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Special concircular normal projective Lie-recurrence in NP-F_n

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Abstract. In this paper we discuss a normal projective Lie-recurrence generated by special concircular vector field and such Lie-recurrence is termed as a special concircular normal projective Lie-recurrence. We have obtained certain results related to a special concircular Lie-recurrence in $NP-F_n$ as well as in birecurrent and bisymmetric $NP-F_n$. It is established that an $NP-F_n$ admitting a special concircular normal projective Lie-recurrence is a Finsler space admitting special concircular H-Lie-recurrence.

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1. Introduction

In 1982, P. N. Pandey [3] introduced the concept of Lie-recurrence in a Finsler space. It is an infinitesimal transformation $\overline{x}^{\,i} = x^i + \epsilon\, v^i(x^j)$ with respect to which the Lie-derivative of curvature tensor is proportional to itself. In 2003, S. P. Singh [12] studied an infinitesimal transformation with respect to which the Lie-derivative of curvature tensor is proportional to itself and called such transformation as curvature inheritance. Obviously a curvature inheritance is nothing but a Lie-recurrence. Shivalika Saxena [4] discussed projective N-curvature inheritance in a Finsler space and obtained some results. C. K. Mishra and D. D. S. Yadav [1] discussed projective curvature inheritance in $NP-F_n$ and obtained some results.

The aim of the present paper is to discuss a normal projective Lie-recurrence generated by special concircular vector field. Such Lie-recurrence is termed as a special concircular normal projective Lie-recurrence. We have obtained certain results related to a special concircular Lie-recurrence in $NP-F_n$ as well as in birecurrent and bisymmetric $NP-F_n$. Apart from other theorems, it is being proved that an $NP-F_n$ admitting a special concircular normal projective Lie-recurrence is a Finsler space admitting special concircular H-Lie-recurrence.

2. Preliminaries

Let F_n be an n-dimensional Finsler space equipped with a metric function F satisfying the requisite conditions [11]. The relation between the metric tensor g_{ij} of the Finsler space F_n and the metric function F are given by

$$\text{(a) } g_{ij} = \frac{1}{2}\dot{\partial}_i\dot{\partial}_j F^2, \qquad \text{(b) } g_{ij}y^iy^j = F^2, \tag{1}$$

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where $\dot{\partial}_i \equiv \frac{\partial}{\partial y^i}$.
K. Yano [14] defined normal projective connection coefficients Π^i_{kh} by

$$\Pi^i_{kh} = G^i_{kh} - \frac{y^i}{n+1} G^r_{khr}, \tag{2}$$

where G^i_{kh} are Berwald connection coefficients. The functions Π^i_{kh}, G^i_{kh} and G^i_{jkh} are symmetric in their lower indices and positively homogeneous of degree 0, 0 and -1 respictively in y^i. The derivatives $\dot{\partial}_j \Pi^i_{kh}$, denoted by Π^i_{jkh}, is given by

$$\Pi^i_{jkh} = G^i_{jkh} - \frac{1}{n+1}(\delta^i_j G^r_{khr} + \dot{x}^i G^r_{jkhr}), \tag{3}$$

where tensor Π^i_{jkh} is symmetric in last two lower indices and positively homogeneous of degree -1 in y^i. The tensor Π^i_{jkh} satisfies the following:

$$\begin{cases} a)\ \Pi^i_{jkh} = \Pi^i_{jhk}, \\ b)\ \Pi^i_{jki} = G^i_{jki}, \\ c)\ \dot{x}^j \Pi^i_{jkh} = 0, \\ d)\ \dot{x}^h \Pi^i_{jkh} = \frac{\dot{x}^i}{n+1} G^r_{jkr}, \\ e)\ \Pi^i_{ikh} = \frac{2}{n+1} G^i_{ikh}. \end{cases} \tag{4}$$

The normal projective covariant derivative of an arbitrary vector field T^i with respect to connection coefficients Π^i_{jk} is given by

$$\nabla_k T^i = \partial_k T^i - (\dot{\partial}_r T^i)\Pi^r_{hk} y^h + T^r \Pi^i_{rk}, \tag{5}$$

where $\partial_k \equiv \frac{\partial}{\partial x^k}$.
The Ricci commutation formula for normal covariant derivative is given by

$$\nabla_k \nabla_h T^i - \nabla_h \nabla_k T^i = T^r N^i_{hkr} - (\dot{\partial}_r T^i) N^r_{hks} y^s, \tag{6}$$

where $N^i_{hkr} = \partial_h \Pi^i_{kr} - (\dot{\partial}_t \Pi^i_{kr})\Pi^t_{sh}\dot{x}^s + \Pi^t_{kr}\Pi^i_{th} - \partial_k \Pi^i_{hr} + (\dot{\partial}_t \Pi^i_{hr})\Pi^t_{sk}\dot{x}^s - \Pi^t_{hr}\Pi^i_{tk}$ are components of the normal projective curvature tensor. This tensor is skew-symmetric in first two lower indices and positively homogeneous of degree zero in y^i. The tensor N_{kh} defined by

$$N_{kh} = N^i_{ikh}, \tag{7}$$

satisfies

$$\text{(a) } N^i_{jih} = -N^i_{ijh} = -N_{jh}, \ \text{(b) } N^i_{jki} = N_{kj} - N_{jk}. \tag{8}$$

P. N. Pandey [5] obtained the following relation between the normal projective curvature tensor N^i_{jkh} and the Berwald curvature tensor H^i_{jkh}:

$$N^i_{jkh} = H^i_{jkh} - \frac{y^i}{n+1} \dot{\partial}_h H^r_{jkr}. \tag{9}$$

Transvecting (9) by y^h and using the fact $y^h \dot{\partial}_h H^r_{jkr} = 0$, we get

$$N^i_{jkh} y^h = H^i_{jk}, \tag{10}$$

where $H^i_{hk} = \partial_k G^i_h - \partial_h G^i_k + G^r_h G^i_{rk} - G^r_k G^i_{rh}$.
The commutation formula for the operators of partial differentiation with respect to y^k and normal covariant differentiation is given by

$$\dot{\partial}_k(\nabla_h T^i) - \nabla_h(\dot{\partial}_k T^i) = T^r \Pi^i_{khr} - (\dot{\partial}_r T^i)\Pi^r_{khs} y^s. \tag{11}$$

Let us consider an infinitesimal transformation

$$\overline{x}^i = x^i + \epsilon v^i(x^j), \tag{12}$$

generated by a contravariant vector field $v^i(x^j)$ which depends on position co-ordinates only. ϵ appearing in (12) is an infinitesimal constant.
The Lie-derivative of an arbitrary tensor T^i_j with respect to the infinitesimal transformation (12) is given by [14]

$$\pounds T^i_j = v^r \nabla_r T^i_j - T^r_j \nabla_r v^i + T^i_r \nabla_j v^r + (\dot{\partial}_r T^i_j)\nabla_s v^r y^s. \tag{13}$$

The commutation formula for the operators $\pounds$ and $\dot{\partial}_h$ is given by

$$\dot{\partial}_h \pounds \Omega - \pounds \dot{\partial}_h \Omega = 0, \tag{14}$$

where Ω is any geometrical object.
An infinitesimal transformation (12) is Lie-recurrence or H-Lie-recurrence if the Lie-derivative of Berwald curvature tensor H^i_{jkh} of the Finsler space satisfies ([13], [6])

$$\pounds H^i_{jkh} = \phi H^i_{jkh}, \tag{15}$$

where ϕ is a non-zero scalar field [3].
In view of this concept, the infinitesimal transformation (12) is called normal projective Lie-recurrence if the Lie-derivative of normal projective curvature tensor satisfies [4]

$$\pounds N^i_{jkh} = \phi N^i_{jkh}, \;\; \phi \neq 0. \tag{16}$$

A vector field v^i in $NP - F_n$ is said to be special concircular if

$$\nabla_k v^i = \rho \delta^i_k, \tag{17}$$

where $\rho = \rho(x)$ [7].

Definition 2.1. A Finsler space F_n with normal projective connection coefficients Π^i_{kh} and the normal projective curvature tensor N^i_{jkh} is termed as *normal projective Finsler space* and usually denoted by $NP - F_n$.

3. Special concircular normal projective Lie-recurrence

Definition 3.1. Let a symmetric $NP - F_n(n > 2)$ be characterized by [2]

$$\nabla_m N^i_{jkh} = 0. \tag{18}$$

Let us consider an $NP - F_n$ admitting the infinitesimal transformation (12) generated by a special concircular vector field $v^i(x^j)$. Suppose that the special concircular transformation (12) is a Lie-recurrence in the $NP - F_n(n > 2)$. Then we have equation (16). In view of equation (13), equation (16) may be written as

$$\begin{aligned} &v^r \nabla_r N^i_{jkh} + (\dot{\partial}_r N^i_{jkh})\nabla_s v^r y^s - N^r_{jkh}\nabla_r v^i \\ &+ N^i_{rkh}\nabla_j v^r + N^i_{jrh}\nabla_k v^r + N^i_{jkr}\nabla_h v^r = \phi N^i_{hjk}. \end{aligned} \tag{19}$$

Using equations (18) and (17) and the fact that the normal projective curvature tensor N^i_{hjk} is positively homogeneous of degree zero in y^j, we get $\phi = 2\rho$ if the space is non-flat. Since ρ is independent of y^i and $\phi = 2\rho$, ϕ is also independent of y^i [4]. Transvecting equation (16) by y^h and using (10), we get

$$\pounds H^i_{jk} = \phi H^i_{jk}. \tag{20}$$

Differentiating (20) partially with respect to y^h and using equation (14), we get $\pounds H^i_{hjk} = \phi H^i_{hjk}$, which shows that the special concircular normal projective Lie-recurrence is an H-Lie-recurrence. Thus we have:

Theorem 3.1. *A special concircular normal projective Lie-recurrence in a non-flat symmetric NP-F_n is an H-Lie-recurrence.*

Suppose the special concircular transformation (12) is a normal projective Lie-recurrence in $NP-F_n$ space, characterised by (16). In view of equation (13), equation (16) may be written as

$$\begin{aligned} v^r\nabla_r N^i_{jkh} + (\dot{\partial}_r N^i_{jkh})\nabla_s v^r y^s - N^r_{jkh}\nabla_r v^i \\ + N^i_{rkh}\nabla_j v^r + N^i_{jrh}\nabla_k v^r + N^i_{jkr}\nabla_h v^r = \phi N^i_{hjk}. \end{aligned} \tag{21}$$

Using equation (17) and degree of homogeneity of N^i_{jkh} in y^i, we get

$$v^r\nabla_r N^i_{jkh} = (\phi - 2\rho)N^i_{jkh}. \tag{22}$$

Differentiating equation (17) covariantly with respect to x^h, we get

$$\nabla_h\nabla_k v^i = \rho_h\delta^i_k, \tag{23}$$

where $\rho_h = \nabla_h\rho$.
Taking skew-symmetric part of equation (23), we get

$$\nabla_h\nabla_k v^i - \nabla_k\nabla_h v^i = \rho_h\delta^i_k - \rho_k\delta^i_h. \tag{24}$$

In view of equation (6), equation (24) becomes

$$v^r N^i_{khr} = \rho_h\delta^i_k - \rho_k\delta^i_h. \tag{25}$$

Contracting the indices i and k in equation (25) and using equation (7), we get

$$v^r N_{hr} = (n-1)\rho_h. \tag{26}$$

Using equation (26) in equation (25), we get

$$((n-1)N^i_{khr} + N_{kr}\delta^i_h - N_{hr}\delta^i_k)v^r = 0. \tag{27}$$

This leads to:

Theorem 3.2. *The normal projective curvature tensor N^i_{jkh} of an $NP-F_n$ admitting a special concircular Lie-recurrence satisfies the identity* (27).

The Lie-derivative of ρ_k is given by

$$\pounds\rho_k = v^r\nabla_r\rho_k + (\dot{\partial}_r\rho_k)\nabla_s v^r y^s + \rho_r\nabla_k v^r. \tag{28}$$

Using equation (17) and fact that ρ_k is homogeneous of degree 0 in y^i, we get

$$\pounds\rho_k = v^r\nabla_r\rho_k + \rho\rho_k. \tag{29}$$

Transvecting equation (22) by v^j, we get

$$v^j v^r\nabla_r N^i_{jkh} = (\phi - 2\rho)N^i_{jkh}v^j. \tag{30}$$

Differentiating equation (25) covariantly with respect to x^m and using (17), we obtain

$$\rho N^i_{khm} + v^r \nabla_m N^i_{khr} = \nabla_m \rho_h \delta^i_k - \nabla_m \rho_k \delta^i_h. \tag{31}$$

Transvecting equation (31) by v^m, we get

$$\rho v^m N^i_{khm} + v^m v^r \nabla_m N^i_{khr} = v^m (\nabla_m \rho_h \delta^i_k - \nabla_m \rho_k \delta^i_h). \tag{32}$$

In view of equation (30), equation (32) implies

$$(\phi - \rho) N^i_{khr} v^r = v^m (\nabla_m \rho_h \delta^i_k - \nabla_m \rho_k \delta^i_h). \tag{33}$$

Using equation (25) in equation (33), we get

$$(\phi - \rho)(\rho_h \delta^i_k - \rho_k \delta^i_h) = v^m (\nabla_m \rho_h \delta^i_k - \nabla_m \rho_k \delta^i_h). \tag{34}$$

Contracting indices i and h in equation (34), we get

$$(\phi - \rho)\rho_k = v^m \nabla_m \rho_k. \tag{35}$$

Using equation (35) in equation (29), we have

$$\pounds \rho_k = \phi \rho_k. \tag{36}$$

This leads to:

Theorem 3.3. *If an NP-F_n admits a special concircular Lie-recurrence characterized by equations* (16) *and* (17)*, the covariant derivative of scalar ρ appearing in equation* (17) *is Lie-recurrent with respect to the Lie-recurrence.*

4. Special concircular normal projective Lie-recurrence in a birecurrent $NP - F_n$

Let us consider a birecurrent $NP - F_n$ characterized by

$$\nabla_m \nabla_l N^i_{jkh} = a_{lm} N^i_{jkh}, \tag{37}$$

where a_{lm} are components of a non-zero covariant tensor of type $(0, 2)$ and $N^i_{jkh} \neq 0$ [10].

Suppose that this space admits a special concircular normal projective Lie-recurrence characterized by equations (16) and (17). Differentiating equation (22) covariantly with respect to x^m, we get

$$\nabla_m v^r \nabla_r N^i_{jkh} + v^r \nabla_m \nabla_r N^i_{jkh} = (\phi_m - 2\rho_m) N^i_{jkh} + (\phi - 2\rho) \nabla_m N^i_{jkh}, \tag{38}$$

where $\phi_m = \nabla_m \phi$. Using (17) and equation (37) in equation (38), we have

$$(v^r a_{rm} - \phi_m + 2\rho_m) N^i_{jkh} = (\phi - 3\rho) \nabla_m N^i_{jkh}. \tag{39}$$

In view of the definition for a birecurrent Finsler space, $N^i_{jkh} \neq 0$. In equation (39), $\nabla_m N^i_{jkh} \neq 0$, for $\nabla_m N^i_{jkh} = 0$ implies $\nabla_l \nabla_m N^i_{jkh} = 0$ i.e. $a_{lm} = 0$, a contradiction. Therefore, equation (39) implies either of the follwing conditions:

(i) $\phi - 3\rho = 0, \;\; v^r a_{mr} - \phi_m + 2\rho_m = 0,$

(ii) $\phi - 3\rho \neq 0, \;\; v^r a_{mr} - \phi_m + 2\rho_m \neq 0.$

We can write the condition (i) as $\phi = 3\rho$, $v^r a_{mr} = \rho_m$.
Let us consider the condition (ii). In this case equation (39) may be written as

$$\nabla_m N^i_{jkh} = \frac{(v^r a_{mr} - \phi_m + 2\rho_m)}{\phi - 3\rho} N^i_{jkh}, \tag{40}$$

which shows that the space is $NPR - F_n$. The second author [5] proved that an $NPR - F_n$ does not admit a special concircular vector field, which implies that an $NPR - F_n$ does not admit a special concircular normal projective Lie-recurrence. Therefore, the pair of conditions (ii) is not possible.
Hence, we may conclude:

Theorem 4.1. *A birecurrent $NP - F_n$ admitting a special concircular normal projective Lie-recurrence necessarily satisfies the conditions $\phi = 3\rho$ and $v^r a_{mr} = \rho_m$.*

Taking skew-symmetric part of equation (37) and using Ricci-commutation formula exhibited by (6), we have

$$N^r_{jkh} N^i_{lmr} - N^i_{rkh} N^r_{jlm} - N^i_{jrh} N^r_{klm} - N^i_{jkr} N^r_{lmh} - (\dot{\partial} N^i_{jkh}) N^r_{lms} y^s = A_{lm} N^i_{jkh}, \tag{41}$$

where $A_{lm} = a_{lm} - a_{ml}$. Operating both sides of equation (41) by operator $\pounds$ and using equation (16), we get

$$\pounds A_{lm} = \phi A_{lm}.$$

This leads to:

Theorem 4.2. *The skew-symmetric part of the recurrence tensor a_{lm} of a birecurrent NP-F_n admitting a special concircular normal projective Lie-recurrence is Lie-recurrent with respect to the Lie-recurrence.*

5. Special concircular normal projective Lie-recurrence in a bisymmetric $NP - F_n$

Let us consider a bisymmetric $NP - F_n$ admitting a special concircular normal projective Lie-recurrence characterized by [8]

$$\nabla_m \nabla_l N^i_{jkh} = 0. \tag{42}$$

In view of equations (17) and (42), equation (38) becomes

$$(\phi - 3\rho) \nabla_m N^i_{jkh} = (2\rho_m - \phi_m) N^i_{jkh}. \tag{43}$$

If $\phi = 3\rho$, equation (43) reduces to $\rho_m N^i_{jkh} = 0$ which implies $N^i_{jkh} = 0$ for $\rho_m \neq 0$. Thus, we conclude:

Theorem 5.1. *A bisymmetric $NP - F_n$ admitting a special concircular normal projective Lie-recurrence with condition $\phi = 3\rho$ is flat.*

If $\phi = 2\rho$ then $\phi_m = 2\rho_m$. Therefore, equation (43) may be written as

$$\nabla_m N^i_{jkh} = 0. \tag{44}$$

This shows that the space is symmetric. Thus, we see that a bisymmetric $NP - F_n$ admitting a special concircular normal projective Lie-recurrence with $\phi = 2\rho$ is a symmetric space admitting a special concircular normal projective Lie-recurrence. In view of a result due to the second author [9], a symmetric $NP - F_n$ admitting special concircular transformation is a Riemannian space.
Thus we conclude:

Theorem 5.2. *A bisymmetric $NP - F_n$ admitting a special concircular normal projective Lie-recurrence with $\phi = 2\rho$ is a Riemannian space with constant Riemannian curvature.*

If $\phi \neq 2\rho$ and $\phi \neq 3\rho$, then equation (43) may be written as

$$\nabla_m N^i_{jkh} = \frac{2\rho_m - \phi_m}{\phi - 3\rho} N^i_{jkh}. \tag{45}$$

This shows that the space is recurrent, but an $NPR - F_n$ admitting a special concircular normal projective Lie-recurrence does not exist.
Thus, we may conclude:

Theorem 5.3. *A bisymmetric $NP - F_n$ can not admit a special concircular normal projective Lie-recurrence if ϕ is neither 2ρ nor 3ρ.*

From Theorems 5.1, 5.2 and 5.3, we may conclude:

Theorem 5.4. *A bisymmetric $NP - F_n$ admitting a special concircular normal projective Lie-recurrence is either flat or a Riemannian space of constant Riemannian curvature.*

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