Annals of the University of Craiova, Mathematics and Computer Science Series
Volume 47(1), 2020, Pages 141–157
ISSN: 1223-6934

Double merging of phase space for differential equations with small stochastic supplements under Levy and Poisson approximation conditions

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Abstract. The paper is devoted to the study of limit theorems of evolving evolutionary systems of "particles" in random environment. Here the term "particle" is used broadly to include molecules in the infected individuals considered in epidemic models, species in logistic growth models, age classes of population in demographics models, etc. The evolutionary system is complicated by the influence of impulse perturbation and non-trivial structure of the random environment. Namely, the the switching Markov process has a split phase space of states. We propose a new approach in construction of the approximation scheme for the impulse perturbation that allows not only to see the averaged and diffusion component of the limit process, but also to preserve Poisson jumps that models catastrophic events like mass extinction, earthquakes, etc. We discuss limit behavior of the generators of the evolutionary systems that allows not only to claim convergence of corresponding distributions, but to use the results obtained for solving the problems of stability and dissipativity of the limit processes.

2010 Mathematics Subject Classification. 60H10, 34F05.
Key words and phrases. stochastic differential equation, Levy and Poisson approximation schemes, double merging, Markov switch.

1. Introduction

During the analysis of complex systems, we often meet difficulties, which are essentially complicated by the phase space of such systems. This situation can lead an investigator to the practical impossibility of a visual representation of the model. The actual problem of modern system theory is the development of mathematically justified methods for constructing simplified models and their analysis does not cause significant difficulties. Of course, most of characteristics of such models can be taken for the corresponding characteristics of real models.

Concerning this problem, in order to be able to give analytical or numerical tractable models, the state space must be simplified via a reduction of the number of states. This is possible when some subsets are connected between them by small transition probabilities and the states within such subsets are asymptotically connected. That is typically the case of reliability - and in most applications involving hitting time models, for which the state space is naturally cut into two subsets (the up states set and the down states set). In this case, transitions between the subsets are slow compared with those within the subsets. In the literature, the reduction of state space is also called aggregation, lumping, or consolidation of state space.

Received June 23, 2018. Accepted May 25, 2019.

For the first time, the algorithm for phase consolidation of states of the system was proposed and described in [1] by Korolyuk V.S. and Turbin A.F. The analysis of the merging system is greatly simplified, but at the same time, with the successful splitting of the phase space, the main characteristics of the simplified system can accurately reflect the corresponding characteristics of the output. In turn, the proximity of real and merging systems also means the proximity of global characteristics, which are determined at increasing intervals of time. An important property of the algorithms for phase consolidation is the possibility of constructing a hierarchy of aggregated systems $\hat{S}$, $\hat{\hat{S}}$, ...

Random evolution in the form of a differential equation with stochastic applications use to describe a wide class of natural processes in many branches of science. An extremely important case is the study of the behavior of similar evolutionary systems in a random environment. The study of such systems is devoted to a large number of famous scientists, including A.V. Skorokhod, M.I. Gichman, M.M. Bogolyubov and others. A detailed bibliography on this subject can be found, for example, in the monographs of V.S. Korolyuk [2], [3]. Particular attention should be paid to the works [4], [5], [7], [6], [8], in which the approaches used in this paper were initiated, in particular to the study of the stability and control of an evolutionary system with diffusion perturbation.

This paper is devoted to the case when the system perturbations are determined by the impulse process in the Levi and Poisson approximation schemes. First of all, we will be interested in the double merging of the phase space of states of such evolutionary models.

2. Double merging of phase space for differential equations with small stochastic supplements under Levi approximation conditions

We investigate the stochastic evolutionary system in ergodic Markov environment

$$du^\varepsilon(t) = C(u^\varepsilon(t), x(t/\varepsilon^3)dt + d\eta^\varepsilon(t),\ u^\varepsilon(t) \in \mathbb{R}, \tag{2.1}$$

where $x^\varepsilon(t), t \geq 0$ is Markov process determined on a standard phase space $(E, \mathcal{E})$ with splitting

$$E = \bigcup_{k=1}^{N} E_k,\ E_k \cap E_{k'} = \emptyset,\ k \neq k'$$

in a series scheme with a small parameter of a series of $\varepsilon \to 0$, $\varepsilon > 0$.

The Markov kernel has the form

$$\mathbf{Q}^\varepsilon(x, B, t) = P^\varepsilon(x, B)[1 - \exp\{-q(x)t\}],\ x \in E,\ B \in \mathcal{E},\ t \geq 0.$$

Let us consider the following assumptions:

1: A kernel describing transition probabilities of imbedded Markov chain $x_n^\varepsilon, n \geq 0$ is defined as follows

$$P^\varepsilon(x, B) = P(x, B) + \varepsilon P_1(x, B).$$

The stochastic kernel $P(x, B)$ on the split phase space is defined as

$$P(x, E_k) = 1_k(x) = \begin{cases} 1,\ x \in E_k, \\ 0,\ x \notin E_k. \end{cases}$$

The stochastic kernel $P(x,B)$ define accompanying Markov chain x_n, $n \geq 0$ on classes E_k, $1 \leq k \leq N$. In addition, the perturbing kernel $P_1(x,B)$ satisfies the condition

$$P_1(x,E) = 0,$$

that is a direct corollary of equality

$$P^\varepsilon(x,E) = P(x,E) = 1.$$

2: Associated Markov process $x^0(t)$, $t \geq 0$, given by the generator

$$\mathbf{Q}\varphi(x) = q(x)\int_E P(x,dy)[\varphi(y) - \varphi(x)],$$

is uniformly ergodic in each of classes E_k, $1 \leq k \leq N$, with stationary distribution $\pi_k(dx)$, $1 \leq k \leq N$, satisfied relation:

$$\pi_k(dx)q(x) = q_k\rho_k(dx),\ q_k := \int_{E_k} \pi_k(dx)q(x).$$

3: Average probability of exit

$$\hat{p}_k := q(x)\int_{E_k} \rho_k(dx)P_1(x, E/E_k) > 0,\ 1 \leq k \leq N.$$

Thus, the perturbing kernel $P_1(x,B)$ determines transitional probabilities between classes E_k, $1 \leq k \leq N$, so, equality

$$P^\varepsilon(x,B) = P(x,B) + \varepsilon P_1(x,B)$$

means that imbedded Markov chain x_n^ε, $n \geq 0$ spends a great deal of time in each of the classes E_k and jumps between classes with low probabilities $\varepsilon P_1(x, E/E_k)$.

Example 2.1. Consider a three state Markov process, $E^0 = \{0,1,2\}$ with generator matrix

$$Q^\varepsilon = \begin{pmatrix} 0 & 0 & 0 \\ \varepsilon\lambda & -(1+\varepsilon)\lambda & \lambda \\ \varepsilon\mu & \mu & -(1+\varepsilon)\mu \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\lambda & \lambda \\ 0 & \mu & -\mu \end{pmatrix} + \varepsilon \begin{pmatrix} 0 & 0 & 0 \\ \lambda & -\lambda & 0 \\ \mu & 0 & -\mu \end{pmatrix}$$

The transition matrix of the embedded Markov chain is

$$P^\varepsilon = \begin{pmatrix} 1 & 0 & 0 \\ \varepsilon & 0 & 1-\varepsilon \\ \varepsilon & 1-\varepsilon & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + \varepsilon \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}.$$

Now, for the ergodic process $x(t)$, $t \geq 0$, taking values in $E = \{1,2\}$, and generator Q, we have $\pi = (\frac{\mu}{\lambda+\mu}, \frac{\lambda}{\lambda+\mu})$. For the ergodic embedded Markov chain x_n, $n \geq 0$, we have $\rho = (1/2, 1/2)$.

Thus, since we have $p(1) = -P_1(1,E) = 1$ and $p(2) = -P_1(2,E) = 1$, the stoppage probability is $p = 1$.

On the other hand, we have $q(1) = \lambda$, and $q(2) = \mu$.

Hence

$$q = \pi_1 q(1) + \pi_2 q(2) = \frac{2\lambda}{\lambda + \mu}$$

and

$$\Lambda = qp = \frac{2\lambda\mu}{\lambda+\mu}.$$

The limit of distribution of the normalized absorption times is

$$\mathbb{P}\{\zeta > t\} = \exp(-\Lambda t).$$

Under conditions ME1–ME3, there is a weak convergence [3]

$$\nu(x^\varepsilon(t)) \Rightarrow \hat{x}(t),\ \varepsilon \to 0,\ \nu(x) = k \in \hat{E} = \{1, ..., N\},\ x \in E_k,\ 1 \le k \le N.$$

Limited Markov process $\hat{x}(t)$, $t \geq 0$ on the enlarged phase space $\hat{E} = \{1, ..., N\}$ determined by the generating matrix

$$\hat{Q}_1 = (\hat{q}_{kr},\ 1 \le k, r \le N),$$

where

$$\hat{q}_{kr} = \hat{q}_k \hat{p}_{kr},\ k \ne r,\ \hat{q}_k = q_k \hat{p}_k,\ 1 \le k \le N,$$

$$\hat{p}_{kr} = p_{kr}/\hat{p}_k,\ p_{kr} = \int\limits_{E_k} \rho_k(dx) P_1(x, E_r),\ 1 \le k, r \le N,\ k \ne r,$$

$$\hat{p}_k = -\int\limits_{E_k} \rho_k(dx) P_1(x, E_k).$$

4: Merging Markov process $\hat{x}(t)$, $t \geq 0$ is ergodic process with stationary distribution $\hat{\pi} = (\pi_k,\ k \in \hat{E})$.

Therefore, $\mathbf{Q}^\varepsilon$ is the operator that can be represented as

$$\mathbf{Q}^\varepsilon = \mathbf{Q} + \varepsilon \mathbf{Q}_1,\ \mathbf{Q}_1(x) = q(x) \int\limits_E P_1(x, dy)\varphi(y),$$

where $\mathbf{Q}^\varepsilon = \mathbf{Q} + \varepsilon \mathbf{Q}_1$

$$\mathbf{Q}(x) = q(x) \int\limits_E P(x, dy)[\varphi(y) - \varphi(x)],\ \mathbf{Q}_1(x) = q_1(x) \int\limits_E P_1(x, dy)\varphi(y).$$

Let Π be the projector to zero-subspace of a reducible-invertible operator $\mathbf{Q}$. Its effect on the test functions is defined as follows:

$$\Pi\varphi(x) = \sum_{k=1}^{N} \hat{\varphi}_k 1_k(x),\ \hat{\varphi}_k := \int\limits_{E_k} \pi_k(dx)\varphi(dx).$$

Reducible operator $\hat{Q}_1$ can be determined by relation

$$\hat{Q}_1 \Pi = \Pi \mathbf{Q}_1 \Pi.$$

Let $\hat{\Pi}$ be the projector to zero-subspace of a reducible-invertible operator $\hat{Q}_1$:

$$\hat{\Pi}\hat{\varphi} := q(x) \sum_{k \in E} \hat{\pi}_k \hat{\varphi}_k.$$

Potential matrix $\hat{R}_0 = [\hat{R}^0_{kj};\ 1 \le k, l \le N]$ can be determined by relation

$$\hat{Q}_1 \hat{R}_0 = \hat{R}_0 \hat{Q}_1 = \hat{\Pi} - E.$$

Impulse perturbation process (IPP) $\eta^\varepsilon(t)$, $t \geq 0$, in the Levy approximative scheme can be determined by relation

$$\eta^\varepsilon(t) = \int_0^t \eta^\varepsilon(ds, x(s/\varepsilon^3)), \tag{2.2}$$

where $\eta^\varepsilon(t,x)$, $t \geq 0$, $x \in X$ is a set of processes with independent increments that can be determined by generators

$$\Gamma^\epsilon(x)\varphi(w) = \varepsilon^{-2} \int_R (\varphi(w+v) - \varphi(w))\Gamma^\varepsilon(dv, x),\ x \in X \tag{2.3}$$

and satisfied to Levy approximation conditions

L1. Approximation of averages

$$\int_R v\Gamma^\varepsilon(dv,x) = \varepsilon a_1(x) + \varepsilon^2(a_2(x) + \theta_a(x)),\ \theta_a(x) \to 0,\ \varepsilon \to 0,$$

and

$$\int_R v^2\Gamma^\varepsilon(dv,x) = \varepsilon(b(x) + \theta_b(x)),\ \theta_b(x) \to 0,\ \varepsilon \to 0,$$

L2. Condition to the distribution function

$$\int_R g(v)\Gamma^\varepsilon(dv,x) = \varepsilon^2(\Gamma_g(x) + \theta_g(x)),\ \theta_g(x) \to 0,\ \varepsilon \to 0,$$

for each $g(v) \in C^2(\mathbb{R})$ (space of bounded functions with values in $\mathbb{R}$ and $g(v)/|v|^2 \to 0$, $|v| \to 0$). Measure $\Gamma_g(x)$ is bounded for each $g(v) \in C2(\mathbb{R})$ and can be determined by relation

$$\Gamma_g(x) = \int_R g(v)\Gamma_0(dv,x),\ g(v) \in C^2(\mathbb{R});$$

L3. Uniform quadratic integrability

$$\sup \lim_{c\to\infty} \int_{|v|>c} v^2\Gamma_0(dv,x) = 0;$$

Let's denote:

$$\Gamma_1(x)\varphi(w) = a(x)\varphi'(w)$$

Let the condition of balance be fulfilled

$$\hat{\Pi}\hat{\Gamma}_1 = 0, \tag{2.4}$$

where $\hat{\Gamma}_1\hat{\varphi}(w) = \Pi\Gamma_1(x)\varphi(w)$.

We shall investigate further the asymptotic properties of the perturbation process.

Theorem 2.1. *Let the balance condition* (2.4) *and L1–L3 hold. Then the weak convergence*

$$\eta^\varepsilon(t) \to \eta^0(t),\ \varepsilon \to 0.$$

holds true.

The limit process $\eta^0(t)$ is determined by generator

$$\hat{\hat{\Gamma}}\varphi(w) = \hat{\hat{a}}_2\varphi'(w) + \frac{1}{2}\hat{\hat{\sigma}}^2\varphi''(w) + \int_R [\varphi(w+v) - \varphi(w)]\hat{\hat{\Gamma}}_0(dv),$$

where

$$\hat{\hat{a}}_2 = \sum_{k\in\hat{E}} \hat{\pi}_k \int_{E_k} \pi_k(dx)(a_2(x) - a_0(x)),$$

$$\hat{\hat{\sigma}}^2 = \sum_{k\in\hat{E}} \hat{\pi}_k \int_{E_k} \pi(dx)(b(x) - b_0(x)) + 2\sum_{k\in\hat{E}} \hat{\pi}_k \int_{E_k} \pi(dx)a_1(x)R_0a_1(x),$$

$$\hat{\hat{a}}_0(x) = \sum_{k\in\hat{E}} \hat{\pi}_k \int_{E_k} v\Gamma_0(dv, x),$$

$$\hat{\hat{b}}_0(x) = \sum_{k\in\hat{E}} \hat{\pi}_k \int_{E_k} v^2\Gamma_0(dv, x),$$

$$\hat{\hat{\Gamma}}_0(v) = \sum_{k\in\hat{E}} \hat{\pi}_k \int_{E_k} \pi(dx)\Gamma_0(v, x)$$

and it is a Levy process that has three components: deterministic shift, diffusion, and Poisson jump part.

Proof. Firstly let's prove some additional propositions.

Lemma 2.2. *Generators of independent increment processes $\eta^\varepsilon(t,x)$, $t \geq 0$, $x \in X$, acting on test functions $\varphi(w) \in C^2(\mathbb{R})$ under assumptions L1–L3 have asymptotic presentation*

$$\Gamma^\varepsilon(x)\varphi(w) = \varepsilon^{-1}\Gamma_1(x)\varphi(w) + \Gamma_2(x)\varphi(w), \tag{2.5}$$

where

$$\begin{aligned}\Gamma_1(x)\varphi(w) =& a_1(x)\varphi'(w),\\ \Gamma_2(x)\varphi(w) =& (a_2(x) - a_0(x))\varphi'(x) + \frac{1}{2}(b(x) - b_0(x))\varphi''(x) + \\ &+ \int_R [\varphi(w+v) - \varphi(v)]\Gamma_0(dv, x).\end{aligned}$$

Proof. We use the expansion of the function $\varphi(w)$ to the Taylor series. Then by (2.3) we obtain:

$$\begin{aligned}\mathbf{\Gamma}^\varepsilon(x)\varphi(w) =& \varepsilon^{-2}\int_R (\varphi(w+v) - \varphi(v))\Gamma^\varepsilon(dv, x)\\ =& \varepsilon^{-2}\int_R (\varphi(w+v) - \varphi(v) - v\varphi'(v) - \frac{1}{2}v^2\varphi''(w))\Gamma^\varepsilon(dv, x) + \\ &+ \varepsilon^{-2}\int_R (v\varphi'(w)\Gamma^\varepsilon(dv, x) + \frac{1}{2}v^2\varepsilon^{-2}\int_R v^2\varphi''(w)\Gamma^\varepsilon(dv, x)\end{aligned}$$

$$= \int_R (\varphi(u+v) - \varphi(v) - v\varphi'(w) - \frac{1}{2}v^2\varphi''(w))\Gamma_0(dv,x)+$$

$$+ \varepsilon^{-1}a_1(x)\varphi'(w) + a_2(x)\varphi'(w) + +\frac{1}{2}(b(x) - b_0(x))\varphi''(w)+$$

$$+ \int_R (\varphi(u+v) - \varphi(v))\Gamma_0(dv,x) + \gamma^\varepsilon(w)\varphi(w),$$

where penultimate equality follows from L1–L3 (we remark also that function $\varphi(w+v) - \varphi(w) - v\varphi'(w) - \frac{1}{2}v^2\varphi''(w) \in C^2(\mathbb{R})$, because it is bounded and $\varphi(w)$ is bounded along with its derivatives, and

$$[\varphi(w+v) - \varphi(w) - v\varphi'(w) - \frac{1}{2}v^2\varphi''(w)]/|v^2| \to 0$$

when $v \to 0$.

We recall that $\gamma^\varepsilon(w)\varphi(w) = o(\varepsilon^2)$, $\varphi(w) \in C^2(\mathbb{R})$, and thus, we obtained the presentation (2.5). □

Lemma 2.3. *Generator of a two-component Markov process* $(\eta^\varepsilon, x(t/\varepsilon^2))$, $t \geq 0$ *can be represented as follows*

$$\hat{\Gamma}^\varepsilon(x)\varphi(w,x) = \varepsilon^{-3}\mathbf{Q}\varphi(w,x) + \varepsilon^{-1}\Gamma_1(x)\varphi(w,x) + \Gamma_2(x)\varphi(w,x) + \gamma^\varepsilon(x)\varphi(w,x), \tag{2.6}$$

where $\Gamma_1(x)$ *and* $\Gamma_2(x)$ *are defined at Lemma 2.1 and remainder term* $\|\gamma^\varepsilon(x)\varphi(w,x)\| \to 0$ *when* $\varepsilon \to 0$, $\varphi(w,\cdot) \in C^2(\mathbb{R})$.

Proof. The lemma's statement follows from the generator of Markov process definition and from the form of the corresponding process generators $\eta^\varepsilon(t,x)$ and $x(t/\varepsilon^2)$. □

The truncated operator has a form

$$\mathbf{L}^\varepsilon = \varepsilon^{-3}\mathbf{Q}^\varepsilon + \varepsilon^{-1}\Gamma_1 + \Gamma_2 = \varepsilon^{-3}\mathbf{Q} + \varepsilon^{-2}\mathbf{Q}_1 + \varepsilon^{-1}\Gamma_1 + \Gamma_2. \tag{2.7}$$

Lemma 2.4. *Under balance condition* (2.4) *the solution of the problem of singular perturbation for cut operator* (2.4) *on test functions*

$$\varphi(u,x) = \varphi(u) + \varepsilon\varphi_1 + \varepsilon^2\varphi_2 + \varepsilon\varphi_3$$

is determined by the relation

$$\Gamma_0^\varepsilon(x)\varphi^\varepsilon(u,x) = \hat{\hat{L}}\varphi(u) + \varepsilon\theta_\eta^\varepsilon(x)\varphi(u), \tag{2.8}$$

where remainder term is uniform bounded with respect to x.

Limiting operator can be represented by the formula

$$\hat{\hat{L}} = \hat{\Pi}\hat{\Gamma}_1\hat{R}_0\hat{\Gamma}_1\hat{\Pi} + \hat{\Pi}\hat{\Gamma}_2\hat{\Pi}. \tag{2.9}$$

Proof. Let's calculate

$$(\varepsilon^{-3}\mathbf{Q} + \varepsilon^{-2}\mathbf{Q}_1 + \varepsilon^{-1}\Gamma_1 + \Gamma_2)(\varphi + \varepsilon\varphi_1 + \varepsilon^2\varphi_2 + \varepsilon^3\varphi_3) =$$
$$= \varepsilon^{-3}\mathbf{Q}\varphi + \varepsilon^{-2}(\mathbf{Q}\varphi_1 + \mathbf{Q}_1\varphi) + \varepsilon^{-1}(\mathbf{Q}\varphi_2 + \mathbf{Q}_1\varphi_1 + \Gamma_1\varphi)+$$
$$+(\mathbf{Q}\varphi_3 + \mathbf{Q}_1\varphi_2 + \Gamma_1\varphi_1 + \Gamma_2\varphi) + o(\varepsilon).$$

From here we get 4 relations:

$$\mathbf{Q}\varphi = 0; \tag{2.10}$$

$$\mathbf{Q}\varphi_1 + \mathbf{Q}_1\varphi = 0; \tag{2.11}$$

$$\mathbf{Q}\varphi_2 + \mathbf{Q}_1\varphi_1 + \Gamma_1\varphi = 0; \tag{2.12}$$

$$\mathbf{Q}\varphi_3 + \mathbf{Q}_1\varphi_2 + \Gamma_1\varphi_1 + \Gamma_2\varphi = \hat{\hat{L}}\varphi. \tag{2.13}$$

Further we define the form of $\hat{\hat{L}}$.

From (2.10) follows $\varphi \in N_Q$;

According to (2.11), since $\varphi \in N_Q$, then from the solvability condition we have

$$\Pi\mathbf{Q}_1\Pi\varphi = 0.$$

Let's denote

$$\Pi\mathbf{Q}_1 = \hat{Q}_1,\ \Pi\varphi = \hat{\varphi}.$$

Then

$$\hat{Q}_1\hat{\varphi} = 0,$$

and further

$$\hat{\varphi} \in N_{\hat{Q}_1}.$$

Let's consider (2.12): from the solvability condition for $\mathbf{Q}$ we obtain

$$\Pi\mathbf{Q}_1\Pi\varphi_1 + \Pi\Gamma_1\Pi\varphi = 0, \tag{2.14}$$

$$\hat{Q}_1\hat{\varphi}_1 + \hat{\Gamma}_1\hat{\varphi} = 0.$$

According to balance condition (2.4) one can see $\hat{\Gamma}_1\hat{\varphi} \in R_Q$, so, the solution

$$\hat{\varphi}_1 = \hat{R}_0\hat{\Gamma}_1\hat{\varphi},$$

where $\hat{R}_0$ – reducible-invertible to $\hat{Q}_1$.

Let's turn to (2.13): from the solvability condition for $\mathbf{Q}$ we obtain

$$\Pi\mathbf{Q}\Pi\varphi_2 + \Pi\Gamma_1\Pi\varphi_1 + \Pi\Gamma_2\Pi\varphi = \Pi\hat{L}\Pi\varphi, \tag{2.15}$$

$$\hat{Q}_1\hat{\varphi}_2 + \hat{\Gamma}_1\hat{\varphi}_1 + \hat{\Gamma}_2\hat{\varphi} = \hat{\hat{L}}\hat{\varphi}.$$

Remembering $\hat{\varphi}_1 = \hat{R}_0\hat{\Gamma}_1\hat{\varphi}$, we have

$$\hat{Q}_1\hat{\varphi}_2 + \hat{\Gamma}_1\hat{R}_0\hat{\Gamma}_1\hat{\varphi} + \hat{\Gamma}_2\hat{\varphi} = \hat{\hat{L}}\hat{\varphi}.$$

Further, from the solvability condition for $\hat{\varphi}_2$

$$\hat{\Pi}\hat{\Gamma}_1\hat{R}_0\hat{\Gamma}_1\hat{\Pi}\varphi + \hat{\Pi}\hat{\Gamma}_2\hat{\Pi}\hat{\varphi} = \hat{\hat{L}}\varphi,$$

where

$$\hat{\hat{L}} = \hat{\Pi}\hat{\Gamma}_1\hat{R}_0\hat{\Gamma}_1\hat{\Pi} + \hat{\Pi}\hat{\Gamma}_2\hat{\Pi},$$

$$\hat{\varphi}_2 = \hat{R}_0[\hat{\Gamma}_1\hat{R}_0\hat{\Gamma}_1 + \hat{\Gamma}_2 - \hat{L}]\hat{\varphi},$$

$$\hat{\varphi}_3 = R_0[\mathbf{Q}_1\varphi_2 + \Gamma_1\varphi_1 + \Gamma_2\varphi - \hat{L}\varphi].$$

The boundedness of $\theta^{\varepsilon}_{\eta}(x)\varphi(w)$ follows from the form of operators Γ_1, Γ_2 and R_0. □

The completion of the Theorem 2.1 proof is carried out using Lemma 2.4 and Theorem 6.3 in [3]. □

Further let's consider asymptotic properties of evolutionary system (2.1).

Theorem 2.5. *Under balance condition* (2.4) *the weak convergence*

$$(u^\varepsilon(t), \eta^\varepsilon(t)) \Rightarrow (\hat{\hat{u}}(t), \eta^0(t)),\ \varepsilon \to 0$$

holds true.

The limit process $(\hat{\hat{u}}(t), \eta^0(t))$ *can be given by generator*

$$\mathbf{L}\varphi(u,w) = \hat{\hat{C}}(u)\varphi'_u(u,w) + \hat{\hat{\Gamma}}{}^u_1\varphi(u,\cdot) + \hat{\hat{\Gamma}}{}^w_1\varphi(\cdot,w)\hat{R}_0\hat{\hat{\Gamma}}{}^w_1\varphi(\cdot,w) + \hat{\hat{\Gamma}}{}^w_2\varphi(\cdot,w) \quad (2.16)$$

where

$$\hat{\hat{C}}(u) = \Pi\mathbf{C}(x) = \int_X \pi(dx)C(u,x);$$

and generators $\hat{\hat{\Gamma}}{}^u_1$ *and* $\hat{\hat{\Gamma}}{}^w_{1,2}$ *are determined in 2.2 and they have the same form, but acting with respect to different values.*

Remark 2.1. The weak convergence of processes $(u^\varepsilon(t), \eta^\varepsilon(t)) \Rightarrow (\hat{\hat{u}}(t), \eta^0(t))$, $\varepsilon \to 0$, follows from convergence of respectiv generators when prelimiting set of processes $u^\varepsilon(t)$ is compact. Theorems about compactness of processes with independent increments in Levy aproximative scheme were proved, in particular, in [4].

Proof of Theorem 2.5.

Lemma 2.6. *The generator of three components Markov process* $(u^\varepsilon(t), \eta^\varepsilon(t), x^\varepsilon(t/\varepsilon^3))$, $t \geq 0$, *can be represented as follows*

$$\begin{aligned}\mathbf{L}^\varepsilon(x)\varphi(u,w,x) =& \varepsilon^{-3}\mathbf{Q}^\varepsilon\varphi(u,w,x) + \Gamma^\varepsilon_u(x)\varphi(u,\cdot,x) + \varepsilon\Gamma^\varepsilon_w(x)\varphi(\cdot,w,x) + \\ &+ \mathbf{C}(x)\varphi(u,w,x) + \theta^\varepsilon_w(x)\varphi(u,w,x),\end{aligned} \quad (2.17)$$

where $\Gamma^\varepsilon(x)$ *– the generator of set of IPP* (2.3),

$$\mathbf{C}(x)\varphi(u,w,x) = C(u,x)\varphi'_u(u,w,x).$$

Remainder term $\|\theta^\varepsilon_w(x)\varphi(u,w,x)\| \to 0$ *when* $\varepsilon \to 0$.

Proof. One can be find in [6]. □

Lemma 2.7. *Generator* $\mathbf{L}^\varepsilon(x)$ *in a case of IPP has asymptotical representation*

$$\begin{aligned}\mathbf{L}^\varepsilon(x)\varphi(u,w,x) =& \varepsilon^{-3}\mathbf{Q}^\varepsilon\varphi(u,w,x) + \varepsilon^{-1}\Gamma^w_1(x)\varphi(u,w,x) + +\Gamma^w_2(x)\varphi(u,w,x) \\ &+ \Gamma^u_1(x)\varphi(u,w,x) + +\mathbf{C}(x)\varphi(u,w,x) + \hat{\theta}^\varepsilon_w\varphi(u,w,x),\end{aligned} \quad (2.18)$$

where

$$\hat{\theta}^\varepsilon_w(x) = \gamma^\varepsilon + \theta^\varepsilon_w(x),$$

$\Gamma^{\cdot}_1(x)$ *and* $\Gamma^{\cdot}_2(x)$ *are given in 2.2*

Remainder term $\|\hat{\theta}^\varepsilon_w(x)\varphi(u,w,x)\| \to 0$ *when* $\varepsilon \to 0$.

Proof. The proof is carried out with the representation of the operator (2.5) and the results of the 2.6. □

Truncate operator has a form:

$$\begin{aligned}\mathbf{L}^\varepsilon(x)\varphi(u,w,x) =& \varepsilon^{-3}\mathbf{Q}^\varepsilon\varphi(u,w,x) + \varepsilon^{-1}\Gamma^w_1(x)\varphi(u,w,x) + \Gamma^w_2(x)\varphi(u,w,x) + \\ &+ \Gamma^u_1(x)\varphi(u,w,x) + \mathbf{C}(x)\varphi(u,w,x)\end{aligned} \quad (2.19)$$

Lemma 2.8. *Under balance condition* (2.4) *the solution of singular perturbation problem for truncate operator* (2.19) *on test functions*

$$\varphi^\varepsilon(w,x) = \varphi(w) + \varepsilon\varphi_1(w,x) + \varepsilon^2\varphi_2(w,x) + \varepsilon^3\varphi_3(w,x)$$

can be found from relation

$$\mathbf{L}_0^\varepsilon(x)\varphi^\varepsilon(w,x) = \mathbf{L}\varphi(w) + \varepsilon^3\theta_w^\varepsilon(x)\varphi(w), \tag{2.20}$$

where remainder term $\theta_w^\varepsilon(x)$ *is uniform bounded with respect to* x.

Limit operator $\mathbf{L}$ *can be given by the formula*

$$\mathbf{L} = \Pi[\hat{\hat{C}} + \hat{\hat{\Gamma}}_1^u + \hat{\hat{\Gamma}}_1^w \hat{R}_0 \hat{\hat{\Gamma}}_1^w + \hat{\hat{\Gamma}}_2^w]\Pi. \tag{2.21}$$

Proof. To perform the equality (2.20) it is necessary that the coefficients at the same degrees ε on the left and the right be equal. Therefore we can calculate:

$$(\varepsilon^{-3}\mathbf{Q} + \varepsilon^{-2}\mathbf{Q}_2 + \varepsilon^{-1}\Gamma_1^u + \Gamma_2^u + \varepsilon^{-1}\Gamma_1^w + \Gamma_2^w + \mathbf{C})(\varphi + \varepsilon\varphi_1 + \varepsilon^2\varphi_2 + \varepsilon^3\varphi_3) =$$
$$= \varepsilon^{-3}\mathbf{Q}\varphi + \varepsilon^{-2}(\mathbf{Q}\varphi_1 + \mathbf{Q}_1\varphi) + \varepsilon^{-1}(\mathbf{Q}\varphi_2 + \mathbf{Q}_1\varphi_1 + \Gamma_1^u\varphi + \Gamma_1^w\varphi) +$$
$$+ (\mathbf{Q}\varphi_3 + \mathbf{Q}_1\varphi_2 + \Gamma_1^u\varphi_1 + \Gamma_1^w\varphi_1 + \Gamma_2^u\varphi + \Gamma_2^w\varphi + \mathbf{C}\varphi) + o(\varepsilon).$$

Again, we get four relations:

$$\mathbf{Q}\varphi = 0; \tag{2.22}$$

$$\mathbf{Q}\varphi_1 + \mathbf{Q}_1\varphi = 0; \tag{2.23}$$

$$\mathbf{Q}\varphi_2 + \mathbf{Q}_1\varphi_1 + \Gamma_1^u\varphi + \Gamma_1^w\varphi = 0; \tag{2.24}$$

$$\mathbf{Q}\varphi_3 + \mathbf{Q}_1\varphi_2 + \Gamma_1^u\varphi_1 + \Gamma_1^w\varphi_1 + \Gamma_2^u\varphi + \Gamma_2^w\varphi + \mathbf{C}\varphi = \hat{\hat{L}}\varphi. \tag{2.25}$$

Let's define the form of $\hat{\hat{L}}$.

From (2.22) follows $\varphi \in N_Q$.

According to (2.23), since $\varphi \in N_Q$, then from the solvability condition we have

$$\Pi\mathbf{Q}_1\Pi\varphi = 0.$$

Let's denote

$$\Pi\mathbf{Q}_1 = \hat{Q}_1,\ \Pi\varphi = \hat{\varphi}.$$

then

$$\hat{Q}_1\hat{\varphi} = 0,$$

and

$$\hat{\varphi} \in N_{\hat{Q}_1}.$$

In (2.24) from the solvability condition for $\mathbf{Q}$ we have

$$\Pi\mathbf{Q}_1\Pi\varphi_1 + \Pi\Gamma_1\Pi\varphi = 0,$$

$$\hat{Q}_1\hat{\varphi}_1 + \hat{\Gamma}_1^u\varphi + \hat{\Gamma}_1^w\hat{\varphi} = 0.$$

From the balance condition (2.4) one can see $\hat{\Gamma}_1^u\hat{\varphi}$, $\hat{\Gamma}_1^u\hat{\varphi} \in N_Q$, then, the solution

$$\hat{\varphi}_1 = \hat{R}_0[\hat{\Gamma}_1^u + \hat{\Gamma}_1^w]\hat{\varphi},$$

where $\hat{R}_0$ – reducible-invertible to $\hat{Q}_1$.

In (2.25): from the solvability condition for $\mathbf{Q}$ we have

$$\Pi\mathbf{Q}_1\Pi\varphi_2 + \Pi\Gamma_1^u\Pi\varphi_1 + \Pi\Gamma_2^u\Pi\varphi + \Pi\Gamma_1^w\Pi\varphi_1 + \Pi\Gamma_2^w\Pi\varphi + \Pi\mathbf{C}\Pi\varphi = \Pi\hat{\hat{L}}\Pi\varphi,$$

$$\hat{Q}_1\hat{\varphi}_2 + \hat{\Gamma}_1^u\hat{\varphi}_1 + \hat{\Gamma}_1^w\hat{\varphi}_1 + \hat{\Gamma}_2^u\hat{\varphi} + \hat{\Gamma}_2^w\hat{\varphi} + \hat{C}\hat{\varphi} = \hat{\hat{L}}\hat{\varphi}.$$

Remembering $\hat{\varphi}_1 = \hat{R}_0\hat{\Gamma}_1^u\hat{\varphi} + \hat{R}_0\hat{\Gamma}_1^w\hat{\varphi}$, we obtain

$$\hat{Q}_1\hat{\varphi}_2 + \hat{\Gamma}_1^u\hat{R}_0\hat{\Gamma}_1^u\hat{\varphi} + \hat{\Gamma}_1^u\hat{R}_0\hat{\Gamma}_1^w\hat{\varphi} +$$

$$+\hat{\Gamma}_1^w\hat{R}_0\hat{\Gamma}_1^u\hat{\varphi} + \hat{\Gamma}_1^w\hat{R}_0\hat{\Gamma}_1^w\hat{\varphi} + \hat{\Gamma}_2^u\hat{\varphi} + \hat{\Gamma}_2^w\hat{\varphi} + \hat{C}\hat{\varphi} = \hat{\hat{L}}\hat{\varphi}.$$

Further, from the solvability condition for $\hat{\varphi}_2$

$$\hat{\Pi}\hat{\Gamma}_1^u\hat{R}_0\hat{\Gamma}_1^u\Pi\hat{\varphi} + \hat{\Pi}\hat{\Gamma}_1^u\hat{R}_0\hat{\Gamma}_1^w\Pi\hat{\varphi} + \hat{\Pi}\hat{\Gamma}_1^w\hat{R}_0\hat{\Gamma}_1^u\Pi\hat{\varphi} +$$

$$+\hat{\Pi}\hat{\Gamma}_1^w\hat{R}_0\hat{\Gamma}_1^w\Pi\hat{\varphi} + \hat{\Pi}\hat{\Gamma}_2^u\Pi\hat{\varphi} + \hat{\Pi}\hat{\Gamma}_2^w\Pi\hat{\varphi} + \hat{\Pi}\hat{C}\hat{\Pi}\hat{\varphi} = \hat{\hat{L}}\hat{\hat{\varphi}}.$$

where

$$\hat{\hat{L}} = \hat{\Pi}\hat{\Gamma}_1\hat{R}_0\hat{\Gamma}_1\hat{\Pi} + \hat{\Pi}\hat{\Gamma}_2\hat{\Pi} + \hat{\Pi}\hat{C}\hat{\Pi},$$

$$\hat{\varphi}_2 = \hat{R}_0[\hat{\hat{L}} - \hat{\Gamma}_1\hat{R}_0\hat{\Gamma}_1 - \hat{\Gamma}_2 - \hat{C}]\hat{\varphi},$$

$$\hat{\varphi}_3 = \hat{R}_0[\hat{\hat{L}} + \mathbf{Q}_1\varphi_2 + \Gamma_1\varphi_1 + \hat{\Gamma}_2\varphi + \mathbf{C}\varphi].$$

Boundedness of $\theta_\eta^\varepsilon(x)\varphi(w)$ follows from the form of the operator $\hat{\Gamma}_1$, $\hat{\Gamma}_2$ and R_0. □

The completion of the Theorem 2.5 proof is carried out using Lemma 2.4 and Theorem 6.3 in [3]. □

3. Double merging of phase space for differential equations with small stochastic supplements under Poisson approximation conditions

In this section we will consider the case, when system perturbation is determined by jumping process under Poisson approximating scheme. In particular, we interest in double merging of phase space for such evolutionary models.

Let's consider an evolutionary system in ergodic Markov environment that has form of stochastic differential equation as follows

$$du^\varepsilon(t) = C(u^\varepsilon(t), x(t/\varepsilon^2))dt + d\eta^\varepsilon(t),\ u^\varepsilon(t) \in \mathbb{R}, \tag{3.1}$$

where Markov process $x^\varepsilon(t), t \geq 0$ is determined in standard phase space $(E, \mathcal{E})$ with splitting

$$E = \bigcup_{k=1}^{N} E_k,\ E_k \cap E_{k'} = \emptyset,\ k \neq k'$$

at a series scheme with a small parameter of a serie $\varepsilon \to 0$ when $\varepsilon > 0$.

Markov kernel has a form

$$\mathbf{Q}^\varepsilon(x, B, t) = P^\varepsilon(x, B)[1 - \exp\{-q(x)t\}],\ x \in E,\ B \in \mathcal{E},\ t \geq 0.$$

Let's conditions ME1–ME4 from previous section hold true.

Impulse perturbation process $\eta^\varepsilon(t)$, $t \geq 0$ under Poison approximative scheme is given by relation

$$\eta^\varepsilon(t) = \int_0^t \eta^\varepsilon(ds, x(s/\varepsilon^3)), \tag{3.2}$$

where set of processes with independent increments $\eta^\varepsilon(t, x)$, $t \geq 0$, $x \in X$, is determined by generators

$$\Gamma^\epsilon(x)\varphi(w) = \varepsilon^{-2}\int_R (\varphi(w + v) - \varphi(w))\Gamma^\varepsilon(dv, x),\ x \in X \tag{3.3}$$

and satisfies the Poison approximation conditions

P1. Approximation of averages

$$\int_R v\Gamma^\varepsilon(dv,x) = \varepsilon(a(x)+\theta_a(x)),\ \theta_a(x)\to 0,\ \varepsilon\to 0,$$

and

$$\int_R v^2\Gamma^\varepsilon(dv,x) = \varepsilon(b(x)+\theta_b(x)),\ \theta_b(x)\to 0,\ \varepsilon\to 0,$$

P2. Condition for the distribution function

$$\int_R g(v)\Gamma^\varepsilon(dv,x) = \varepsilon(\Gamma_g(x)+\theta_g(x)),\ \theta_g(x)\to 0,\ \varepsilon\to 0,$$

for each $g(v)\in C^2(\mathbb{R})$ (space of real values bounded functions that $g(v)/|v|^2\to 0$, $|v|\to 0$). Measure $\Gamma_g(x)$ is bounded for each $g(v)\in C^2(\mathbb{R})$ and is determined by relation

$$\Gamma_g(x) = \int_R g(v)\Gamma_0(dv,x),\ g(v)\in C^2(\mathbb{R});$$

P3. Uniform quadratic integrability

$$\sup \lim_{c\to\infty} \int_{|v|>c} v^2\Gamma_0(dv,x) = 0;$$

P4. Absence of a diffusion component

$$b(x) = \int_R v^2\Gamma_0(dv,x).$$

Let's denote:

$$\Gamma_1(x)\varphi(w) = a(x)\varphi'(w) + \int_R [\varphi(w+v)-\varphi(v)-v\varphi'(w)]\Gamma_0(dv,x).$$

Firstly we investigate asymptotic properties of perturbation process.

Theorem 3.1. *Under conditions P1–P4 the weak convergence*

$$\eta^\varepsilon(t)\to\eta^0(t),\ \varepsilon\to 0.$$

holds true.

Limit process $\eta^0(t)$ is determined by generator

$$\hat{\hat{\Gamma}}\varphi(w) = \hat{\Pi}\hat{\Gamma}_1(x)\varphi(w) = \hat{\hat{a}}\varphi'(w) + \int_R [\varphi(w+v)-\varphi(w)-v\varphi'(w)]\hat{\hat{\Gamma}}_0(dv),$$

where

$$\hat{\hat{a}} = \sum_{k\in\hat{E}} \hat{\pi}_k \int_{\hat{E}_k} \pi(dx)(a_(x)),$$

$$\hat{\hat{\Gamma}}_0(v) = \sum_{k\in\hat{E}} \hat{\pi}_k \int_{\hat{E}_k} \pi(dx)\Gamma_0(v,x)$$

and it is process with independent increments, deterministic shift and Poison jumping component.

Proof. We begin from obtaining of some propositions.

Lemma 3.2. *Generators of processes with independent increments $\eta^\varepsilon(t,x)$, $t \geq 0$, $x \in X$, on test functions $\varphi(w) \in C^2(\mathbb{R})$ under Poison approximative conditions P1–P4 have asymptotical representation*

$$\mathbf{\Gamma}^\varepsilon(x)\varphi(w) = \Gamma_1(x)\varphi(w) + \gamma^\varepsilon(x)\varphi(w), \tag{3.4}$$

where

$$\Gamma_1(x)\varphi(w) = a(x)\varphi'(w) + \int_R [\varphi(w+v) - \varphi(v) - v\varphi'(w)]\Gamma_0(dv,x),$$

and remainder term $\|\gamma^\varepsilon(x)\varphi(w)\| \to 0$ when $\varepsilon \to 0$, $\varphi(w,\cdot) \in C^2(\mathbb{R})$.

Proof. Using a Taylor series of functions $\Gamma_1(x)\varphi(w)$ we can transform generator (3.3):

$$\begin{aligned}
\mathbf{\Gamma}^\varepsilon(x)\varphi(w) =&\varepsilon^{-1}\int_R (\varphi(w+v) - \varphi(v))\Gamma^\varepsilon(dv,x) = \\
=&\varepsilon^{-1}\int_R (\varphi(w+v) - \varphi(v) - v\varphi'(v) - \frac{1}{2}v^2\varphi''(w))\Gamma^\varepsilon(dv,x) + \\
&+ \varepsilon^{-1}\int_R (v\varphi'(w)\Gamma^\varepsilon(dv,x) + \frac{1}{2}v^2\varepsilon^{-1}\int_R v^2\varphi''(w)\Gamma^\varepsilon(dv,x) = \\
=&\int_R (\varphi(u+v) - \varphi(v) - v\varphi'(w) - \frac{1}{2}v^2\varphi''(w))\Gamma_0(dv,x) + \\
&+ a(x)\varphi'(w) + \frac{1}{2}b(x)\varphi''(w) + \gamma^\varepsilon(x)\varphi(w) = \\
=&\int_R (\varphi(u+v) - \varphi(v) - \varphi'(w))\Gamma_0(dv,x) + a(x)\varphi'(w) + \gamma^\varepsilon(w)\varphi(w),
\end{aligned}$$

where preemption equality follows from conditions P1–P4 (we remark that function

$$\varphi(w+v) - \varphi(w) - v\varphi'(w) - \frac{1}{2}v^2\varphi''(w) \in C^2(\mathbb{R}),$$

since it is bounded because $\varphi(w)$ and its derivatives are bounded and relation

$$[\varphi(w+v) - \varphi(w) - v\varphi'(w) - \frac{1}{2}v^2\varphi''(w)]/|v^2| \to 0$$

holds true when $v \to 0$.

Remembering $\gamma^\varepsilon(w)\varphi(w) = o(\varepsilon^2)$, $\varphi(w) \in C^2(\mathbb{R})$, we obtain (2.4). □

Lemma 3.3. *Generator of two-component Markov process$(\eta^\varepsilon, x(t/\varepsilon^2))$, $t \geq 0$ has a form as follows*

$$\hat{\Gamma}^\varepsilon(x)\varphi(u,w,x) = \varepsilon^{-2}\mathbf{Q}^\varepsilon\varphi(u,w,x) + \gamma^\varepsilon(x)\varphi(u,w,x), \tag{3.5}$$

where $\Gamma_1(x)$ is determined at Lemma 3.2, and remainder term $\|\gamma^\varepsilon(x)\varphi(u,w,x)\| \to 0$ when $\varepsilon \to 0$, $\varphi(u,w,\cdot) \in C^2(\mathbb{R})$.

Proof. The statement of the lemma follows from definition of Markov process generator and from respective forms of generators $\eta^\varepsilon(t,x)$ i $x(t/\varepsilon^2)$.

Truncate generator has a form

$$\mathbb{L}^\varepsilon\varphi(u,w,x) = \varepsilon^{-2}\mathbf{Q}\varphi(u,w,x) + \Gamma_1(x)\varphi(u,w,x). \tag{3.6}$$

□

Lemma 3.4. *The solution of singular perturbation problem for truncate operator* (3.6) *on test functions*

$$\varphi^\varepsilon(u,w,x) = \varphi(u,w) + \varepsilon\varphi_1(u,w,x) + \varepsilon^2\varphi_2(u,w,x)$$

can be given by relation

$$\Gamma_0^\varepsilon(x)\varphi^\varepsilon(u,x) = \hat{\hat{L}}\varphi(u) + \varepsilon\theta_\eta^\varepsilon(x)\varphi(u), \tag{3.7}$$

where remainder term is uniform bounded with respect to x.

Limit operator can be determined by formula

$$\hat{\hat{L}} = \hat{\Pi}\hat{\Gamma}_1\hat{\Pi} \tag{3.8}$$

Proof. Let's calculate

$$(\varepsilon^{-2}\mathbf{Q} + \varepsilon^{-1}\mathbf{Q}_1 + \Gamma_1)(\varphi(u) + \varepsilon\varphi_1 + \varepsilon^2\varphi_2) =$$
$$= \varepsilon^{-2}\mathbf{Q}\varphi + \varepsilon^{-1}(\mathbf{Q}\varphi_1 + \mathbf{Q}_1\varphi) + (\mathbf{Q}\varphi_2 + \mathbf{Q}_1\varphi_1 + \Gamma_1\varphi) + o(\varepsilon).$$

We have three relations:

$$\mathbf{Q}\varphi = 0; \tag{3.9}$$
$$\mathbf{Q}\varphi_1 + \mathbf{Q}_1\varphi = 0; \tag{3.10}$$
$$\mathbf{Q}\varphi_2 + \mathbf{Q}_1\varphi_1 + \Gamma_1\varphi = \hat{\hat{L}}\varphi. \tag{3.11}$$

Further we find a form of $\hat{\hat{L}}$.

From (3.9) follows $\varphi \in N_Q$;

In (3.10), since $\varphi \in N_Q$, from solvability condition we obtain

$$\Pi\mathbf{Q}_1\Pi\varphi = 0.$$

Let's denote

$$\Pi\mathbf{Q}_1 = \hat{Q}_1, \ \Pi\varphi = \hat{\varphi}.$$

Then

$$\hat{Q}_1\hat{\varphi} = 0,$$

where

$$\hat{\varphi} \in N_{\hat{Q}_1}.$$

Let's consider (3.11): from solvability condition for $\mathbf{Q}$ we obtain

$$\Pi\mathbf{Q}_1\Pi\varphi_1 + \Pi\Gamma_1\Pi\varphi = \Pi\hat{\hat{L}}\Pi\varphi,$$

where

$$\hat{Q}_1\hat{\varphi}_1 + \hat{\Gamma}_1\hat{\varphi} = \hat{\hat{L}}\hat{\varphi}.$$

Further, from solvability condition for $\hat{\varphi}_2$ we obtain

$$\hat{\Pi}\hat{\Gamma}_1\hat{\Pi}\hat{\varphi} = \hat{\hat{L}}\hat{\hat{\varphi}},$$

where

$$\hat{\hat{L}} = \hat{\Pi}\hat{\Gamma}_1\hat{\Pi},$$

$$\hat{\varphi}_1 = \hat{R}_0[\hat{\Gamma}_1 - \hat{\hat{L}}]\hat{\varphi},$$
$$\hat{\varphi}_2 = R_0[\mathbf{Q}_1\varphi_1 + \Gamma_1\varphi - \hat{\hat{L}}\varphi].$$

Boundedness of $\theta^\varepsilon_\eta(x)\varphi(w)$ follows from the form of operators Γ_1 and R_0. □

The completion of the Theorem 3.1 proof is carried out using 3.4 and Theorem 6.3 in [3]. □

Further we investigate asymptotic properties of evolutionary system (26).

Theorem 3.5. *Under conditions P1–P4 weak convergence*

$$(u^\varepsilon(t), \eta^\varepsilon(t)) \Rightarrow (\hat{u}(t), \eta^0(t)),\ \varepsilon \to 0.$$

holds true.

Limit process $(\hat{u}(t), \eta^0(t))$ *is can be determined by generator*

$$\mathbf{L}\varphi(u, w) = \hat{\hat{C}}(u)\varphi'_u(u, w) + \hat{\hat{\Gamma}}^w\varphi(\cdot, w), \tag{3.12}$$

where

$$\hat{\hat{C}}(u) = \Pi\mathbf{C}(x) = \sum_{k\in\hat{E}} \hat{\pi}_k \int_{\hat{E}_k} \pi(dx)C(u, x);$$

and generator $\hat{\hat{\Gamma}}^w$ *is defined in Theorem 3.1, acting at variable* w.

Remark 3.1. Weak convergence of processes $u^\varepsilon(t) \Rightarrow \hat{u}(t)$, $\varepsilon \to 0$, follows from convergence of respective generators under assumption of compactness prelimiting set of processes $u^\varepsilon(t)$. Respective theorems one can read, in particulary, in [3].

Lemma 3.6. *Generator of three-component Markov process* $(u^\varepsilon(t), \eta^\varepsilon(t), x^\varepsilon(t/\varepsilon^3))$, $t \geq 0$, *has a representation*

$$\begin{aligned}\mathbf{L}^\varepsilon(x)\varphi(u, w, x) =& \varepsilon^{-2}Q^\varepsilon\varphi(u, w, x) + \varepsilon\Gamma^\varepsilon_u(x)\varphi(u, \cdot, x) + \Gamma^\varepsilon_w(x)\varphi(\cdot, w, x)+\\ &+ \mathbf{C}(x)\varphi(u, w, x) + \theta^\varepsilon_w(x)\varphi(u, w, x),\end{aligned} \tag{3.13}$$

where $\Gamma^\varepsilon_\cdot(x)$ *is IPP generator* (3.3),

$$\mathbf{C}(x)\varphi(u, w, x) = C(u, x)\varphi'_u(u, w, x).$$

Remainder term $\|\theta^\varepsilon_w(x)\varphi(u, w, x)\| \to 0$ *when* $\varepsilon \to 0$.

Proof the reader can see in [3].

Lemma 3.7. *Generator* $\mathbf{L}^\varepsilon(x)$ *in a case of IPP has asymptotic representation*

$$\begin{aligned}\mathbf{L}^\varepsilon(x)\varphi(u, w, x) &= \varepsilon^{-2}\mathbf{Q}^\varepsilon\varphi(u, w, x) + \varepsilon\Gamma^u_1(x)\varphi(u, w, x)+\\ &+\Gamma^w_1(x)\varphi(u, w, x) + \mathbf{C}(x)\varphi(u, w, x) + \hat{\theta}^\varepsilon_w\varphi(u, w, x),\end{aligned} \tag{3.14}$$

where

$$\hat{\theta}^\varepsilon_w(x) = \gamma^\varepsilon + \theta^\varepsilon_w(x),$$

$\Gamma^\cdot_1(x)$ *is defined in 3.2*

Remainder term $\|\hat{\theta}^\varepsilon_w(x)\varphi(u, w, x)\| \to 0$ *when* $\varepsilon \to 0$.

Proof. follows from representation of generator (3.4) and 3.6

Truncate operator has a form:

$$\mathbf{L}^\varepsilon_0(x)\varphi = \varepsilon^{-2}\mathbf{Q}^\varepsilon\varphi + \Gamma^w_1(x)\varphi + \mathbf{C}(x)\varphi \tag{3.15}$$

□

Lemma 3.8. *The solving of singular perturbation problem for truncate operator* (3.15) *on test functions*

$$\varphi^\varepsilon(w,x) = \varphi(w) + \varepsilon\varphi_1(w,x) + \varepsilon^2\varphi_2(w,x)$$

holds by relation

$$\mathbf{L}_0^\varepsilon(x)\varphi^\varepsilon(w,x) = \mathbf{L}\varphi(w) + \varepsilon^3\theta_w^\varepsilon(x)\varphi(w), \tag{3.16}$$

where remainder term $\theta_w^\varepsilon(x)$ *is uniform bounded at* x.

Limit operator $\mathbf{L}$ *is defined by formula*

$$\mathbf{L} = \hat{\hat{C}} + \hat{\hat{\Gamma}}_1^w. \tag{3.17}$$

Proof. In order to satisfy (3.16), it is necessary that the coefficients at the same degrees ε of the left and right be equal. For this purpose we calculate

$$(\varepsilon^{-2}\mathbf{Q} + \varepsilon^{-1}\mathbf{Q}_1 + \Gamma_1^w + \mathbf{C})(\varphi + \varepsilon\varphi_1 + \varepsilon^2\varphi_2) =$$
$$= \varepsilon^{-2}\mathbf{Q}\varphi + \varepsilon^{-1}(\mathbf{Q}\varphi_1 + \mathbf{Q}_1\varphi) + (\mathbf{Q}\varphi_2 + \mathbf{Q}_1\varphi_1 + \Gamma_1^w\varphi + \mathbf{C}\varphi) + o(\varepsilon).$$

and again we have three relations:

$$\mathbf{Q}\varphi = 0; \tag{3.18}$$

$$\mathbf{Q}\varphi_1 + \mathbf{Q}_1\varphi = 0; \tag{3.19}$$

$$\mathbf{Q}\varphi_2 + \mathbf{Q}_1\varphi_1 + \Gamma_1^w\varphi = \hat{\hat{L}}\varphi. \tag{3.20}$$

We define the form of $\hat{\hat{L}}$.

From (3.18) it follows that $\varphi \in N_Q$.

In (3.19), since $\varphi \in N_Q$, according to solvability condition we obtain

$$\Pi\mathbf{Q}_1\Pi\varphi = 0.$$

Let's denote

$$\Pi\mathbf{Q}_1 = \hat{Q}_1,\ \Pi\varphi = \hat{\varphi}.$$

Then

$$\hat{Q}_1\hat{\varphi} = 0,$$

where

$$\hat{\varphi} \in N_{\hat{Q}_1}.$$

Let's consider (3.20): from solvability condition for $\mathbf{Q}$ we have

$$\Pi\mathbf{Q}_1\Pi\varphi_1 + \Pi\Gamma_1^w\Pi\varphi + \Pi\mathbf{C}\Pi\varphi = \Pi\hat{\hat{L}}\Pi\varphi. \tag{3.21}$$

Further

$$\hat{Q}_1\hat{\varphi}_1 + \hat{\Gamma}_1^w\hat{\varphi} + \hat{C}\hat{\varphi} = \hat{\hat{L}}\hat{\varphi}.$$

Again, from the solvability condition for $\hat{\varphi}_2$ we obtain

$$\hat{\Pi}\hat{C}\hat{\Pi}\hat{\varphi} + \hat{\Pi}\hat{\Gamma}_1^w\hat{\Pi}\hat{\varphi} = \hat{\hat{L}}\hat{\hat{\varphi}},$$

where

$$\hat{\hat{L}} = \hat{\Pi}\hat{C}\hat{\Pi} + \hat{\Pi}\hat{\Gamma}_1^w\hat{\Pi},$$

$$\hat{\varphi}_1 = \hat{R}_0[\hat{\hat{L}} - \hat{\Gamma}_1^w\varphi_1 - \hat{C}]\hat{\varphi},$$

$$\hat{\varphi}_2 = \hat{R}_0[\hat{\hat{L}}\varphi + Q_1\varphi_1 + \Gamma_1^w\varphi + \mathbf{C}\varphi].$$

The boundedness of $\theta_\eta^\varepsilon(x)\varphi(w)$ follows from the form of operators $\hat{\Gamma}_1$ and R_0. $\square$

The completion of the Theorem 3.5 proof is carried out using Lemma 3.4 and Theorem 6.3 in [3]. □

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