta f(\frac{x+2\mathfrak{n}-1}{2^{\mathfrak{z}}})\frac{dx}{2^{\mathfrak{z}}}, \tag{5.5}$$

On the other hand, we can also use $U_{\mathfrak{m}}(x)dx = \frac{1}{\mathfrak{m}+1}dT_{\mathfrak{m}+1}(x)$ where $T_{\mathfrak{m}}(x)$ is the Chebyshev polynomials of the first kind. Then

$$\begin{aligned}
\mathfrak{c}_{\mathfrak{nm}} =& \frac{1}{(\mathfrak{m}+1)\times 2^{\frac{\mathfrak{z}}{2}}}\sqrt{\frac{2}{\pi}}\int_{-1}^1 \mathfrak{W}(x){}_0^C\mathbb{D}_t^\beta f(\frac{x+2\mathfrak{n}-1}{2^{\mathfrak{z}}})dT_{\mathfrak{m}+1}(x) \\
=& \frac{1}{(\mathfrak{m}+1)\times 2^{\frac{\mathfrak{z}}{2}}}\sqrt{\frac{2}{\pi}}\Bigg[\mathfrak{W}(x){}_0^C\mathbb{D}_t^\beta f(\frac{x+2\mathfrak{n}-1}{2^{\mathfrak{z}}})T_{\mathfrak{m}+1}(x)\Big|_{-1}^{+1} \\
&- \int_{-1}^1 \left(\mathfrak{W}'(x){}_0^C\mathbb{D}_t^\beta f(\frac{x+2\mathfrak{n}-1}{2^{\mathfrak{z}}}) + \frac{1}{2^{\mathfrak{z}}}\mathfrak{W}(x){}_0^C\mathbb{D}_t^{\beta+1} f(\frac{x+2\mathfrak{n}-1}{2^{\mathfrak{z}}})\right)T_{\mathfrak{m}+1}(x)dx\Bigg],
\end{aligned}$$

The first sentence of the above bracket is zero. Now, we use $T_{\mathfrak{m}+1}(x)dx = \frac{1}{\mathfrak{m}+1}\big((x^2-1)dU_{\mathfrak{m}}(x) + xU_{\mathfrak{m}}(x)dx\big)$. Then

$$\begin{aligned}
\mathfrak{c}_{\mathfrak{nm}} = \frac{-1}{(\mathfrak{m}+1)^2\times 2^{\frac{\mathfrak{z}}{2}}}\sqrt{\frac{2}{\pi}}\Bigg[&\int_{-1}^1 (x^2-1)\times\mathfrak{W}'(x){}_0^C\mathbb{D}_t^\beta f(\frac{x+2\mathfrak{n}-1}{2^{\mathfrak{z}}})dU_{\mathfrak{m}}(x) \\
&+ \int_{-1}^1 x\times\mathfrak{W}'(x){}_0^C\mathbb{D}_t^\beta f(\frac{x+2\mathfrak{n}-1}{2^{\mathfrak{z}}})U_{\mathfrak{m}}(x)dx \\
&+ \frac{1}{2^{\mathfrak{z}}}\int_{-1}^1 (x^2-1)\times\mathfrak{W}(x){}_0^C\mathbb{D}_t^{\beta+1} f(\frac{x+2\mathfrak{n}-1}{2^{\mathfrak{z}}})dU_{\mathfrak{m}}(x) \\
&+ \frac{1}{2^{\mathfrak{z}}}\int_{-1}^1 x\times\mathfrak{W}(x){}_0^C\mathbb{D}_t^{\beta+1} f(\frac{x+2\mathfrak{n}-1}{2^{\mathfrak{z}}})U_{\mathfrak{m}}(x)dx\Bigg].
\end{aligned}$$

With simplifying the above equation and using (5.5), we obtain

$$\mathfrak{c}_{\mathfrak{n}\mathfrak{m}} = \frac{1}{1+(\mathfrak{m}+1)^2}\sqrt{\frac{2}{\pi}}\Bigg[\frac{3}{2^{\frac{3\mathfrak{z}}{2}}}\int_{-1}^{1} x\times\mathfrak{W}(x){}_0^C\mathbb{D}_t^{\beta+1}f(\frac{x+2\mathfrak{n}-1}{2^{\mathfrak{z}}})U_{\mathfrak{m}}(x)dx - \frac{1}{2^{\frac{5\mathfrak{z}}{2}}}\int_{-1}^{1}\mathfrak{W}^3(x){}_0^C\mathbb{D}_t^{\beta+2}f(\frac{x+2\mathfrak{n}-1}{2^{\mathfrak{z}}})U_{\mathfrak{m}}(x)dx\Bigg].$$

Thus

$$|\mathfrak{c}_{\mathfrak{n}\mathfrak{m}}| \leq \frac{1}{1+(\mathfrak{m}+1)^2}\sqrt{\frac{2}{\pi}}\Bigg[\frac{3}{2^{\frac{3\mathfrak{z}}{2}}}\int_{-1}^{1} |x\times\mathfrak{W}(x)||{}_0^C\mathbb{D}_t^{\beta+1}f(\frac{x+2\mathfrak{n}-1}{2^{\mathfrak{z}}})||U_{\mathfrak{m}}(x)|dx + \frac{1}{2^{\frac{5\mathfrak{z}}{2}}}\int_{-1}^{1}|\mathfrak{W}^3(x)||{}_0^C\mathbb{D}_t^{\beta+2}f(\frac{x+2\mathfrak{n}-1}{2^{\mathfrak{z}}})||U_{\mathfrak{m}}(x)|dx\Bigg].$$

If we set $\mathfrak{W}(x) \leq 1$, $Max|{}_0^C\mathbb{D}_t^{\beta+1}f(\frac{x+2\mathfrak{n}-1}{2^{\mathfrak{z}}})| \leq \mathcal{M}'$ and $Max|{}_0^C\mathbb{D}_t^{\beta+2}f(\frac{x+2\mathfrak{n}-1}{2^{\mathfrak{z}}})| \leq \mathcal{M}''$ for all $x \in [-1, 1]$, we have

$$|\mathfrak{c}_{\mathfrak{n}\mathfrak{m}}| \leq \frac{1}{1+(\mathfrak{m}+1)^2}\sqrt{\frac{2}{\pi}}\Bigg[\frac{3\mathcal{M}'}{2^{\frac{3\mathfrak{z}}{2}}}\int_{-1}^{1}|U_{\mathfrak{m}}(x)|dx + \frac{\mathcal{M}''}{2^{\frac{5\mathfrak{z}}{2}}}\int_{-1}^{1}|U_{\mathfrak{m}}(x)|dx\Bigg].$$

Since $\int_{-1}^{1}|U_{\mathfrak{m}}(x)|dx = \int_0^{\pi}|sin(\mathfrak{m}+1)\theta|d\theta \leq \frac{2}{\mathfrak{m}+1}$ then we get

$$\begin{aligned} |\mathfrak{c}_{\mathfrak{n}\mathfrak{m}}| &\leq \frac{1}{(1+(\mathfrak{m}+1)^2)\times(m+1)}\sqrt{\frac{2}{\pi}}\Bigg[\frac{6\mathcal{M}'}{2^{\frac{3\mathfrak{z}}{2}}} + \frac{2\mathcal{M}''}{2^{\frac{5\mathfrak{z}}{2}}}\Bigg] \\ &\leq \sqrt{\frac{2}{\pi}}\times\frac{1}{\mathfrak{m}^3}\Bigg[\frac{6\mathcal{M}'}{(2\mathfrak{n})^{\frac{3}{2}}} + \frac{2\mathcal{M}''}{(2\mathfrak{n})^{\frac{5}{2}}}\Bigg] = \frac{1}{\sqrt{\pi}}\times\frac{1}{\mathfrak{m}^3}\Bigg[\frac{3\mathcal{M}'}{\mathfrak{n}^{\frac{3}{2}}} + \frac{\mathcal{M}''}{4\mathfrak{n}^{\frac{5}{2}}}\Bigg]. \end{aligned} \tag{5.6}$$

Therefore, we have

$$|\mathfrak{R}_{\mathfrak{n}\mathfrak{m}}f(t)| = \Bigg|\sum_{\mathfrak{n}=2^{\mathfrak{z}}}^{+\infty}\sum_{\mathfrak{m}=M}^{+\infty}\mathfrak{c}_{\mathfrak{n}\mathfrak{m}}\psi_{\mathfrak{n},\mathfrak{m}}(t)\Bigg| \leq \sum_{\mathfrak{n}=2^{\mathfrak{z}}}^{+\infty}\sum_{\mathfrak{m}=M}^{+\infty}\Big|\mathfrak{c}_{\mathfrak{n}\mathfrak{m}}\Big|\Big|\psi_{\mathfrak{n},\mathfrak{m}}(t)\Big|. \tag{5.7}$$

Using the Theorem 5.4 and substituting the relation (5.6) in (5.7), we get

$$\begin{aligned} |\mathfrak{R}_{\mathfrak{n}\mathfrak{m}}f(t)| &\leq \sum_{\mathfrak{n}=2^{\mathfrak{z}}}^{+\infty}\sum_{\mathfrak{m}=M}^{+\infty}\Big|\mathfrak{c}_{\mathfrak{n}\mathfrak{m}}\Big|\Big|\psi_{\mathfrak{n},\mathfrak{m}}(t)\Big| \leq c2^{-1-\varepsilon}\sum_{\mathfrak{n}=2^{\mathfrak{z}}}^{+\infty}\sum_{\mathfrak{m}=M}^{+\infty}\Big|\mathfrak{c}_{\mathfrak{n}\mathfrak{m}}\Big| \\ &\leq \frac{c2^{-1-\varepsilon}}{\sqrt{\pi}}\times\sum_{\mathfrak{m}=M}^{+\infty}\frac{1}{\mathfrak{m}^3}\Bigg[3\mathcal{M}'\times\sum_{\mathfrak{n}=2^{\mathfrak{z}}}^{+\infty}\frac{1}{\mathfrak{n}^{\frac{3}{2}}} + \frac{\mathcal{M}''}{4}\sum_{\mathfrak{n}=2^{\mathfrak{z}}}^{+\infty}\frac{1}{\mathfrak{n}^{\frac{5}{2}}}\Bigg] \\ &\leq \frac{c2^{-1-\varepsilon}}{\sqrt{\pi}}\times(\frac{1}{2}PolyGamma[2,M])\Big[3\mathcal{M}'\times\zeta(\frac{3}{2},2^{\mathfrak{z}}) + \frac{\mathcal{M}''}{4}\times\zeta(\frac{5}{2},2^{\mathfrak{z}})\Big], \end{aligned} \tag{5.8}$$

where $\zeta(s,q) = \sum_{n=0}^{\infty}\frac{1}{(q+n)^s}$ is the Hurwitz zeta function and the PolyGamma function of order M is the $(M+1)$nd derivative of the logarithm of the gamma function. We can see that $|\mathfrak{R}_{\mathfrak{n}\mathfrak{m}}f(t)| \to 0$ when M is fixed and $\mathfrak{z} \to +\infty$. $\square$

For the convergence results of the current method, we prove the following theorem.

Theorem 5.4. *Let $\mathfrak{P}_{\mathfrak{J}}(f(t)) = \tilde{f}_{\mathfrak{nm}}(t)$ from (3.8) be approximate answer $f_{\mathfrak{nm}}(t)$ from 3.7. Then $\tilde{f}_{\mathfrak{nm}}(t)$ converges uniformaly to $f(t)$. That is*

$$||f(t) - \tilde{f}_{\mathfrak{nm}}(t)|| \leq c2^{-\mathfrak{J}a}, \tag{5.9}$$

where $c, a > 0$ are constant.

Proof. In (1.1), we apply the derivative of Caputo then using delay condition we have

$${}_0^c\mathbb{D}_t^{\beta} f(t) = \frac{1}{\Gamma(n-\beta)} \int_0^t (t-\tau)^{n-\beta-1} f^{(n)}(\tau) d\tau = \mathfrak{F}(t, \phi(t)) + a(t) f(t),$$

The above relation can be written as follows for $0 < \beta < 1$

$$f(t) - \int_0^t k(t,\tau) df(\tau) = -F(t), \quad 0 \leq t \leq \zeta.$$

where $F(t) = \frac{1}{a(t)} \mathfrak{F}(t, \phi(t))$ and $k(t,\tau) = \frac{(t-\tau)^{-\beta}}{a(t) \times \Gamma(1-\beta)}$. So, we can inscribe the problem (1.1) as the below operator:

$$f(t) - \mathcal{I} f(t) = -F(t), \quad 0 \leq t \leq \zeta, \tag{5.10}$$

where $\mathcal{I} f(t) = \int_0^t k(t,\tau) df(\tau)$. Assume that $f_{\mathfrak{nm}}(t)$ is the approximation of function $f(t)$ and also, $\tilde{f}_{\mathfrak{nm}}(t)$ is a projection of $f_{\mathfrak{nm}}(t)$ in subspace $\mathcal{V}_{\mathfrak{J}}$. Therefore, they apply in operator of (5.12), so

$$f_{\mathfrak{nm}} - \mathcal{I}\mathfrak{P}_{\mathfrak{J}} f = -F_{\mathfrak{J}}, \qquad \tilde{f}_{\mathfrak{nm}} - \mathcal{I}\tilde{f}_{\mathfrak{nm}} = -F_{\mathfrak{J}}. \tag{5.11}$$

Then we obtain

$$\begin{aligned} f_{\mathfrak{nm}} - \tilde{f}_{\mathfrak{nm}} &= \mathcal{I}\mathfrak{P}_{\mathfrak{J}} f - \mathcal{I}\tilde{f}_{\mathfrak{nm}} + \mathcal{I} f_{\mathfrak{nm}} - \mathcal{I} f_{\mathfrak{nm}} = \mathcal{I}(\mathfrak{P}_{\mathfrak{J}} f - f_{\mathfrak{nm}}) + \mathcal{I}(f_{\mathfrak{nm}} - \tilde{f}_{\mathfrak{nm}}), \\ \Rightarrow f_{\mathfrak{nm}} - \tilde{f}_{\mathfrak{nm}} &= (1-\mathcal{I})^{-1}(\mathcal{I}\mathfrak{P}_{\mathfrak{J}} f - \mathcal{I} f_{\mathfrak{nm}}) = (1-\mathcal{I})^{-1}(\mathcal{I}\mathfrak{P}_{\mathfrak{J}} f - \mathcal{I} f + \mathcal{I} f - \mathcal{I} f_{\mathfrak{nm}}). \end{aligned} \tag{5.12}$$

Using Theorem 5.4 and assuming this theorem, we have

$$\begin{aligned} &||\mathcal{I}\mathfrak{P}_{\mathfrak{J}} f - \mathcal{I} f|| \leq ||\mathcal{I}\mathfrak{P}_{\mathfrak{J}} - \mathcal{I}|| ||f|| \leq c_1 2^{-\mathfrak{J}a} ||f||, \\ &\exists \mathcal{M} > 0, \quad |(1-\mathcal{I})^{-1}| \leq \mathcal{M}. \end{aligned} \tag{5.13}$$

Then we get

$$\begin{aligned} ||f_{\mathfrak{nm}} - \tilde{f}_{\mathfrak{nm}}|| &\leq |(1-\mathcal{I})^{-1}| \Big(||\mathcal{I}\mathfrak{P}_{\mathfrak{J}} f - \mathcal{I} f|| + ||\mathcal{I} f - \mathcal{I} f_{\mathfrak{nm}}|| \Big), \\ &\leq \mathcal{M} \Big(c_1 2^{-\mathfrak{J}a} ||f|| + c_2 2^{-\mathfrak{J}a} \Big) = c_3 2^{-\mathfrak{J}a}. \end{aligned} \tag{5.14}$$

This yields

$$||f - \tilde{f}_{\mathfrak{nm}}|| \leq ||f - f_{\mathfrak{nm}}|| + ||f_{\mathfrak{nm}} - \tilde{f}_{\mathfrak{nm}}|| \leq c_4 2^{-\mathfrak{J}a} + c_3 2^{-\mathfrak{J}a} = c_5 2^{-\mathfrak{J}a}, \tag{5.15}$$

where $\{c_i\}_{i=1}^5$ are positive constant. □

6. Numerical conclusions

We indicate the accuracy of the proposed method with numerical problems. In order to accomplish the aforementioned, we investigate five examples whose exact answer is recognized. In these examples, all the calculations accomplished using the Mathematica software and numerical results compared with paper [6], [7], [10] and [23]. Note that in all tables, we use the abbreviations RE, AE and MAE that means the relative error with Euclidean norm, the absolute error of $y(1)$ and maximum absolute error of all collocation points in $[0,1]$ respectively.

Example 6.1. Here, we present the following problem definition where the values of fractional order are in $(0,1)$. This is an example from articles [6, 23], which the method presented is compared with these articles.

$$
\begin{aligned}
&{}_0^C\mathbb{D}_t^{\alpha} y(t) = \frac{2}{\Gamma(3-\alpha)} t^{2-\alpha} - \frac{1}{\Gamma(2-\alpha)} t^{1-\alpha} + 2t\tau - \tau^2 - \tau - y(t) + y(t-\tau), t \in (0,1),\\
&y(t) = t^2 - t, \qquad t \in [-1,0],
\end{aligned}
$$

The exact solution to this equation is $y(t) = t^2 - t$ for $\alpha = 0.7$ and τ denotes the delay constant or time-varying. The proposed manner is applied and illustrated the numerical results in the Table 1 by applying different values of $\mathfrak{m} = 4$, scale degree of $\mathfrak{J}$ and the values of fractional order $\alpha = 0.7$. We conclude that the numerical errors achieved using current method better than the numerical methods of [6, 23]. Absolute errors for different values of α are plotted in Figure 1 for $\mathfrak{m} = 4, \mathfrak{J} = 4$. We can see that the minimum error belongs to the values of fractional order $\alpha = 0.7$ in the interval $[0,1]$.

TABLE 1. Approximate errors for test Example 6.1, for $\mathfrak{m} = 4, \mathfrak{J} = 1,2,3,4$.

$\mathfrak{J}$	*RE*	*AE of* $y(1)$	*MAE of [0,1]*
1	2.34759×10^{-14}	7.51652×10^{-14}	7.51652×10^{-14}
2	2.15417×10^{-14}	2.20018×10^{-14}	2.20018×10^{-14}
3	1.06051×10^{-15}	3.33126×10^{-16}	3.46945×10^{-16}
4	5.88070×10^{-16}	1.55138×10^{-16}	2.77556×10^{-16}

Example 6.2. Suppose the following FDDE in terms of the Caputo fractional derivative. This is an example from article [7], which is compared with the presented method in this article

$$
\begin{aligned}
&{}_0^C\mathbb{D}_t^{0.5} y(t) = y(t-1) - y(t) + 2t - 1 + \frac{\Gamma(3)}{\Gamma(2.5)} t^{1.5}, \qquad t \in [0,1],\\
&y(t) = t^2, \qquad t \in [-1,0],
\end{aligned}
$$

The analytical solution is $y(t) = t^2$. In this article, the stated manner is employed and the shown measures are tabulated in Table 2 by utilizing the value of $\mathfrak{m} = 4$ and a scale degree of $\mathfrak{J}$.

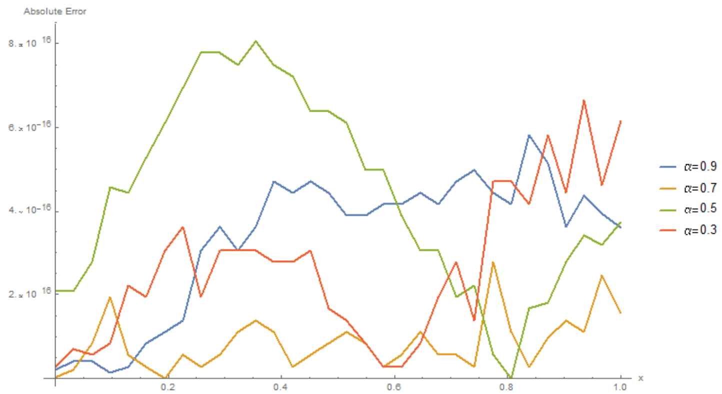

FIGURE 1. Absolute error of all collocation points of example 6.1 for $\alpha = 0.3, 0.5, 0.7 and\, 0.9$ with $\mathfrak{m} = 4$ and $\mathfrak{J} = 4$.

TABLE 2. Approximate errors for test Example 6.2, for $\mathfrak{m} = 4, \mathfrak{J} = 1, 2, 3, 4$.

	Current method			*Absolute error in* [7]	
$\mathfrak{J}$	*RE*	*AE of* $y(1)$	*MAE of[0,1]*	h	*AE*
1	1.01218×10^{-13}	1.01252×10^{-13}	1.01252×10^{-13}	$\frac{1}{10}$	4.91843×10^{-2}
2	1.97579×10^{-14}	1.27676×10^{-14}	1.27676×10^{-14}	$\frac{1}{20}$	2.76172×10^{-2}
3	2.77029×10^{-15}	8.88178×10^{-16}	2.16494×10^{-15}	$\frac{1}{40}$	1.46507×10^{-2}
4	2.70476×10^{-15}	2.22045×10^{-16}	3.74003×10^{-15}	$\frac{1}{80}$	7.56493×10^{-3}

Example 6.3. Suppose the following FDDE problem:

$$
\begin{aligned}
&{}_{0}^{C}\mathbb{D}_{t}^{0.3} y(t) = y(t-1) - y(t) + 3t^2 - 3t + 1 + \frac{2000}{1071\Gamma(0.7)} t^{2.7}, \qquad t \in [0,1],\\
&y(t) = t^3, \qquad\qquad t \in [-1, 0],
\end{aligned}
$$

The exact solution is $y(t) = t^3$. In Table 3, we reported the numerical conclusions by applying the value of $\mathfrak{m} = 5$ and the scale degree of $\mathfrak{J}$. Also, the errors of the current method are compared with absolute error in paper in paper [7].

TABLE 3. Approximate errors for test Example 6.3, for $\mathfrak{m} = 5, \mathfrak{J} = 1, 2, 3, 4$.

	Current method			*Absolute error in* [7]	
$\mathfrak{J}$	*RE*	*AE of* $y(1)$	*MAE of[0,1]*	h	*AE*
1	1.31288×10^{-13}	1.30340×10^{-13}	1.30340×10^{-13}	$\frac{1}{10}$	7.10508×10^{-2}
2	8.78671×10^{-14}	4.95159×10^{-14}	4.95159×10^{-14}	$\frac{1}{20}$	4.11543×10^{-2}
3	3.34876×10^{-15}	8.88178×10^{-16}	2.46331×10^{-15}	$\frac{1}{40}$	2.19612×10^{-2}
4	3.04858×10^{-15}	1.99840×10^{-16}	2.10942×10^{-15}	$\frac{1}{80}$	1.13125×10^{-2}

Example 6.4. We examine the below FDDE's problem with the anti-periodic condition of the article [10].

$$
\begin{aligned}
&{}_{0}^{C}\mathbb{D}_t^{0.2}y(t) + y(t-1) = \frac{\Gamma(3)t^{1.8}}{\Gamma(2.8)} - \frac{\Gamma(2)t^{0.8}}{\Gamma(1.8)} + t^2 - 3t + 1, \qquad t \in [0,2],\\
&y(t) = t^2 - t - 1, \qquad t \in [-1,0],\\
&y(0) = -y(2),
\end{aligned}
$$

The exact solution is $y(t) = t^2 - t - 1$. By using several values $\mathfrak{m}$ and scale degree of $\mathfrak{J}$, the errors of the current method are reported in Table 4. Also, the errors of the current method are compared with absolute error in paper[10].

TABLE 4. Approximate errors for test Example 6.4, for $\mathfrak{m} = 5, \mathfrak{J} = 1, 2, 3, 4$.

	Current method			*Absolute error in* [10]	
$\mathfrak{J}$	*RE*	*AE of* $y(1)$	*MAE of[0,1]*	h	*AE of* $y(1)$
1	1.31288×10^{-13}	1.30340×10^{-13}	1.30340×10^{-13}	$\frac{1}{10}$	7.10508×10^{-2}
2	8.78671×10^{-14}	4.95159×10^{-14}	4.95159×10^{-14}	$\frac{1}{20}$	4.11543×10^{-2}
3	3.34876×10^{-15}	8.88178×10^{-16}	2.46331×10^{-15}	$\frac{1}{40}$	2.19612×10^{-2}
4	3.04858×10^{-15}	1.99840×10^{-16}	2.10942×10^{-15}	$\frac{1}{80}$	1.13125×10^{-2}

Example 6.5. We examine the below FDDE's problem with the periodic condition of the article [10].

$$
\begin{aligned}
{}_{0}^{C}\mathbb{D}_t^{0.4}y(t) + {}_{0}^{C}\mathbb{D}_t^{0.3}y(t) + y(t-1) &= \frac{\Gamma(3)t^{1.6}}{\Gamma(2.6)} - \frac{\Gamma(2)t^{0.6}}{\Gamma(1.6)} + \frac{\Gamma(3)t^{1.7}}{\Gamma(2.7)} - \frac{\Gamma(2)t^{0.7}}{\Gamma(1.7)},\\
&+ t^2 - 3t + 2, \qquad t \in [0,1],\\
y(t) = t^2 - t, \qquad t \in [-1,0],&\\
y(0) = y(1),&
\end{aligned}
$$

The correct solution is $y(t) = t^2 - t$. In this article, we solve this problem with the current method by using the value $\mathfrak{m} = 5$ and scale degree of $\mathfrak{J}$. The errors of the current method are compared with the absolute error in paper[10] in Table 5.

TABLE 5. Approximate errors for test Example 6.5, for $\mathfrak{m} = 5, \mathfrak{J} = 1, 2, 3, 4$.

	Current method			*Absolute error in* [10]	
$\mathfrak{J}$	*RE*	*AE of* $y(1)$	*MAE of[0,1]*	h	*AE of* $y(1)$
1	2.43545×10^{-14}	8.79850×10^{-15}	8.79850×10^{-15}	0.1	1.55923×10^{-3}
2	1.64905×10^{-14}	3.39941×10^{-15}	4.02456×10^{-15}	0.05	5.10935×10^{-4}
3	2.75184×10^{-15}	5.06073×10^{-16}	8.60423×10^{-16}	0.01	3.67922×10^{-5}
4	2.02906×10^{-15}	4.54160×10^{-16}	5.55112×10^{-16}	0.005	1.18623×10^{-5}

Figures 2 and 3 represent the Absolute error for all test examples with $\mathfrak{m} = 5$ and $\mathfrak{J} = 4, 3$ in all collocation points of $[0, 1]$. In these graphs, the person can see that the error approximated of the collocation points in the stated method is very close to zero.

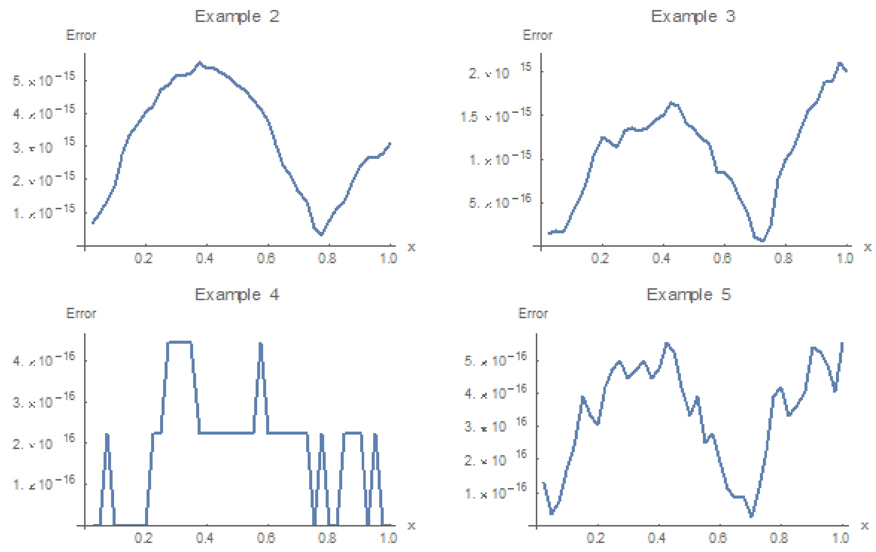

FIGURE 2. Absolute Error of all collocation points of all examples with $\mathfrak{m} = 5$ and $\mathfrak{J} = 4$.

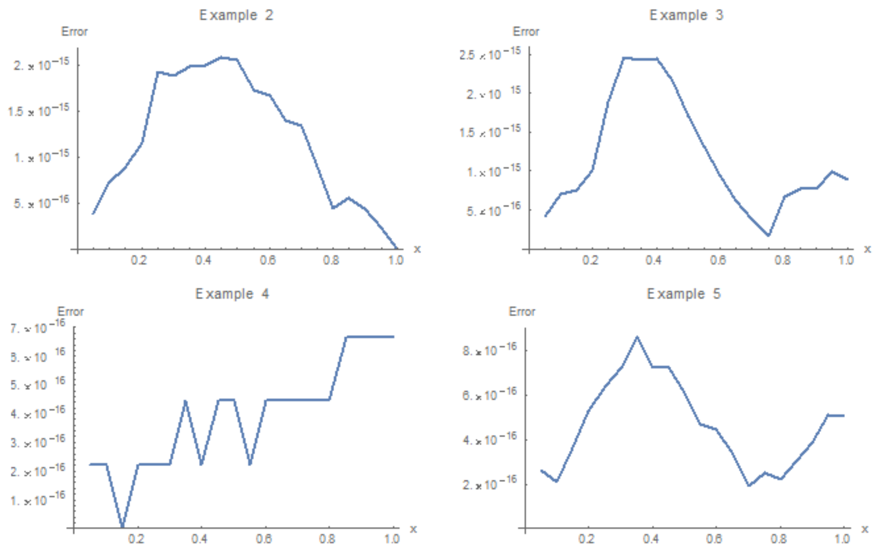

FIGURE 3. Absolute Error of all collocation points of all examples with $\mathfrak{m} = 5$ and $\mathfrak{J} = 3$.

7. Conclusion

In this article, the collocation scheme of the second kind Chebyshev wavelet has been used to solving the fractional delay differential equations on $[0, 1]$ and the approximation solution acquired by using the linear system. The Chebyshev wavelet basis can construct a sparse coefficients matrix because they are orthonormal and have small intervals of support. The convergence analysis of the collocation scheme is presented. In Theorem 5.3, studied the error bound of the Caputo fractional derivative of the Chebyshev wavelet of the second kind and in Theorem 5.4 is shown that the approximate solutions of FDDE's are uniformly convergent to the analytic solution. The proof of the Theorem 5.3 is based on the first-order Chebyshev polynomial. Using this method, several examples are tested and all numerical conclusions confirm the accuracy of the methods.

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(Hamid Safdari, Hamid Mesgarani, Yones Esmaeelzade) Department of Mathematics, Shahid Rajaee Teacher Training University, Tehran, Iran
E-mail address: `Hsafdari@sru.ac.ir, Hmesgarani@sru.ac.ir, Yonesesmaeelzade@gmail.com`

(Mohamad Javidi) Department of Mathematics, Faculty of Mathematical Sciences, University of Tabriz, Tabriz, Iran
E-mail address: `Correspondence : mo_javidi@tabrizu.ac.ir`