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# Ground state solutions and concentration phenomena in nonlinear eigenvalue problems with variable exponents

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Abstract. In this paper we consider a class of nonlinear eigenvalue problems that involves a particular nonhomogeneous operator with variable growth condition and multiple variable exponents. The main results establish the existence of minimum action solutions and the concentration of the spectrum in a bounded interval.

The proofs rely on variational arguments based on the Mountain-Pass Theorem and the Nehari manifold technique.

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## 1. Introduction

The study of differential equations involving $p(x)$-growth conditions was motivated by their applications. Some materials with inhomogeneities, for instance electrorheological fluids, required that the exponents of classical Lebesgue and Sobolev spaces $L^p$ and $W^{1,p}$ (where $1 \leq p \leq \infty$) to be able to vary. For example, electrorheological fluids have an extensive application in robotics, aircraft and aerospace.

Recently, the applications of variable exponent analysis in image restoration attracted more and more attention. To this end we may refer to [10] and [11].

Therefore, the need to study variable exponent Lebesgue and Sobolev spaces.

In this paper we study the following quasilinear elliptic equation involving several variable exponents:

$$(P) \qquad \begin{cases} -\mathrm{div}\,[\phi(x,|\nabla u|)\nabla u] = \lambda \left(g(x)|u|^{q(x)-2} + |u|^{r(x)-2}\right) u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega$ is a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary and the function $\phi(x,t)$ is of the type $|t|^{p(x)-2}$, where the function $p : \overline{\Omega} \to (1,\infty)$ is a continuous function.

The operator $-\mathrm{div}\,[\phi(x,|\nabla u|)\nabla u]$ was firstly introduced by I. H. Kim and Y.H. Kim in [14], and it enables the understanding of problems with possible lack of uniform convexity.

For the more strictly case when $\phi(x,t) = |t|^{p(x)-2}$ our operator becomes the $p(x)$-Laplacian, i.e., $-\Delta_{p(x)}u := -\mathrm{div}\left(|\nabla u|^{p(x)-2}\nabla u\right)$. This operator was intensely studied in the last decade in various ways. For more details we refer to [19] and [21].

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Our abstract setting includes also the case

$$\phi(x,t) = (1+|t|^2)^{\frac{p(x)-2}{2}},$$

which corresponds to the generalized mean curvature operator:

$$\operatorname{div}\left[(1+|\nabla u|^2)^{\frac{p(x)-2}{2}}\nabla u\right]$$

and our equation becomes the capillary surface equation described by the operator

$$\operatorname{div}\left[\left(1+\frac{|\nabla u|^{p(x)}}{\sqrt{1+|\nabla u|^{2p(x)}}}\right)|\nabla u|^{p(x)-2}\nabla u\right].$$

In a recent monograph V. Rădulescu and D. Repovš [20] studied various problems involving the $p(x)$-Laplacian and $p(x)$-Laplacian type operators. Some significant ideas are also detailed by Y. Fu, Y. Shan in [9] and G. Molica Bisci, D. Repovš in [17].

In the present paper we extend the study from [14] by using a nonlinearity which involves several variable exponents on the right-hand side of the equation. This fact leads us to new results which consists in finding an other type of weak solutions which minimizes the functional of the action on the set of all weak solutions.

We also point out that we obtain the concentration of the spectrum in an interval away from the origin, which is bounded. Due to the new type of nonlinearities we will also observe later on this paper that we can not establish the coercivity of the associated energy functional and we could not apply the Direct Method in Calculus of Variations in order to determine its critical points like I. H. Kim, Y.H. Kim in [14] in the proof of Theorem 4.6.

The first result is concerned with the revealing of the first eigenvalue associated to this problem and the existence of its eigenfunction. The spectrum of $p(x)$-Laplacian type operators was also studied by M. Mihăilescu and V. Rădulescu in [15], M. Mihăilescu, V. Rădulescu, D. Repovš in [16], X. Fan, Q. Zhang and D. Zhao in [8], I. H. Kim, Y.H. Kim in [14] and I. Stăncuţ, I. Stîrcu in [22]. A special topic in the study of the eigenvalue problems is constituted by the perturbed eigenvalue problems and in this case we may refer to [4]. The difficulties in the study of spectral properties of these operators comes from the existence of the gap between the infimum and the supremum of the nonconstant exponents.

In the next sections of this paper we will prove the existence of the solutions for our problem. Our first existence result state that the problem $(P)$ admits at least one nontrivial weak solution due to the mountain pass theory. X. Fan, V. Rădulescu and D. Repovš studied also the mountain-pass type solutions in [7], respectively in [20].

The second existence result lets us know that the problem $(P)$ admits a ground state solution via some theory founded in a recent work of A. Azzollini, P. d'Avenia, A. Pomponio [2].

In our last section, we will study the concentration phenomena related to the spectrum and the fact that each eigenvalue of our problem is strictly positive.

## 2. The functional framework

With the emergence of nonlinear problems in applied sciences, standard Lebesgue and Sobolev spaces demonstrated their limitations in the applications.

In this section we will introduce the necessary functional framework needed in the study of the problem $(P)$, i.e., the Lebesgue and Sobolev spaces with variable exponent, and we will recall their main properties. To this end we studied the following books by J. Musielak [18], L. Diening, P. Harjulehto, P. Hästö, M. Ruzicka [5], V. Rădulescu and D. Repovš [20].

Let us assume that $\Omega$ is a bounded domain in $\mathbb{R}^N$. We define

$$C_+(\Omega) = \left\{ p \in C(\overline{\Omega}) : \min_{x\in\Omega} p(x) > 1 \right\}.$$

For any continuous function $p : \overline{\Omega} \to (1, \infty)$ one have

$$p^- = \inf_{x\in\Omega} p(x) \qquad \text{and} \qquad p^+ = \sup_{x\in\Omega} p(x).$$

The condition $p^+ < +\infty$ is known to imply many desirable features for the associated *variable exponent Lebesgue space*, which is defined by:

$$L^{p(x)}(\Omega) = \left\{ u : \Omega \to \mathbb{R} \text{ a measurable function} : \int_\Omega |u|^{p(x)} dx < \infty \right\}.$$

Equipped with the so called *Luxemburg norm*

$$|u|_{p(x)} = \inf \left\{ \mu > 0 : \int_\Omega \left| \frac{u(x)}{\mu} \right|^{p(x)} dx \le 1 \right\},$$

$L^{p(x)}(\Omega)$ becomes a Banach space.

This space is a special case of an Orlicz-Musielak space and its dual space is $L^{p'(x)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$.

For $1 < p(x) < +\infty$, $L^{p(x)}(\Omega)$ is reflexive, uniformly convex Banach space, and for any measurable bounded exponent $p$, $L^{p(x)}(\Omega)$ is separable.

If $0 < |\Omega| < +\infty$ and $p(x)$, $q(x)$ are two variable exponents so that $p(x) \le q(x)$ almost everywhere in $\Omega$, then there exists the continuous embedding

$$L^{q(x)}(\Omega) \hookrightarrow L^{p(x)}(\Omega).$$

For the next conjugate variable exponents, if we have $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$ then the *Hölder type inequality* holds:

$$\left| \int_\Omega uv\, dx \right| \le \left( \frac{1}{p^-} + \frac{1}{p'^-} \right) |u|_{p(x)} |v|_{p'(x)} \le 2 |u|_{p(x)} |v|_{p'(x)}. \tag{1}$$

We define by $\rho_{p(x)} : L^{p(x)}(\Omega) \to \mathbb{R}$, the *modular* of the space $L^{p(x)}(\Omega)$ such that:

$$\rho_{p(x)}(u) = \int_\Omega |u(x)|^{p(x)} dx.$$

If $p(x) \not\equiv$ *constant* in $\Omega$, and $u$, $\{u_n\} \in L^{p(x)}(\Omega)$, then the following relations hold true:

$$|u|_{p(x)} < 1 \Rightarrow \ |u|^{p^+}_{p(x)} \le \rho_{p(x)}(u) \le |u|^{p^-}_{p(x)}, \tag{2}$$

$$|u|_{p(x)} > 1 \Rightarrow \ |u|^{p^-}_{p(x)} \le \rho_{p(x)}(u) \le |u|^{p^+}_{p(x)}, \tag{3}$$

$$|u|_{p(x)} = 1 \Rightarrow \ \rho_{p(x)}(u) = 1, \tag{4}$$

$$|u_n - u|_{p(x)} \to 0 \Leftrightarrow \ \rho_{p(x)}(u_n - u) \to 0. \tag{5}$$

We define in what follows the *variable exponent Sobolev space* $W^{1,p(x)}(\Omega)$ by

$$W^{1,p(x)}(\Omega) = \left\{ u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega) \right\}.$$

On $W^{1,p(x)}(\Omega)$ we may consider the following equivalent norms:

$$\|u\|_{p(x)} = |u|_{p(x)} + |\nabla u|_{p(x)}$$

and

$$\|u\| = \inf \left\{ \mu > 0 : \int_\Omega \left( \left| \frac{\nabla u(x)}{\mu} \right|^{p(x)} + \left| \frac{u(x)}{\mu} \right|^{p(x)} \right) dx \leq 1 \right\}.$$

We define $W_0^{1,p(x)}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{p(x)}$ or

$$W_0^{1,p(x)}(\Omega) = \left\{ u; u|_{\partial\Omega} = 0, u \in L^{p(x)}(\Omega), |\nabla u| \in L^{p(x)}(\Omega) \right\}.$$

Taking account of [14] for $p \in C_+(\overline{\Omega})$ we have the $p(\cdot)$ – *Poincaré type inequality*

$$|u|_{p(x)} \leq C \|\nabla u\|_{p(x)}, \tag{6}$$

where $C > 0$ is a constant which depends on $p$ and $\Omega$.

For $\Omega \subset \mathbb{R}^N$ a bounded domain and $p$ a global log-Hölder continuous function, on $W_0^{1,p(x)}(\Omega)$ we can work with the norm $|\nabla u|_{p(x)}$ equivalent with $\|u\|_{p(x)}$.

**Remark 2.1.** [20] If $p, q : \Omega \to (1, \infty)$ are Lipschitz continuous, $p^+ < N$ and $p(x) \leq q(x) \leq p^*(x)$, for any $x \in \Omega$, where $p^*(x) = \frac{Np(x)}{N-p(x)}$, the embedding

$$W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$$

is compact and continuous.

## 3. Existence of solutions

**3.1. Basic hypotheses.** We will study the problem

$$(P) \qquad \begin{cases} -\mathrm{div}\,[\phi(x, |\nabla u|)\nabla u] = \lambda \left( g(x)|u|^{q(x)-2} + |u|^{r(x)-2} \right) u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

In order to state more precisely our results we have that:
$(\phi_1)$ $\phi : \Omega \times [0, \infty) \to [0, \infty)$ fulfill the following assumptions:
$\to \phi(\cdot, t)$ is measurable on $\Omega$ for all $t \geq 0$;
$\to \phi(x, \cdot)$ is locally absolutely continuous on $[0, \infty)$ for almost all $x \in \Omega$.
$(\phi_2)$ There exist a function $\alpha \in L^{p'(x)}(\Omega)$ and a positive constant $\beta$ such that

$$|\phi(x, |t|)t| \leq \alpha(x) + \beta |t|^{p(x)-1}$$

for almost all $x \in \Omega$ and for all $t \in \mathbb{R}^N$.
$(\phi_3)$ There is a positive constant $c$ such that the following hypotheses hold for almost all $x \in \Omega$:
$\to \phi(x, t) \geq ct^{p(x)-2}$;
$\to t\dfrac{\partial \phi}{\partial t}(x, t) + \phi(x, t) \geq ct^{p(x)-2}$;
for almost all $t > 0$.

**Definition 3.1.** We say that $u \in W_0^{1,p(x)}(\Omega)$ is a weak solution of the problem $(P)$ if:

$$\int_\Omega \phi(x,|\nabla u(x)|)\nabla u(x)\cdot\nabla v(x)dx = \lambda\int_\Omega g(x)|u(x)|^{q(x)-2}u(x)v(x)+|u(x)|^{r(x)-2}u(x)v(x)dx$$

for all $v \in W_0^{1,p(x)}(\Omega)$.

In what follows we set

$$\Phi_0(x,s) = \int_0^s \phi(x,t)tdt,$$

and we define the functional

$$\Phi : W_0^{1,p(x)}(\Omega) \to \mathbb{R}$$

by

$$\Phi(u) = \int_\Omega \Phi_0(x,|\nabla u(x)|)dx.$$

**Lemma 3.2.** [14] Assume that $(\phi_1)$ and $(\phi_2)$ hold. Then $\Phi \in C^1(W_0^{1,p(x)}(\Omega),\mathbb{R})$ and its Frèchet derivative is given by

$$\langle \Phi'(u), v\rangle = \int_\Omega \phi(x,|\nabla u(x)|)\nabla u(x)\cdot v(x)dx.$$

**Lemma 3.3.** [14] Assume that $(\phi_1)-(\phi_3)$ hold. Then the operator $\Phi' : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ is a strictly monotone on $W_0^{1,p(x)}(\Omega)$ and a mapping of type $(S_+)$, i.e., if $u_n \rightharpoonup u$ in $W_0^{1,p(x)}(\Omega)$ as $n \to \infty$ and $\limsup_{n\to\infty}\langle \Phi'(u_n)-\Phi'(u), u_n - u\rangle \le 0$, then $u_n \to u$ in $W_0^{1,p(x)}(\Omega)$, as $n \to \infty$.

By the definition of $\Phi_0$ and $\phi$ we can assert that one have:

$(\phi_4)$ For every $x \in \overline{\Omega}$, and every $t \in \mathbb{R}^N$ one have: $0 \le a(x,t)\cdot t \le p^+\Phi_0(x,|t|)$, where $a(x,t) = \phi(x,|t|)t$.

Before giving our results we must state some conditions on our exponents, and our domain.

We consider $\Omega \subset \mathbb{R}^N$ $(N \ge 3)$ is a bounded domain with smooth boundary, $\lambda > 0$ is a real number and $p, q, r \in C_+(\Omega)$ such that:

$$1 < p^- \le p(x) \le p^+ < q^- \le q(x) \le q^+ < r^- \le r(x) \le r^+ < p^*(x) = \tfrac{Np(x)}{N-p(x)}, \quad \text{and } p(x) < N \tag{7}$$

for all $x \in \overline{\Omega}$.

Furthermore, we assume that the function $g(x)$ satisfies the hypotheses:

$(g_1)$ $g : \overline{\Omega} \to [0,\infty)$, $g \in L^\infty(\overline{\Omega})$.

$(g_2)$ $g(x) > 0$ for any $x \in \overline{\Omega}$ and $\|g\|_\infty \le C$, where $C > 0$ is a constant.

Denote the following functional:

$$J(u) := \int_\Omega \Phi_0(x,|\nabla u|)dx - \lambda\left(\int_\Omega \frac{g(x)}{q(x)}|u|^{q(x)}dx + \int_\Omega \frac{1}{r(x)}|u|^{r(x)}dx\right).$$

Set

$$\Gamma(u) := \int_\Omega \frac{g(x)}{q(x)}|u|^{q(x)}dx + \int_\Omega \frac{1}{r(x)}|u|^{r(x)}dx.$$

We can say from now on, that $J(u) := \Phi(u) - \lambda\Gamma(u)$.

In the study of our results we will further use the following assumption related to the $\Gamma$ functional:
$(\Gamma_1)$ There exist the constants $A_1 > 0$ and $\omega > 0$ such that $\omega > p^+$, and

$$0 < \int_\Omega \frac{g(x)}{q(x)}|u|^{q(x)}dx + \int_\Omega \frac{1}{r(x)}|u|^{r(x)}dx \le \int_\Omega \frac{g(x)}{\omega}|u|^{q(x)}dx + \int_\Omega \frac{1}{\omega}|u|^{r(x)}dx$$

for all $|u| > A_1$.

Using the properties founded in [14] and in the Chapter 3, Section 3.2 of [20] we can say that $J(u)$ is well defined on $W_0^{1,p(x)}(\Omega)$, it is a $C^1$ functional and for every $v \in W_0^{1,p(x)}(\Omega)$ we have:

$$\begin{aligned} J'(u)(v) &= \Phi'(u)(v) - \lambda\Gamma'(u)(v) \\ &= \int_\Omega \phi(x, |\nabla u(x)|)\nabla u(x)\cdot\nabla v(x)dx - \\ &-\lambda\left(\int_\Omega g(x)|u(x)|^{q(x)-2}u(x)\cdot v(x)dx + \int_\Omega |u(x)|^{r(x)-2}u(x)\cdot v(x)dx\right). \end{aligned}$$

Therefore we observe that critical points of our energy functional are weak solutions for the problem $(P)$.

**3.2. On the first eigenvalue of the problem** $(P)$**.** Let us define the following quantity

$$\lambda_1 := \inf_{u\in W_0^{1,p(x)}(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega \Phi_0(x, |\nabla u|)dx}{\displaystyle\int_\Omega \frac{g(x)}{q(x)}|u|^{q(x)}dx + \int_\Omega \frac{1}{r(x)}|u|^{r(x)}dx}.$$

In order to prove that $\lambda_1$ is an eigenvalue for the problem $(P)$ we firstly assert some important lemmas.

**Lemma 3.4.** Assume that $(\phi_3)$, (7), $(g_1)$ and $(g_2)$ hold. Then the functionals $\Phi$ and $\Gamma$ satisfy the following relation:

$$\lim_{\|u\|_{p(x)}\to 0} \frac{\Phi(u)}{\Gamma(u)} = +\infty.$$

*Proof.* Taking use of the properties of the function $g$ we obtain that

$$\begin{aligned} |\Gamma(u)| &= \left|\int_\Omega \frac{g(x)}{q(x)}|u|^{q(x)}dx + \int_\Omega \frac{1}{r(x)}|u|^{r(x)}dx\right| \\ &\le \left|\int_\Omega \frac{g(x)}{q(x)}|u|^{q(x)}dx\right| + \left|\int_\Omega \frac{1}{r(x)}|u|^{r(x)}dx\right| \\ &\le \frac{\|g\|_\infty}{q^-}\int_\Omega |u|^{q(x)}dx + \frac{1}{r^-}\int_\Omega |u|^{r(x)}dx. \end{aligned}$$

Let $u \in W_0^{1,p(x)}(\Omega)$, with $\|u\|_{p(x)} < 1$.

Now, using the fact that $p(x) < q(x) < r(x) < p^*(x)$ we have the following continuous embeddings:

$$W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega) \ \Rightarrow\ \exists\, C_q > 0 \text{ such that } |u|_{q(x)} \le C_q\|u\|_{p(x)}$$

and

$$W_0^{1,p(x)}(\Omega) \hookrightarrow L^{r(x)}(\Omega) \ \Rightarrow\ \exists\, C_r > 0 \text{ such that } |u|_{r(x)} \le C_r\|u\|_{p(x)}.$$

Taking account of relations (2) and (3) we have that:

$$\int_\Omega |u|^{q(x)}dx \leq |u|^{q^-}_{q(x)} + |u|^{q^+}_{q(x)},$$

$$\int_\Omega |u|^{r(x)}dx \leq |u|^{r^-}_{r(x)} + |u|^{r^+}_{r(x)}.$$

We can say now that

$$|\Gamma(u)| \leq \frac{C}{q^-}\left(C_q^{q^-}\|u\|^{q^-}_{p(x)} + C_q^{q^+}\|u\|^{q^+}_{p(x)}\right) + \frac{1}{r^-}\left(C_r^{r^-}\|u\|^{r^-}_{p(x)} + C_r^{r^+}\|u\|^{r^+}_{p(x)}\right). \tag{8}$$

It follows using $(\phi_3)$, (8) and (2) that

$$\left|\frac{\Phi(u)}{\Gamma(u)}\right| \geq \frac{\frac{c}{p^+}\|u\|^{p^+}_{p(x)}}{\frac{C}{q^-}\left(C_q^{q^-}\|u\|^{q^-}_{p(x)} + C_q^{q^+}\|u\|^{q^+}_{p(x)}\right) + \frac{1}{r^-}\left(C_r^{r^-}\|u\|^{r^-}_{p(x)} + C_r^{r^+}\|u\|^{r^+}_{p(x)}\right)}.$$

Since our exponents follow relation (7) we conclude that:

$$\frac{\Phi(u)}{\Gamma(u)} \to +\infty, \text{ as } \|u\|_{p(x)} \to 0.$$

□

**Lemma 3.5.** [14] Assume that $(\phi_1) - (\phi_3)$ and (7) hold. Then $\Phi$ is weakly lower semicontinuous, i.e., $u_n \rightharpoonup u$ in $W_0^{1,p(x)}(\Omega)$ as $n \to \infty$ implies that

$$\Phi(u) \leq \liminf_{n\to\infty} \Phi(u_n).$$

The following remark is a consequence of Lemma 21, Chapter 3 in [20] combined with some details from the Section 2 of [6].

**Remark 3.1.** Assume that (7) holds, then $\Gamma$ is weakly-strongly continuous, i.e., $u_n \rightharpoonup u$ in $W_0^{1,p(x)}(\Omega)$ as $n \to \infty$, implies that $\Gamma(u_n) \to \Gamma(u)$ as $n \to \infty$.

**Theorem 3.6.** The quantity $\lambda_1$ is an eigenvalue for the problem $(P)$.

*Proof.* Let $\{u_n\} \subseteq W_0^{1,p(x)}(\Omega) \setminus \{0\}$ be a minimizing sequence for $\lambda_1$, such that

$$\lambda_1 = \lim_{n\to\infty} \frac{\Phi(u_n)}{\Gamma(u_n)}. \tag{9}$$

Taking account of Lemma 3.4, we note that $\{u_n\}$ is bounded in $W_0^{1,p(x)}(\Omega)$ and so, one have $u_n \rightharpoonup u$ as $n \to \infty$, for some nonzero element $u \in W_0^{1,p(x)}(\Omega)$.

Arguing by contradiction we suppose that $u \equiv 0$. Since $\Gamma$ is weakly-strongly continuous, we obtain that $\Gamma(u_n) \to 0$ as $n \to \infty$. By (9) we have that

$$\lim_{n\to\infty} \Phi(u_n) = \lim_{n\to\infty} \frac{\Phi(u_n)}{\Gamma(u_n)}\Gamma(u_n) = 0.$$

Since $\Phi(u_n) \geq \tilde{C}\|u_n\|^{p^+}_{p(x)}$ or $\Phi(u_n) \geq \tilde{C}\|u\|^{p^-}_{p(x)}$ for some positive constant $\tilde{C}$, using $(\phi_3)$, one have $\|u_n\|_{p(x)} \to 0$ as $n \to \infty$. Applying Lemma 3.4 we deduce that

$$\lim_{n\to\infty} \frac{\Phi(u_n)}{\Gamma(u_n)} = \infty,$$

which is a contradiction.

Hence, we have $u \not\equiv 0$.

Since $u_n \rightharpoonup u$ as $n \to \infty$ we get by Lemma 3.5 and Remark 3.1 that

$$\Phi(u) \leq \liminf_{n\to\infty} \Phi(u_n) \qquad \text{and} \qquad \Gamma(u_n) \to \Gamma(u). \tag{10}$$

In conclusion by the definition of $\lambda_1$ and (10) we have that $\Phi(u) = \lambda_1 \Gamma(u)$ so, $\lambda_1$ is an eigenvalue of the problem (P).

□

**3.3. Mountain-Pass type solution.** In order to show the existence of the critical points of the energy functional $J$, and so, the existence of weak solutions for for our problem we will use the following version of mountain pass lemma of A. Ambrosetti and P. Rabinowitz and we can also refer to [3].

**Theorem 3.7.** [1] Let $\mathcal{X}$ be a real Banach space and assume that $J : \mathcal{X} \to \mathbb{R}$ is a $C^1$-functional that satisfies the following geometric hypotheses:
$(i)$ $J(0) = 0$ and there exist positive numbers $c_0$ and $R$ such that $J(u) \geq c_0$ for all $u \in \mathcal{X}$ with $\|u\| = R$;
$(ii)$ there exists $e \in \mathcal{X}$ with $\|e\| > R$ such that $J(e) < 0$.

Set

$$\mathcal{P} := \{p \in C([0,1]; \mathcal{X});\ p(0) = 0,\ p(1) = e\}$$

and

$$\underline{C} := \inf_{p\in\mathcal{P}} \sup_{t\in[0,1]} J(p(t)).$$

Then there exists a sequence $(u_n) \subset \mathcal{X}$ such that

$$\lim_{n\to\infty} J(u_n) = \underline{C} \quad \text{and} \quad \lim_{n\to\infty} \|J'(u_n)\|_{\mathcal{X}^*} = 0.$$

Moreover, if $J$ satisfies the Palais-Smale condition at the level $\underline{C}$, then $\underline{C}$ is a critical value of $J$.

**Theorem 3.8.** Assume that hypotheses $(\phi_1) - (\phi_4)$, $(g_1)$, $(g_2)$, $\lambda > 0$ and (7) hold. Then the problem $(P)$ admits at least one weak solution.

We will proof the Theorem 3.8 using the following results.

**Proposition 3.9.** Suppose that the hypotheses of Theorem 3.8 are fulfilled, then the functional $J : W_0^{1,p(x)}(\Omega) \to \mathbb{R}$ possesses a mountain-pass geometry.

*Proof.* Our aim is to prove the existence of a "mountain and a valley".

We may assume, without lose of generality, that $\|u\|_{p(x)} < 1$, then we have

$$\begin{aligned} J(u) \quad &= \Phi(u) - \lambda\Gamma(u) \\ &= \int_\Omega \Phi_0(x, |\nabla u|)dx - \lambda \left( \int_\Omega \frac{g(x)}{q(x)} |u|^{q(x)} dx + \int_\Omega \frac{1}{r(x)} |u|^{r(x)} dx \right). \end{aligned}$$

Using $(\phi_4)$ we obtain

$$J(u) \geq \int_\Omega \frac{1}{p^+} \phi(x, |\nabla u|) \cdot \nabla u dx - \lambda \left( \int_\Omega \frac{g(x)}{q(x)} |u|^{q(x)} dx + \int_\Omega \frac{1}{r(x)} |u|^{r(x)} dx \right).$$

By $(\phi_3)$, and the fact that $q(x)$ and $r(x) \in C_+(\Omega)$, we obtain that

$$J(u) \geq \frac{c}{p^+} \int_\Omega |\nabla u|^{p(x)} dx - \lambda \left( \frac{1}{q^-} \int_\Omega g(x)|u|^{q(x)} dx + \frac{1}{r^-} \int_\Omega |u|^{r(x)} dx \right).$$

Now by the fact that
$$1 < p(x) < q(x) < r(x) < p^*(x)$$
the embeddings $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{r(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ are continuous and compact, hence it yields that
$$\begin{aligned} J(u) \geq & \frac{c}{p^+}\|u\|_{p(x)}^{p^+} - \lambda \left[ \frac{\|g\|_\infty}{q^-} \left( C_q^{q^-} \|u\|_{p(x)}^{q^-} + C_q^{q^+} \|u\|_{p(x)}^{q^+} \right) \right. \\ & \left. + \frac{1}{r^-} \left( C_r^{r^-} \|u\|_{p(x)}^{r^-} + C_r^{r^+} \|u\|_{p(x)}^{r^+} \right) \right]. \end{aligned}$$
Since $g \in L^\infty(\Omega)$, we have that, there exists a constant $C > 0$ such that $\|g\|_\infty \leq C$, so we have
$$\begin{aligned} J(u) \geq & \frac{c}{p^+}\|u\|_{p(x)}^{p^+} - \lambda \left[ \frac{C}{q^-} \left( C_q^{q^-} \|u\|_{p(x)}^{q^-} + C_q^{q^+} \|u\|_{p(x)}^{q^+} \right) \right. \\ & \left. + \frac{1}{r^-} \left( C_r^{r^-} \|u\|_{p(x)}^{r^-} + C_r^{r^+} \|u\|_{p(x)}^{r^+} \right) \right]. \end{aligned}$$
So, by the assumption
$$1 < p^- \leq p(x) \leq p^+ < q^- \leq q(x) \leq q^+ < r^- \leq r(x) \leq r^+ < p^*(x)$$
taking $\|u\|_{p(x)} = R > 0$ small enough we have that
$$J(u) \geq c_0 > 0.$$
This shows the existence of a "mountain" around the origin.

Next, we prove the existence of a "valley" over the "chain of mountains".

The hypothesis $(\phi_4)$ implies that for all $\gamma \geq 1$ and $x \in \overline{\Omega}$, $t \in \mathbb{R}^N$,
$$\Phi_0(x, \gamma|t|) \leq \gamma^{p^+} \Phi_0(x, |t|). \tag{11}$$
For now, we choose $u_0 \in W_0^{1,p(x)}(\Omega)$, $u_0 > 0$ in $\Omega$ and $t \in \mathbb{R}$ with $t > 0$.
From a straightforward computation we get
$$J(tu_0) = \int_\Omega \Phi_0(x, t|\nabla u_0|)dx - \lambda \left( \int_\Omega \frac{t^{q(x)} g(x)}{q(x)} |u_0|^{q(x)} dx + \int_\Omega \frac{t^{r(x)}}{r(x)} |u_0|^{r(x)} dx \right).$$
Since $p^+ < q^- \leq q^+ < r^- \leq r^+$, $g(x) > 0$ for all $x \in \overline{\Omega}$ and $g \in L^\infty(\Omega)$, does not depends on $t$, we obtain that
$$J(tu_0) \to -\infty, \quad \text{as } t \to \infty.$$
Therefore we can consider $v = tu_0$ with $\|v\|_{p(x)} = t\|u_0\|_{p(x)} > R$ such that $J(v) < 0$, for $t$ chosen sufficiently large.

Hence, we have checked that our functional $J : W_0^{1,p(x)}(\Omega) \to \mathbb{R}$ possesses a mountain-pass geometry. □

**Proposition 3.10.** Suppose that $(\phi_1) - (\phi_4)$, $(g_1)$, $(g_2)$, $(\Gamma_1)$ and (7) hold. The the functional $J$ verifies the Palais-Smale condition for all $\lambda > 0$.

*Proof.* We have $J(u) := \Phi(u) - \lambda\Gamma(u)$.

Note that using Lemma 21 from chapter 3 in [20] we have that $\Gamma'$ is of type $(S_+)$, since $\Gamma'$ is compact (due to the compact Sobolev embeddings, $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{r(x)}(\Omega)$, $g \in L^\infty(\Omega)$.

Let $\{u_n\}$ be a (PS)-sequence in $W_0^{1,p(x)}(\Omega)$, i.e.,

$$J(u_n) \to \underline{C} \quad \text{and} \quad J'(u_n) \to 0, \quad \text{as } n \to \infty.$$

Due to Lemma 3.3 we know that $\Phi' : W_0^{1,p(x)}(\Omega) \to W^{1,p'(x)}(\Omega)$ is of type $(S_+)$, so $J'$ is of type $(S_+)$.

Since $J'$ is of type $(S_+)$ and $W_0^{1,p(x)}(\Omega)$ is reflexive, it is enough to show that $\{u_n\}$ is bounded in $W_0^{1,p(x)}(\Omega)$.

Arguing by contradiction, we suppose that $\|u_n\|_{p(x)} \to \infty$ (passing eventually to a subsequence).

Using the hypothesis $(\Phi_4)$ we obtain

$$\begin{aligned} J(u_n)-\frac{1}{\omega}J'(u_n)(u_n) &= \int_\Omega \left(\Phi_0(x,|\nabla u_n|) - \frac{1}{\omega}\phi(x,|\nabla u_n|)\nabla u_n \cdot \nabla u_n\right) dx + \\ &\quad + \lambda \int_\Omega \left(\frac{g(x)}{\omega}|u_n|^{q(x)} - \frac{g(x)}{q(x)}|u_n|^{q(x)} + \frac{1}{\omega}|u_n|^{r(x)} - \frac{1}{r(x)}|u_n|^{r(x)}\right) dx \\ &\geq \left(1 - \frac{p^+}{\omega}\right) \int_\Omega \Phi_0(x,|\nabla u_n|)dx + \\ &\quad + \lambda \int_\Omega \left(\frac{g(x)}{\omega} - \frac{g(x)}{q(x)}\right)|u_n|^{q(x)} + \left(\frac{1}{\omega} - \frac{1}{r(x)}\right)|u_n|^{r(x)}dx. \end{aligned}$$

Let us consider

$$A = \sup\left\{\left|\left(\frac{g(x)}{\omega} - \frac{g(x)}{q(x)}\right)|u|^{q(x)} + \left(\frac{1}{\omega} - \frac{1}{r(x)}\right)|u|^{r(x)}\right| \,:\, x \in \overline{\Omega}, |u| \leq A_1\right\}.$$

Using the condition $(\Gamma_1)$ we get that

$$\begin{aligned} &\left(1 - \frac{p^+}{\omega}\right) \int_\Omega \Phi_0(x,|\nabla u_n|)dx \leq J(u_n) - \frac{1}{\omega}J'(u_n)(u_n) - \\ &- \lambda \int_{\{x\in\Omega:|u_n(x)|>A_1\}} \left(\frac{g(x)}{\omega}|u_n|^{q(x)} - \frac{g(x)}{q(x)}|u_n|^{q(x)} + \frac{1}{\omega}|u_n|^{r(x)} - \frac{1}{r(x)}|u_n|^{r(x)}\right) dx \\ &+ \lambda A|\Omega| \leq J(u_n) - \frac{1}{\omega}J'(u_n)(u_n) + \lambda A|\Omega|, \end{aligned}$$

where $|\Omega|$ is the Lebesgue measure of the domain $\Omega$.

Since we supposed that $\|u_n\|_{p(x)} \to \infty$, we may say that for $n$ taken large enough we have $\|u_n\|_{p(x)} > 1$. Now using $(\phi_3)$, $(\phi_4)$ and (3) we obtain that:

$$\left(1 - \frac{p^+}{\omega}\right)\frac{c}{p^+}\|u_n\|_{p(x)}^{p^-} \leq J(u_n) + \frac{1}{\omega}\|J'(u_n)\|_{W^{-1,p'(x)}(\Omega)}\|u_n\|_{p(x)} + \lambda A|\Omega|.$$

Using the fact that $\omega > p^+$ and $p^- > 1$ we obtain a contradiction.

Therefore $J : W^{1,p(x)_0(\Omega)} \to \mathbb{R}$ fulfills the Palais-Smale condition. □

Finally we will conclude the affirmation stated by the Theorem 3.8.

**Conclusion of Theorem 3.8.** Since $J(0) = 0$, using the Propositions 3.9 and 3.10 we observe that $J$ has at least one critical point via Theorem 3.7, therefore our problem $(P)$ has at least one nontrivial weak solution.

**3.4. Existence of a ground state solution.** To find a ground state solution in $W_0^{1,p(x)}(\Omega)$ we will develop some ideas founded in [2].

Firstly, we define $\mathcal{W}_s$, as the set of all nontrivial weak solutions of $(P)$ in $W_0^{1,p(x)}(\Omega)$ like that

$$\mathcal{W}_s := \{u \in W_0^{1,p(x)}(\Omega) \setminus \{0\} : J'(u) = 0\}.$$

Using Theorem 3.8 we observe that $\mathcal{W}_s$ is not empty.

We will show that the following lemmas hold on the set $\mathcal{W}_s$.

**Lemma 3.11.** There exists $C_w > 0$ such that $\|u\|_{p(x)} \geq C_w$, for all $u \in \mathcal{W}_s$.

*Proof.* Assume that $\|u\|_{p(x)} < 1$. The case $\|u\|_{p(x)} > 1$ is irrelevant.

$$\begin{aligned}\lambda\left(\int_\Omega g(x)|u|^{q(x)} + |u|^{r(x)}dx\right) &= \int_\Omega \phi(x,|\nabla u|)\cdot|\nabla u|^2 dx\\ &\geq c\int_\Omega |\nabla u|^{p(x)}dx.\end{aligned}$$

Next, using $(g_1)$, $(g_2)$, $(\phi_3)$ and relation (7), we obtain

$$\lambda\left[C\cdot\left(C_q^{q^-}\|u\|_{p(x)}^{q^-} + C_q^{q^+}\|u\|_{p(x)}^{q^+}\right) + \left(C_r^{r^-}\|u\|_{p(x)}^{r^-} + C_r^{r^+}\|u\|_{p(x)}^{r^+}\right)\right] \geq c\|u\|_{p(x)}^{p^+}.$$

Hence, one have

$$\lambda C_M \|u\|_{p(x)}^{q^-} \geq c\|u\|_{p(x)}^{p^+}.$$

Therefore we obtain $\|u\|_{p(x)}^{q^- - p^+} \geq \frac{c}{\lambda C_M}$, so

$$\|u\|_{p(x)} \geq \left(\frac{c}{\lambda C_M}\right)^{\frac{1}{q^- - p^+}},$$

where $C_M = C(C_q^{q^-} + C_q^{q^+}) + C_r^{r^-} + C_r^{r^+}$. □

**Lemma 3.12.** There exists a positive constant $\overline{c} > 0$, such that $J(u) \geq \overline{c}$, for all $u \in \mathcal{W}_s$ and for all $p^+ < \omega$(where $\omega$ is the same from the assumption $(\Gamma_1)$).

*Proof.* Let $u \in \mathcal{W}_s$. Using the terms of Proposition 3.10, and by Lemma 3.11 we have:

$$\begin{aligned}&J(u) = J(u) - \frac{1}{\omega}J'(u)(u) \geq\\ &\geq \left(1 - \frac{p^+}{\omega}\right)\int_\Omega \Phi_0(x,|\nabla u|)dx + \lambda\int_\Omega\left(\frac{g(x)}{\omega} - \frac{g(x)}{q(x)}\right)|u|^{q(x)} + \left(\frac{1}{\omega} - \frac{1}{r(x)}\right)|u|^{r(x)}dx\\ &\geq \left(1 - \frac{p^+}{\omega}\right)\int_\Omega \Phi_0(x,|\nabla u|)dx + \lambda\left[\left(\frac{C}{\omega} - \frac{C}{q^-}\right)\int_\Omega |u|^{q(x)}dx + \left(\frac{1}{\omega} - \frac{1}{r^-}\right)\int_\Omega |u|^{r(x)}dx\right]\\ &\geq \overline{c}.\end{aligned}$$

In the previous estimates we used the Sobolev embeddings allowed by relation (7) and the properties of the function $g$. □

Let us now indicate the Nehari manifold associated to the functional $J$, i.e.,

$$\mathcal{N}_\lambda := \{u \in W_0^{1,p(x)}(\Omega) \setminus \{0\} \ : \ J'(u)(u) = 0\}.$$

We note that Lemmas 3.11 and 3.12 hold also for $\mathcal{N}_\lambda$.

By Lemma 3.12, we infer that

$$\xi = \inf_{u\in\mathcal{W}_s} J(u) > 0.$$

We will show by our next theorem that this infimum is achieved.

**Theorem 3.13.** Assuming that $(\phi_1)-(\phi_4)$, $(g_1)-(g_2)$, $(\Gamma_1)$ and relation (7) hold, then in the space $W_0^{1,p(x)}(\Omega)$, there exists a ground state solution for the problem $(P)$, i.e.,

$$J(\underline{u}) = \min_{u\in\mathcal{W}_s} J(u).$$

*Proof.* Let $\{u_n\}\subset\mathcal{W}_s$ be a minimizing sequence, i.e.,

$$J(u_n)\to\xi \quad \text{and} \quad J'(u_n)=0.$$

Then $\{u_n\}$ is Palais-Smale sequence for $J$ and we showed its existence in the proof of Proposition 3.10. □

## 4. Spectrum consisting in a bounded interval

In this section we will establish a concentration phenomena related to the spectrum which consists in an interval which is bounded by two Rayleigh-type quotients, while no eigenvalues exists outside this interval.

We define the following Rayleigh-type quotient:

$$\lambda_0 := \inf_{u\in W_0^{1,p(x)}(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega \phi(x,|\nabla u|)\cdot|\nabla u|^2dx}{\displaystyle\int_\Omega g(x)|u|^{q(x)}dx + \int_\Omega |u|^{r(x)}dx}$$

and we remind that:

$$\lambda_1 := \inf_{u\in W_0^{1,p(x)}(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega \Phi_0(x,|\nabla u|)dx}{\displaystyle\int_\Omega \frac{g(x)}{q(x)}|u|^{q(x)}dx + \int_\Omega \frac{1}{r(x)}|u|^{r(x)}dx}.$$

Before we can study the concentration of the spectrum we will give some important lemmas.

**Lemma 4.1.** We have that $\lambda_1 > \lambda_0 > 0$.

*Proof.* Using the assumption $(\phi_4)$ we find that

$$\int_\Omega \Phi_0(x,|\nabla u|)dx \geq \frac{1}{p^+}\int_\Omega \phi(x,|\nabla u|)\cdot|\nabla u|^2dx.$$

By hypothesis $(\Gamma_1)$ we deduce that for all $u\in W_0^{1,p(x)}(\Omega)$, with $|u|>0$ we have:

$$\frac{\displaystyle\int_\Omega \Phi_0(x,|\nabla u|)dx}{\displaystyle\int_\Omega \frac{g(x)}{q(x)}|u|^{q(x)}dx + \int_\Omega \frac{1}{r(x)}|u|^{r(x)}dx} \geq \frac{\omega}{p^+}\cdot\frac{\displaystyle\int_\Omega \phi(x,|\nabla u|)\cdot|\nabla u|^2dx}{\displaystyle\int_\Omega g(x)|u|^{q(x)}dx + \int_\Omega |u|^{r(x)}dx}.$$

Taking the inequality to the infimum for $u \in W_0^{1,p(x)}(\Omega) \setminus \{0\}$ we obtain using $(\Gamma_1)$ and $(\phi_4)$ that

$$\lambda_1 \geq \frac{\omega}{p^+}\lambda_0 > \lambda_0.$$

We proceed now to show that $\lambda_0 > 0$. Firstly we suppose that $\|u\|_{p(x)} < 1$. Hence, using $(\phi_3)$ and $(g_2)$ we obtain

$$\begin{aligned}
\frac{\displaystyle\int_\Omega \phi(x,|\nabla u|)\cdot|\nabla u|^2dx}{\displaystyle\int_\Omega g(x)|u|^{q(x)}dx + \int_\Omega |u|^{r(x)}dx} &\geq \frac{\displaystyle c\int_\Omega |\nabla u|^{p(x)}dx}{\displaystyle\|g\|_\infty \int_\Omega |u|^{q(x)}dx + \int_\Omega |u|^{r(x)}dx}\\
&\geq \frac{c\|u\|_{p(x)}^{p^+}}{C(C_q^{q^-}\|u\|_{p(x)}^{q^-} + C_q^{q^+}\|u\|_{p(x)}^{q^+}) + (C_r^{r^-}\|u\|_{p(x)}^{r^-} + C_r^{r^+}\|u\|_{p(x)}^{r^+})}\\
&\geq \frac{c\|u\|_{p(x)}^{q^-}}{[C(C_q^{q^-} + C_q^{q^+}) + (C_r^{r^-} + C_r^{r^+})]\|u\|_{p(x)}^{q^-}}\\
&\geq \frac{c}{C(C_q^{q^-} + C_q^{q^+}) + (C_r^{r^-} + C_r^{r^+})} > 0.
\end{aligned}$$

where $c$ is the constant given by $(\phi_3)$, $C$ is the constant from $(g_2)$ and $C_q$ and $C_r$ are positive constants resulted from the continuous embeddings:

$$W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega) \quad \text{and} \quad W_0^{1,p(x)}(\Omega) \hookrightarrow L^{r(x)}(\Omega).$$

Taking the infimum in the previous inequality we obtain that $\lambda_0 > 0$.

Due to the fact that relation (7) implies that $p^- < q^- \leq q^+ < r^- \leq r^+$ we are not interested in the weak solutions which have $\|\cdot\|_{p(x)} > 1$, and their associated eigenvalues, because we can not show that the functional $J$ is coercive and we can not apply the Direct Method in the Calculus of Variations in order to point out its critical points. $\square$

**Theorem 4.2.** The problem $(P)$ has no solution for every $\lambda < \lambda_0$.

*Proof.* We recall that:

$$\begin{aligned}
\lambda_0 \quad &:= \inf_{u\in W_0^{1,p(x)}(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega \phi(x,|\nabla u|)\cdot|\nabla u|^2dx}{\displaystyle\int_\Omega g(x)|u|^{q(x)}dx + \int_\Omega |u|^{r(x)}dx}\\
&:= \inf_{u\in W_0^{1,p(x)}(\Omega)\setminus\{0\}} \frac{P(x,u)}{Q(x,u)},
\end{aligned}$$

where $P(x,u) := \displaystyle\int_\Omega \phi(x,|\nabla u|)\cdot|\nabla u|^2dx$ and $Q(x,u) := \displaystyle\int_\Omega g(x)|u|^{q(x)}dx + \int_\Omega |u|^{r(x)}dx$.

Fix $\lambda < \lambda_0$. We argue by contradiction, and assume that $\lambda$ is an eigenvalue of the problem $(P)$. Therefore we may find $u \in W_0^{1,p(x)}(\Omega) \setminus \{0\}$ such that

$$\int_\Omega \phi(x,|\nabla u|)\cdot\nabla u\cdot\nabla\varphi dx = \lambda\left(\int_\Omega g(x)|u|^{q(x)-2}u\varphi dx + \int_\Omega |u|^{r(x)-2}u\varphi dx\right),$$

for all $\varphi \in W_0^{1,p(x)}(\Omega)$, thus we have

$$P'_u(x,u) = \lambda Q'_u(x,u).$$

For now on, taking $\varphi = u$ we get that $P(x,u) = \lambda Q(x,u)$. Hence,

$$\lambda = \frac{P(x,u)}{Q(x,u)} \geq \inf_{u \in W_0^{1,p(x)}(\Omega)\setminus\{0\}} \frac{P(x,u)}{Q(x,u)} = \lambda_0$$

which contradicts de choice of $\lambda$. □

**Theorem 4.3.** Every $\lambda \in (\lambda_0, \lambda_1]$ is an eigenvalue of the problem $(P)$.

*Proof.* As we pointed out by Theorem 3.6 , $\lambda_1$ is an eigenvalue of the problem $(P)$. It remains to show that every $\lambda \in (\lambda_0, \lambda_1)$ is an eigenvalue for our problem.

As we proved in the previous section there exists $\underline{u} \in \mathcal{N}_\lambda$ which is a global minimizer of $J$, such that

$$J(\underline{u}) = \min_{u \in \mathcal{W}_s} J(u) = \xi > 0.$$

We argue in what follows to reveal that $\underline{u} \neq 0$. Indeed, using the definition of $\lambda_1$, and the fact that $\lambda < \lambda_1$, we may find $v \in W_0^{1,p(x)}(\Omega) \setminus \{0\}$ such that

$$\lambda < \frac{\Phi(v)}{\Gamma(v)}.$$

Then $J(v) > 0$.

Since $\underline{u}$ is a global minimum point of $J$, it follows that $J(v) \geq J(\underline{u}) > 0$, which implies $\underline{u} \neq 0$.

We conclude that $\lambda$ is an eigenvalue for the problem $(P)$ with the corresponding eigenfunction $\underline{u}$. □

**Theorem 4.4.** For all $\lambda > \lambda_1$, problem $(P)$ does not have a solution.

*Proof.* Suppose that $\lambda$ is an eigenvalue of $(P)$. Set $\lambda > \lambda_1$. Using the definition of $\lambda_1$ and the fact that $\lambda > \lambda_1$, we may find $v \in W_0^{1,p(x)}(\Omega)\setminus\{0\}$ such that $\Phi(v) - \lambda\Gamma(v) < 0$, so $J(v) < 0$.

So we can say that

$$\inf_{u \in W_0^{1,p(x)}(\Omega)\setminus\{0\}} J(u) < 0,$$

which leads us to the following alternative:
$(i)$ $u \in \mathcal{W}_s$ and $\inf_{u \in \mathcal{W}_s} J(u) < 0$, which is a contradiction with the fact that $\inf_{u \in \mathcal{W}_s} J(u) = \xi > 0$;
$(ii)$ $u \in W_0^{1,p(x)}(\Omega) \setminus \{0\}$ and $\inf_{u \in W_0^{1,p(x)}(\Omega)\setminus\{0\}} J(u) < 0$, hence $u \notin \mathcal{W}_s$, fact that contradicts the hypothesis that $\lambda$ is an eigenvalue for the problem $(P)$.

Therefore, we can conclude that for any $\lambda \in (\lambda_1, +\infty)$, the problem $(P)$ does not admit a solution. □

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