Annals of the University of Craiova, Mathematics and Computer Science Series
Volume 47(1), 2020, Pages 132–140
ISSN: 1223-6934

Hypersurfaces of a Finsler space with exponential form of (α, β)- metric

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Abstract. In the present paper we have studied the Finslerian hypersurfaces of a Finsler space with the special form of exponential (α, β)- metric such as $L(\alpha, \beta) = \alpha e^{\frac{\beta}{\alpha}} + \beta e^{-\frac{\beta}{\alpha}}$. We also examined the hypersurfaces of this special exponential metric as a hyperplane of first, second and third kinds.

2010 Mathematics Subject Classification. 53B40, 53C60.
Key words and phrases. Finslerian hypersurface, exponential (α, β)-metric,Induced Cartan Connection,Hyperplane of first,hyperplane of second and hyperplane of third kind.

1. Introduction

We consider an n-dimensional Finsler space $F^n = (M^n, L)$, i.e., a pair consisting of an n-dimensional differential manifold M^n equipped with a fundamental function $L(x, y)$.

Definition 1.1. A Finsler space F^n of dimension n is a differentiable manifold such that the length s of the curve $x^i(t)$ of F^n is defined by $S = \int L(x, y)dt$.The fundamental function $L(x, y) = L(x^1, ...x^n, y^1, ...y^n)$ is supposed to be differentiable for $y \neq (0)$ and satisfies the following condition:-
(i) Positively homogeneous: $L(x, py) = pL(x, y)$, $p > 0$.
(ii) Positive: $L(x, y) > 0$, $y \neq 0$.
(iii)Positive definite metric $g_{ij} = \frac{1}{2}\frac{\partial^2 L}{\partial y^i \partial y^j} > 0$.

The interesting and important examples of an (α, β)- metric are Randers metric $\alpha + \beta$, Kropina metric $\frac{\alpha^2}{\beta}$ and Matsumoto metric [9] $\frac{\alpha^2}{(\alpha-\beta)}$. The notion of an (α, β)-metric was introduced by M. Matsumoto [7] and has been studied by many authors.

Definition 1.2. A Finsler metric $L(\alpha, \beta)$ in a differentiable manifold M^n is called an (α, β)-metric, if L is a positively homogeneous function of degree one of a Riemannian metric $\alpha = (a_{ij}(x)y^i y^j)^{\frac{1}{2}}$ and a one-form $\beta = b_i(x)y^i$ on M^n.

Definition 1.3. A Hypersurface is a generalization of the concept of hyperplane. Suppose an enveloping manifold M has n dimension, then any submanifold of M of $(n - 1)$ dimension is a hypersurface. The co-dimension of hypersurface is one. If $m > n$, then V_n is called a subspace of V_m. V_m is also enveloping space of V_n if $m > n$. In particular if $m = n + 1$, then V_n is called hypersurface of the enveloping space V_{n+1}.

Received June 20, 2018. Accepted May 25, 2019.

A hypersurface M^{n-1} of the M^n may be represented parametrically by the equation $x^i = x^i(u^\alpha), \alpha = 1, 2, 3, ..., (n-1)$, where u are Gaussian coordinates on M^{n-1}. If the supporting element y^i at a point u^α of M^{n-1} is assumed to be tangential to M^{n-1}, we may then write $y^i = B^i_\alpha(u)v^\alpha$, so that v^α thought of as the supporting element of M^{n-1} at a point (u^α). Since the function $\underline{L}(u,v) := L(x(u), y(u,v))$ gives rise to a Finsler metric of M^{n-1}, we get an $(n-1)$-dimensional Finsler space $F^{(n-1)} = (M^{n-1}, \underline{L}(u,v))$.

The concept of Finslerian hypersurface is first introduced by Matsumoto in the year 1985 and further he defined three types of hypersurfaces that were called a hyperplane of the first, second and third kinds. Further many authors studied these hyperplanes in different changes of the Finsler metric [2, 3, 4, 5, 6, 10, 11] and obtained different results.

In the present paper we consider exponential form of special Finsler (α, β)-metric and determine Induced Riemannian metric,second fundamental v tensor also examined the hypersurfaces of this special metric as a hyperplane of first, second and third kinds.

2. Preliminaries

In the present paper we consider an n-dimensional Finsler space $F^n = \{M^n, L(\alpha, \beta)\}$, that is, a pair consisting of an n- dimensional differentiable manifold M^n equipped with a Fundamental function L as a special Finsler Space with the metric

$$L(\alpha, \beta) = \alpha e^{\frac{\beta}{\alpha}} + \beta e^{-\frac{\beta}{\alpha}} \tag{1}$$

Differentiating equation (1) partially with respect to α and β are given by

$$L_\alpha = e^{\frac{\beta}{\alpha}} - \frac{\beta}{\alpha} e^{\frac{\beta}{\alpha}} + \frac{\beta^2}{\alpha^2} e^{-\frac{\beta}{\alpha}}, \qquad L_\beta = e^{\frac{\beta}{\alpha}} - e^{-\frac{\beta}{\alpha}} - \frac{\beta}{\alpha} e^{-\frac{\beta}{\alpha}}$$
$$L_{\alpha\alpha} = \frac{\beta^2}{\alpha^3} e^{\frac{\beta}{\alpha}} - \frac{2\beta^2}{\alpha^3} e^{-\frac{\beta}{\alpha}} + \frac{\beta^3}{\alpha^4} e^{-\frac{\beta}{\alpha}}, \qquad L_{\beta\beta} = \frac{1}{\alpha} e^{\frac{\beta}{\alpha}} - \frac{2}{\alpha} e^{-\frac{\beta}{\alpha}} + \frac{\beta}{\alpha^2} e^{-\frac{\beta}{\alpha}}$$
$$L_{\alpha\beta} = \frac{\beta}{\alpha^2} e^{-\frac{\beta}{\alpha}} - \frac{\beta^2}{\alpha^3} e^{-\frac{\beta}{\alpha}},$$

where $L_\alpha = \frac{\partial L}{\partial \alpha}, \quad L_\beta = \frac{\partial L}{\partial \beta}, \quad L_{\alpha\alpha} = \frac{\partial L\alpha}{\partial \alpha}, \quad L_{\beta\beta} = \frac{\partial L_\beta}{\partial \beta}, \quad L_{\alpha\beta} = \frac{\partial L_\alpha}{\partial \beta}$.

In Finsler space $F^n = \{M^n, L(\alpha, \beta)\}$ the normalized element of support $l_i = \dot{\partial}_i L$ and angular metric tensor h_{ij} are given by [8]:

$$l_i = \alpha^{-1} L_\alpha Y_i + L_\beta b_i$$
$$h_{ij} = p a_{ij} + q_0 b_i b_j + q_{-1}(b_i Y_j + b_j Y_i) + q_{-2} Y_i Y_j$$

where $Y_i = a_{ij} y^j$. For the fundamental metric function (1) above constants are

$$p = \frac{\beta}{\alpha} + (1 - \frac{\beta}{\alpha}) e^{\frac{2\beta}{\alpha}} + \frac{\beta^3}{\alpha^3} e^{-\frac{2\beta}{\alpha}},$$
$$q_0 = -2 + \frac{2\beta}{\alpha} + e^{\frac{2\beta}{\alpha}} - (2 - \frac{\beta}{\alpha}) \frac{\beta}{\alpha} e^{-\frac{2\beta}{\alpha}},$$
$$q_{-1} = (1 - \frac{\beta}{\alpha}) \{ \frac{\beta}{\alpha^2} + \frac{\beta^2}{\alpha^3} e^{-\frac{2\beta}{\alpha}} \},$$
$$q_{-2} = \frac{1}{\alpha^6} \{ 2\alpha\beta^3 - \alpha^3\beta - 2\alpha^2\beta^2 + (\alpha^2\beta^2 - \alpha^4 + \alpha^3\beta) e^{\frac{2\beta}{\alpha}} + (\beta^4 - 3\alpha\beta^3) e^{-\frac{2\beta}{\alpha}} \}. \tag{2}$$

Fundamental metric tensor $g_{ij} = \frac{1}{2}\dot{\partial}_i\dot{\partial}_j L^2$ and its reciprocal tensor g^{ij} for $L = L(\alpha, \beta)$ are given by [8]

$$g_{ij} = pa_{ij} + p_0 b_i b_j + p_{-1}(b_i Y_j + b_j Y_i) + p_{-2} Y_i Y_j, \tag{3}$$

where

$$\begin{aligned} p_0 &= q_0 + L_\beta^2, \\ p_{-1} &= q_{-1} + L^{-1} p L_\beta, \\ p_{-2} &= q_{-2} + p^2 L^{-2}. \end{aligned} \tag{4}$$

The reciprocal tensor g^{ij} of g_{ij} is given by

$$g^{ij} = p^{-1}a^{ij} - s_0 b^i b^j - s_{-1}(b^i y^j + b^j y^i) - s_{-2} y^i y^j, \tag{5}$$

where $b^i = a^{ij} b_j$ *and* $b^2 = a_{ij} b^i b^j$

$$\begin{aligned} s_0 &= \frac{1}{\tau p}\{pp_0 + (p_0 p_{-2} - p_{-1}^2)\alpha^2\}, \\ s_{-1} &= \frac{1}{\tau p}\{pp_{-1} + (p_0 p_{-2} - p_{-1}^2)\beta\}, \\ s_{-2} &= \frac{1}{\tau p}\{pp_{-2} + (p_0 p_{-2} - p_{-1}^2)b^2\}, \\ \tau &= p(p + p_0 b^2 + p_{-1}\beta) + (p_0 p_{-2} - p_{-1}^2)(\alpha^2 b^2 - \beta^2). \end{aligned} \tag{6}$$

The hv-torsion tensor $C_{ijk} = \frac{1}{2}\dot{\partial}_k g_{ij}$ is given by [11]

$$2pC_{ijk} = p_{-1}(h_{ij} m_k + h_{jk} m_i + h_{ki} m_j) + \gamma_1 m_i m_j m_k, \tag{7}$$

where

$$\gamma_1 = p\frac{\partial p_0}{\partial \beta} - 3p_{-1} q_0, \quad m_i = b_i - \alpha^{-2}\beta Y_i. \tag{8}$$

Here m_i is a non-vanishing covariant vector orthogonal to the element of support y^i.

Let $\{^{\,i}_{jk}\}$ be the component of Christoffel symbols of the associated Riemannian space R^n and $\bigtriangledown_k$ be the covariant derivative with respect to x^k relative to this Christoffel symbol. Now we define,

$$2E_{ij} = b_{ij} + b_{ji}, \qquad 2F_{ij} = b_{ij} - b_{ji} \tag{9}$$

where $b_{ij} = \bigtriangledown_j b_i$.

Let $C\Gamma = (\Gamma^{*i}_{jk}, \Gamma^{*i}_{0k}, \Gamma^{i}_{jk})$ be the Cartan connection of F^n. The difference tensor $D^i_{jk} = \Gamma^{*i}_{jk} - \{^{\,i}_{jk}\}$ of the special Finsler space F^n is given by

$$\begin{aligned} D^i_{jk} =& B^i E_{jk} + F^i_k B_j + F^i_j B_k + B^i_j b_{0k} + B^i_k b_{0j} - b_{0m} g^{im} B_{jk} - C^i_{jm} A^m_k - C^i_{km} A^m_j \\ &+ C_{jkm} A^m_s g^{is} + \lambda^s (C^i_{jm} C^m_{sk} + C^i_{km} C^m_{sj} - C^m_{jk} C^i_{ms}) \end{aligned} \tag{10}$$

where

$$\begin{aligned} &B_k = p_0 b_k + p_{-1} Y_k, \quad B^i = g^{ij} B_j, \quad F^k_i = g^{kj} F_{ji}, \\ &B_{ij} = \frac{1}{2}\left[p_{-1}(a_{ij} - \alpha^{-2} Y_i Y_j) + \frac{\partial p_0}{\partial \beta} m_i m_j\right], \quad B^k_i = g^{kj} B_{ji}, \\ &A^m_k = B^m_k E_{00} + B^m E_{k0} + B_k F^m_0 + B_0 F^m_k, \\ &\lambda^m = B^m E_{00} + 2B_0 F^m_0, \quad B_0 = B_i y^i, \end{aligned} \tag{11}$$

where $'0'$ denote contraction with y^i except for the quantities p_0, q_0 and s_o.

3. Induced Cartan Connection

Let F^{n-1} be a hypersurface of F^n given by the equation $x^i = x^i(u^\alpha)$ where $\{\alpha = 1, 2, 3...(n-1)\}$. The element of support y^i of F^n is to be taken tangential to F^{n-1}, that is [8],

$$y^i = B^i_\alpha(u)v^\alpha \tag{12}$$

the metric tensor $g_{\alpha\beta}$ and hv-tensor $C_{\alpha\beta\gamma}$ of F^{n-1} are given by

$$g_{\alpha\beta} = g_{ij}B^i_\alpha B^j_\beta, \quad C_{\alpha\beta\gamma} = C_{ijk}B^i_\alpha B^j_\beta B^k_\gamma,$$

and at each point (u^α) of F^{n-1}, a unit normal vector $N^i(u, v)$ is defined by

$$g_{ij}\{x(u, v), y(u, v)\}B^i_\alpha N^j = 0, \quad g_{ij}\{x(u, v), y(u, v)\}N^i N^j = 1.$$

Angular metric tensor $h_{\alpha\beta}$ of the hypersurface are given by

$$h_{\alpha\beta} = h_{ij}B^i_\alpha B^j_\beta, \quad h_{ij}B^i_\alpha N^j = 0, \quad h_{ij}N^i N^j = 1, \tag{13}$$

(B^α_i, N_i) inverse of (B^i_α, N^i) is given by

$$B^\alpha_i = g^{\alpha\beta}g_{ij}B^j_\beta, \quad B^i_\alpha B^\beta_i = \delta^\beta_\alpha, \quad B^\alpha_i N^i = 0, \quad B^i_\alpha N_i = 0,$$

$$N_i = g_{ij}N^j, \quad B^k_i = g^{kj}B_{ji}, \quad B^i_\alpha B^\alpha_j + N^i N_j = \delta^i_j.$$

The induced connection $IC\Gamma = (\Gamma^{*\alpha}_{\beta\gamma}, G^\alpha_\beta, C^\alpha_{\beta\gamma})$ of F^{n-1} from the Cartan's connection $C\Gamma = (\Gamma^{*i}_{jk}, \Gamma^{*i}_{0k}, C^{*i}_{jk})$ is given by [8].

$$\Gamma^{*\alpha}_{\beta\gamma} = B^\alpha_i(B^i_{\beta\gamma} + \Gamma^{*i}_{jk}B^j_\beta B^k_\gamma) + M^\alpha_\beta H_\gamma,$$

$$G^\alpha_\beta = B^\alpha_i(B^i_{0\beta} + \Gamma^{*i}_{0j}B^j_\beta), \quad C^\alpha_{\beta\gamma} = B^\alpha_i C^i_{jk}B^j_\beta B^k_\gamma,$$

where

$$M_{\beta\gamma} = N_i C^i_{jk}B^j_\beta B^k_\gamma, \quad M^\alpha_\beta = g^{\alpha\gamma}M_{\beta\gamma}, \quad H_\beta = N_i(B^i_{0\beta} + \Gamma^{*i}_{oj}B^j_\beta),$$

and

$$B^i_{\beta\gamma} = \frac{\partial B^i_\beta}{\partial u^\gamma}, \quad B^i_{0\beta} = B^i_{\alpha\beta}v^\alpha.$$

The quantities $M_{\beta\gamma}$ and H_β are called the second fundamental v-tensor and normal curvature vector respectively [8]. The second fundamental h-tensor $H_{\beta\gamma}$ is defined as [8]

$$H_{\beta\gamma} = N_i(B^i_{\beta\gamma} + \Gamma^{*i}_{jk}B^j_\beta B^k_\gamma) + M_\beta H_\gamma, \tag{14}$$

where

$$M_\beta = N_i C^i_{jk}B^j_\beta N^k. \tag{15}$$

The relative h and v-covariant derivatives of projection factor B^i_α with respect to $IC\Gamma$ are given by

$$B^i_{\alpha|\beta} = H_{\alpha\beta}N^i, \quad B^i_\alpha|_\beta = M_{\alpha\beta}N^i.$$

It is obvious form the equation (14) that $H_{\beta\gamma}$ is generally not symmetric and

$$H_{\beta\gamma} - H_{\gamma\beta} = M_\beta H_\gamma - M_\gamma H_\beta. \tag{16}$$

The above equation yield

$$H_{0\gamma} = H_\gamma, \quad H_{\gamma 0} = H_\gamma + M_\gamma H_0. \tag{17}$$

We shall use following lemmas which are due to Matsumoto [8] in the coming section

Lemma 3.1. *The normal curvature* $H_0 = H_\beta v^\beta$ *vanishes if and only if the normal curvature vector* H_β *vanishes.*

Lemma 3.2. *A hypersurface* $F^{(n-1)}$ *is a hyperplane of the first kind with respect to connection* $C\Gamma$ *if and only if* $H_\alpha = 0$.

Lemma 3.3. *A hypersurface* $F^{(n-1)}$ *is a hyperplane of the second kind with respect to connection* $C\Gamma$ *if and only if* $H_\alpha = 0$ *and* $H_{\alpha\beta} = 0$.

Lemma 3.4. *A hypersurface* $F^{(n-1)}$ *is a hyperplane of the third kind with respect to connection* $C\Gamma$ *if and only if* $H_\alpha = 0$ *and* $H_{\alpha\beta} = M_{\alpha\beta} = 0$.

4. Hypersurface $F^{(n-1)}(c)$ of a exponential form of (α, β)-metric in Finsler space

Let us consider a Finsler space with exponential form of (α, β)-metric $L(\alpha, \beta) = \alpha e^{\frac{\beta}{\alpha}} + \beta e^{-\frac{\beta}{\alpha}}$, where, $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and vector field $b_i(x) = \frac{\partial b}{\partial x^i}$ is a gradient of some scalar function $b(x)$. Now we consider a hypersurface $F^{(n-1)}(c)$ given by equation $b(x) = c$, a constant [11].

From the parametric equation $x^i = x^i(u^\alpha)$ of $F^{n-1}(c)$, we get

$$\frac{\partial b(x)}{\partial u^\alpha} = 0, \qquad \frac{\partial b(x)}{\partial x^i}\frac{\partial x^i}{\partial u^\alpha} = 0, \qquad b_i B^i_\alpha = 0.$$

Above shows that $b_i(x)$ are covariant component of a normal vector field of hypersurface $F^{n-1}(c)$. Further, we have

$$b_i B^i_\alpha = 0 \quad \text{and} \quad b_i y^i = 0 \quad \text{i.e.} \quad \beta = 0, \tag{18}$$

and induced matric $L(u, v)$ of $F^{n-1}(c)$ is given by

$$L(u, v) = a_{\alpha\beta} v^\alpha v^\beta, a_{\alpha\beta} = a_{ij} B^i_\alpha B^j_\beta \tag{19}$$

which is a Riemannian metric.

Writing $\beta = 0$ in the equations (2), (3) and (5) we get

$$\begin{aligned} &p = 1, \quad q_0 = -1, \quad q_{-1} = 0, \quad q_{-2} = -\alpha^{-2}, \\ &p_0 = 3, \quad p_{-1} = 2\alpha^{-1}, \quad p_{-2} = 0, \quad \tau = 1 - b^2, \\ &s_0 = -\frac{1}{1-b^2}, \quad s_{-1} = \frac{2}{\alpha(1-b^2)} \quad s_{-2} = -\frac{4b^2}{\alpha^2(1-b^2)}, \end{aligned} \tag{20}$$

from (4) we get,

$$g^{ij} = a^{ij} + \frac{1}{1-b^2} b^i bj - \frac{2}{\alpha(1-b^2)}(b^i y^j + b^j y^i) - \frac{4b^2}{\alpha^2(1-b^2)} y^i y^j \tag{21}$$

thus along $F^{n-1}(c)$, (21) and (18) leads to

$$g^{ij} b_i b_j = \frac{b^2}{(1-b^2)}.$$

So we get

$$b_i(x(u)) = \sqrt{\frac{b^2}{(1-b^2)}} N_i, \quad b^2 = a^{ij} b_i b_j, \tag{22}$$

where b is the length of the vector b^i.

Again from (21) and (22), we get

$$b^i = a^{ij}b_j = (1-b^2)N^i + \frac{2b^2y^i}{\alpha}, \tag{23}$$

thus we have,

Theorem 4.1. *The exponential form of (α,β)-metric, $L(\alpha,\beta) = \alpha e^{\frac{\beta}{\alpha}} + \beta e^{-\frac{\beta}{\alpha}}$ in a Finsler hypersurface $F^{(n-1)}(c)$, the Induced Riemannian metric is given by (19) and the scalar function $b(x)$ is given by (22) and (23).*

Now the angular metric tensor h_{ij} and metric tensor g_{ij} of F^n are given by

$$h_{ij} = a_{ij} - b_ib_j - \frac{1}{\alpha^2}Y_iY_j \quad and \quad g_{ij} = a_{ij} + 3b_ib_j + \frac{2}{\alpha}(b_iY_j + b_jY_i) \tag{24}$$

From equation (18), (24) and (13) it follows that if $h^{(a)}_{\alpha\beta}$ denote the angular metric tensor of the Riemannian $a_{ij}(x)$ then we have along $F^{n-1}_{(c)}$, $h_{\alpha\beta} = h^{(a)}_{\alpha\beta}$.

Thus along $F^{n-1}_{(c)}$, $\frac{\partial p_0}{\partial\beta} = -\frac{2}{\alpha}$, from equation (8) we get

$$r_1 = \frac{2}{\alpha}, \quad m_i = b_i,$$

then hv-torsion tensor becomes

$$C_{ijk} = \frac{1}{\alpha}(h_{ij}b_k + h_{jk}b_i + h_{ki}b_j) + \frac{1}{\alpha}b_ib_jb_k \tag{25}$$

in the exponential form of (α,β)-metric of a Finsler hypersurface $F^{(n-1)}_{(c)}$. Due to fact from (13), (14), (15), (18) and (25) we have

$$M_{\alpha\beta} = \frac{1}{\alpha}\sqrt{\frac{b^2}{(1-b^2)}}h_{\alpha\beta} \quad \text{and} \quad M_\alpha = 0. \tag{26}$$

Therefore from equation (17) it follows that $H_{\alpha\beta}$ is symmetric. Thus we have

Theorem 4.2. *The second fundamental v-tensor of the exponential form of (α,β)-metric with Finsler hypersurface $F^{(n-1)}_{(c)}$ is given by (26) and the second fundamental h-tensor $H_{\alpha\beta}$ is symmetric.*

Now from (18) we have $b_iB^i_\alpha = 0$. Then we have

$$b_{i|\beta}B^i_\alpha + b_iB^i_{\alpha|\beta} = 0.$$

Therefore, from (16) and using $b_{i|\beta} = b_{i|j}B^j_\beta + b_i|_jN^jH_\beta$, we have

$$b_{i|j}B^i_\alpha B^j_\beta + b_{i|j}B^i_\alpha N^jH_\beta + b_iH_{\alpha\beta}N^i = 0, \tag{27}$$

since $b_i|_j = -b_hC^h_{ij}$, we get

$$b_{i|j}B^i_\alpha N^j = 0.$$

Therefore from equation (27) we have,

$$\sqrt{\frac{b^2}{(1-b^2)}}H_{\alpha\beta} + b_{i|j}B^i_\alpha B^j_\beta = 0 \tag{28}$$

because $b_{i|j}$ is symmetric. Now contracting (28) with v^{β} and using (12) we get

$$\sqrt{\frac{b^2}{(1-b^2)}}H_{\alpha}+b_{i|j}B^i_{\alpha}y^j=0. \tag{29}$$

Again contracting by v^{α} equation (29) and using (12), we have

$$\sqrt{\frac{b^2}{(1-b^2)}}H_0+b_{i|j}y^iy^j=0. \tag{30}$$

From Lemma 3.1 and 3.2, it is clear that exponential (α,β)-metric with Finsler hypersurface $F^{(n-1)}_{(c)}$ is a hyperplane of first kind if and only if $H_0=0$. Thus from (29) it is obvious that $F^{n-1}_{(c)}$ is a hyperplane of first kind if and only if $b_{i|j}y^iy^j=0$. This $b_{i|j}$ being the covariant derivative with respect to $C\Gamma$ of F^n defined on y^i, but $b_{ij}=\bigtriangledown_j b_i$ is the covariant derivative with respect to Riemannian connection $\{^{\,i}_{jk}\}$ constructed from $a_{ij}(x)$. Hence b_{ij} does not depend on y^i. We shall consider the difference $b_{i|j}-b_{ij}$ where $b_{ij}=\bigtriangledown_j b_i$ in the following. The difference tensor $D^i_{jk}=\Gamma^{*i}_{jk}-\{^{\,i}_{jk}\}$ is given by (10). Since b_i is a gradient vector, then from (9) we have

$$E_{ij}=b_{ij},\quad F_{ij}=0,\quad \text{and}\quad F^i_j=0.$$

Thus (10) reduces to

$$\begin{aligned} D^i_{jk} \;=\;& B^ib_{jk}+B^i_jb_{0k}+B^i_kb_{0j}-b_{0m}g^{im}B_{jk}-C^i_{jm}A^m_k-C^i_{km}A^m_j\\ &+C_{jkm}A^m_sg^{is}+\lambda^s(C^i_{jm}C^m_{sk}+C^i_{km}C^m_{sj}-C^m_{jk}C^i_{ms}), \end{aligned} \tag{31}$$

where

$$\begin{aligned} B_i&=3b_i+2\alpha^{-1}Y_i,\quad B^i=-\frac{1}{(1-b^2)}b^i+\frac{2(1-8b^2)}{\alpha(1-b^2)}y^i,\\ B_iB^i&=\frac{4-35b^2}{1-b^2},\lambda^m=B^mb_{00},\quad B_{ij}=\frac{1}{\alpha}(a_{ij}-\frac{Y_iY_j}{\alpha^2})-\frac{1}{\alpha}b_ib_j,\\ B^i_j&=\frac{1}{\alpha}(\delta^i_j-\alpha^{-2}Y^iY_j)-\frac{4}{\alpha^2}b^iY_j-\frac{2(1+b^2)}{\alpha^2}b_jy^i-\frac{8b^2}{\alpha^3(1-b^2)}y^iY_j,\\ A^m_k&=B^m_kb_{00}+B^mb_{k0}. \end{aligned} \tag{32}$$

In view of (20) and (21), the relation in (11) becomes to by virtue of (32) we have $B^i_0=0,\ B_{i0}=0$ which leads $A^m_0=B^mb_{00}$.

Now contracting (31) by y^k we get

$$D^i_{j0}=B^ib_{j0}+B^i_jb_{00}-B^mC^i_{jm}b_{00}.$$

Again contracting the above equation with respect to y^j we have

$$D^i_{00}=B^ib_{00}=\{-\frac{1}{(1-b^2)}b^i+\frac{2(1-8b^2)}{\alpha(1-b^2)}y^i\}b_{00}.$$

Paying attention to (18), along $F^{(n-1)}_{(c)}$, we get

$$b_iD^i_{j0}=-\frac{b^2}{(1-b^2)}b_{j0}+\frac{1}{\alpha}b_jb_{00}-\frac{4b^2}{\alpha^2(1-b^2)}Y_jb_{00}-\frac{b^2}{1-b^2}b_ib^mC^i_{jm}b_{00}. \tag{33}$$

Now we contract (33) by y^j we have

$$b_i D^i_{00} = -\frac{5b^2}{1-b^2} b_{00}. \tag{34}$$

From (14), (22), (23), (26) and $M_\alpha = 0$, we have

$$b_i b^m C^i_{jm} B^j_\alpha = b^2 M_\alpha = 0.$$

Thus the relation $b_{i|j} = b_{ij} - b_r D^r_{ij}$ the equation (33) and (34) gives

$$b_{i|j} y^i y^j = b_{00} - b_r D^r_{00} = \frac{1+4b^2}{1-b^2} b_{00}$$

Consequently (29) and (30) may be written as

$$\sqrt{\frac{b^2}{1-b^2}} H_\alpha + \frac{1+4b^2}{1-b^2} b_{i0} B^i_\alpha = 0, \quad \sqrt{\frac{b^2}{1-b^2}} H_0 + \frac{1+4b^2}{1-b^2} b_{00} = 0. \tag{35}$$

Thus the condition $H_0 = 0$ is equivalent to $b_{00} = 0$. Using the fact $\beta = b_i y^i = 0$ the condition $b_{00} = 0$ can be written as $b_{ij} y^i y^j = b_i y^i b_j y^j$ for some $c_j(x)$. Thus we can write,

$$2b_{ij} = b_i c_j + b_j c_i. \tag{36}$$

Now from (18) and (36) we get

$$b_{00} = 0, \quad b_{ij} B^i_\alpha B^j_\beta = 0, \quad b_{ij} B^i_\alpha y^j = 0.$$

Hence from (35) we get $H_\alpha = 0$, again from (36) and (32) we get $b_{i0} b^i = \frac{c_0 b^2}{2}$, $\lambda^m = 0$, $A^i_j B^j_\beta = 0$ and $B_{ij} B^i_\alpha B^j_\beta = \frac{1}{\alpha} h_{\alpha\beta}$.

Now we use equation (14), (21), (22), (23), (26) and (31) then we have

$$b_r D^r_{ij} B^i_\alpha B^j_\beta = -\frac{c_0 b^3}{2\alpha(1-b^2)^{3/2}} h_{\alpha\beta}. \tag{37}$$

Thus the equation (28) reduces to

$$H_{\alpha\beta} + \frac{c_0 b^2}{2\alpha(1-b^2)} h_{\alpha\beta} = 0. \tag{38}$$

Hence the hypersurface $F^{n-1}_{(c)}$ is umbilical.

Theorem 4.3. *The necessary and sufficient condition for a exponential form of (α, β)-metric in Finsler hypersurface $F^{(n-1)}_{(c)}$ to be a hyperplane of first kind is (36). In this case the second fundamental tensor of $F^{n-1}_{(c)}$ is proportional to its angular metric tensor.*

Now from Lemma 3.3, $F^{(n-1)}_{(c)}$ is a hyperplane of second kind if and only if $H_\alpha = 0$ and $H_{\alpha\beta} = 0$. Thus from (37), we get

$$c_0 = c_i(x) y^i = 0.$$

Therefore there exists a function $\psi(x)$ such that

$$c_i(x) = \psi(x) b_i(x).$$

Therefore (36) we get

$$2b_{ij} = b_i(x)\psi(x)b_j(x) + b_j(x)\psi(x)b_i(x).$$

This can also be written as

$$b_{ij} = \psi(x) b_i b_j.$$

Theorem 4.4. *The necessary and sufficient condition for a exponential form of (α, β)-metric in Finsler hypersurface $F_{(c)}^{(n-1)}$ to be a hyperplane of second kind is (38).*

Again Lemma 4.4, together with (26) and $M_\alpha = 0$ shows that $F_{(c)}^{n-1}$ does not become a hyperplane of third kind.

Theorem 4.5. *The exponential form of (α, β)-metric in a Finsler hypersurface $F_{(c)}^{(n-1)}$ is not a hyperplane of the third kind.*

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