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Stress formulation for the blocking property of the inhomogeneous Bingham fluid

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Abstract. This work is concerned with the flow of a viscous plastic fluid. After setting the general threedimensional problem, the blocking property is introduced. We then focus on necessary and sufficient conditions such that blocking of the fluid occurs. The anti-plane flow in twodimensional case is considered. A variational formulation in terms of stresses is deduced. Some properties dealing with local stagnant regions as well as local regions are obtained.
Résumé. Nous considérons le problème de l'écoulement d'un fluide viscoplastique. Nous nous intéressons ensuite aux conditions nécessaires et suffisantes de blocage du fluide. Le problème antiplan est considéré et une formulation variationnelle en contraintes est obtenue et utilisée. Des propriétés concernant les zones stagnantes du fluide ainsi que celles où le fluide se comporte localement comme un corps rigide sont établies.

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1. Introduction

Due the importance of evaluation of landslide risk, great efforts have been devoted to analyzing, modeling, and predicting such phenomena in the last decades. A stability analysis, which treats the geologic material as a rigid viscoplastic body, may provide information on the safety factor of stable mass of soil. One of the simplest and convenient viscoplastic constitutive relation is the one modeling a Bingham fluid, exhibiting viscosity and yield stress.

Recently, the inhomogeneous (or density-dependent) Bingham fluid was considered in landslides modeling [2, 1, 3, 6]. In this work, the inhomogeneous yield limit is a key point in describing a natural slope. Indeed, due to their own weight, the geomaterials are compacted (*i.e.*, their density increase with depth), so that the mechanical properties also vary with depth. Therefore the choice of a Bingham model in which the yield limit g and the viscosity coefficient η vary with density is motivated.

A particularity of the Bingham model lies in the presence of rigid zones located in the interior of the flow of the Bingham solid/fluid. As the yield limit g increases, these rigid zones become larger and may completely block the flow. This property is called the *blocking property.* When considering oil transport in pipelines, in the process of oil drilling or in the case of metal forming, the blocking of the solid/fluid is a catastrophic event to be avoided. In a completely opposite context, when modeling landslides, the solid is blocked in its natural configuration and the beginning of a flow can be seen as a disaster.

This paper deals with some boundary-value problems describing the flow of an inhomogeneous Bingham fluid through a bounded domain in $\mathbb{R}^3$. We focus on the

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blocking phenomenon, the description of the rigid zones and also the stagnant regions (*i.e.*, the zones near the boundary of the domain where the fluid does not move). More precisely, we study the link between the yield limit distribution and the external forces distribution (or the mass density distribution) for which the flow of the Bingham fluid is blocked or exhibits rigid zones. In opposition to the previous works dealing only with homogeneous Bingham fluids [5, 7, 8], we are interested in a fluid whose yield limit is inhomogeneous.

An outline of the paper is as follows. The equations modeling the flow of a Bingham fluid are introduced in section 2 and the corresponding variational formulation is recalled. Section 3 is concerned with the blocking property in the threedimensional context. There we give a necessary and sufficient condition which characterizes the blocking property in the inhomogeneous case. The stationary anti-plane problem (twodimensional) is considered in section 4. We obtain a variational formulation in terms of stresses which is useful in the description of the rigid zones.

2. The mechanical problem

We consider here the evolution equations in the time interval $(0,T)$, $T>0$ describing the flow of an inhomogeneous Bingham fluid in a domain $\mathcal{D}\subset\mathbb{R}^3$ with a smooth boundary $\partial\mathcal{D}$. The notation $\boldsymbol{u}$ stands for the velocity field, $\boldsymbol{\sigma}$ denotes the Cauchy stress tensor field, $p=-\operatorname{trace}(\boldsymbol{\sigma})/3$ represents the pressure and $\boldsymbol{\sigma'}=\boldsymbol{\sigma}+pI$ is the deviatoric part of the stress tensor. The conservation of mass reads

$$\frac{\partial\rho}{\partial t}+\boldsymbol{u}\cdot\nabla\rho=0 \quad \text{in } \mathcal{D}\times(0,T). \tag{1}$$

where $\rho=\rho(t,x)\geq\underline{\rho}>0$ is the mass density distribution. We suppose that the we deal with an incopressible flow

$$div\ \boldsymbol{u}=0, \quad \text{in } \mathcal{D}\times(0,T). \tag{2}$$

If we denote by $D(\boldsymbol{u})=(\boldsymbol{\nabla u}+\boldsymbol{\nabla}^T\boldsymbol{u})/2$ the rate deformation tensor, the constitutive equation of the Bingham fluid can be written as follows:

$$\boldsymbol{\sigma'}=2\eta D(\boldsymbol{u})+g\frac{D(\boldsymbol{u})}{|D(\boldsymbol{u})|} \qquad \text{if } |D(\boldsymbol{u})|\neq 0, \tag{3}$$

$$|\boldsymbol{\sigma'}|\leq g \qquad \text{if } |D(\boldsymbol{u})|=0, \tag{4}$$

where $\eta\geq\eta_0>0$ is the viscosity distribution and $g\geq 0$ is a nonnegative continuous function which stands for the yield limit distribution in $\mathcal{D}$. The type of behavior described by equations (3–4) can be observed in the case of some oils or sediments used in the process of oil drilling. The Bingham model, also denominated "Bingham solid" (see for instance was considered in order to describe the deformation of many solid bodies. Recently, the inhomogeneous (or density-dependent) Bingham fluid was chosen in landslides modeling [2, 1, 3, 6].

When considering a density-dependent model, the viscosity coefficient η and the yield limit g depend on the density ρ through two constitutive functions, *i.e.*,

$$\eta=\eta(\rho(t,x)), \quad g=g(\rho(t,x)). \tag{5}$$

We assume that $\partial\mathcal{D}$ is divided into two disjoint parts so that $\partial\mathcal{D}=\partial_0\mathcal{D}\cup\partial_1\mathcal{D}$ and

$$\boldsymbol{u}=0 \quad \text{on} \quad \partial_0\mathcal{D}\times(0,T), \qquad \boldsymbol{\sigma n}=0 \quad \text{on} \quad \partial_1\mathcal{D}\times(0,T), \tag{6}$$

where $\boldsymbol{n}$ stands for the outward unit normal on $\partial\mathcal{D}$.

Setting

$$\mathcal{V} = \{\boldsymbol{v} \in H^1(\mathcal{D})^3,\ \mathrm{div}\ \boldsymbol{v} = 0 \ \text{ in } \mathcal{D}, \boldsymbol{v} = 0 \ \text{ on } \partial_0\mathcal{D}\},$$

we give the variational formulation for the velocity field is

$$\begin{cases} \forall t \in (0,T), & \boldsymbol{u}(t,\cdot) \in \mathcal{V}, \\ \forall \boldsymbol{v} \in \mathcal{V}, & \displaystyle\int_{\mathcal{D}} \rho\Big(\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\Big)\cdot(\boldsymbol{v}-\boldsymbol{u}) \\ & \displaystyle + \int_{\mathcal{D}} 2\eta D(\boldsymbol{u}) : (D(\boldsymbol{v}) - D(\boldsymbol{u})) \\ & \displaystyle + \int_{\mathcal{D}} g|D(\boldsymbol{v})| - \int_{\mathcal{D}} g|D(\boldsymbol{u})| \geq \int_{\mathcal{D}} \rho \boldsymbol{f}\cdot(\boldsymbol{v}-\boldsymbol{u}), \end{cases} \tag{7}$$

where $\boldsymbol{f}$ denotes the body forces.

Finally the initial conditions are given by

$$\boldsymbol{u}|_{t=0} = \boldsymbol{u}_0, \quad \rho|_{t=0} = \rho_0. \tag{8}$$

and the problem of the flow of a inhomogeneous Bingham fluid becomes:

Find the velocity field $\boldsymbol{u}$ *and the mass density field* ρ *such that conditions (1), (5), (8) and (7) hold.*

As far as we know there does not exist any uniqueness result for this problem. For the Navier-Stokes model (*i.e.*, when $g = 0$) existence results can be found in [4, 11, 10].

3. The blocking property

When considering a viscoplastic model of Bingham type, one can observe rigid zones (*i.e.*, zones where $D(\boldsymbol{u}) = 0$) in the interior of the flow of the solid/fluid. When g increases, the rigid zones are growing and if g becomes sufficiently large, the fluid stops flowing. Commonly called the *blocking property*, such a behavior can lead to unfortunate consequences in oil transport in pipelines, in the process of oil drilling or in the case of metal forming. On the contrary, in landslides modeling, it is precisely the blocking phenomenon which ensures stability of the slope.

We suppose in what follows that the volume forces are independent of time, *i.e.* $\boldsymbol{f} = \boldsymbol{f}(x)$. We say that the Bingham fluid is blocked if $\boldsymbol{u} \equiv 0$ satisfies equations and conditions (1), (5), (8), (7). One can easily check that the fluid is blocked if and only if the density has no time evolution (*i.e.*, $\rho(t,x) = \rho_0(x)$) and fulfills:

$$\int_{\mathcal{D}} g(\rho_0(x))|D(\boldsymbol{v}(x))|\, dx \geq \int_{\mathcal{D}} \rho_0(x)\boldsymbol{f}(x)\cdot\boldsymbol{v}(x)\, dx, \quad \forall \boldsymbol{v} \in \mathcal{V}.$$

Hence the study of the blocking property consists in finding the link between ρ_0 and $\boldsymbol{f}$ such that the above inequality holds. Since in landslides modeling the yield limit $g = g(\rho)$ depends also on some other parameters (as water concentration), another formulation of the blocking property is more adequate. Indeed if we define

$$\boldsymbol{b}(x) = \rho_0(x)\boldsymbol{f}(x), \quad g(x) = g(\rho_0(x)),$$

then the blocking of the Bingham fluid can be characterized by:

$$\int_{\mathcal{D}} g(x)|D(\boldsymbol{v}(x))|\, dx \geq \int_{\mathcal{D}} \boldsymbol{b}(x)\cdot\boldsymbol{v}(x)\, dx, \quad \forall \boldsymbol{v} \in \mathcal{V}. \tag{9}$$

Now the main problem consists in finding properties on $\boldsymbol{b}$ and g such that inequality (9) holds.

We suppose in what follows that

$$\boldsymbol{b} \in L^\infty(\mathcal{D})^3, \quad \text{and} \quad \int_{\mathcal{D}} \boldsymbol{b} \cdot \boldsymbol{r} = 0, \quad \forall \boldsymbol{r} \in \mathcal{R} \cap \mathcal{V}, \tag{10}$$

where $\mathcal{R} = \ker D = \{\boldsymbol{r} : \mathcal{D} \to \mathbb{R}^3\,;\, \boldsymbol{r}(x) = m + n \wedge x\}$ is the set of rigid motions. The first condition in (10) is a natural assumption for the body forces. The second one, which is implied by (9), is always satisfied if $\partial_0 \mathcal{D} \neq \emptyset$.

In the homogeneous case, it is easy to check that the condition $g \geq g^*_{\text{hom}}$, with

$$g^*_{\text{hom}} := \sup_{\boldsymbol{v} \in \mathcal{V} \setminus \mathcal{R}} \frac{\displaystyle\int_{\mathcal{D}} \boldsymbol{b} \cdot \boldsymbol{v}}{\displaystyle\int_{\mathcal{D}} |D(\boldsymbol{v})|} < +\infty,$$

is a complete characterization of the blocking property. More precisely we have that if (10) holds then $g^*_{\text{hom}} < +\infty$ and if $g(x) \geq g^*_{\text{hom}}$, a.e. $x \in \mathcal{D}$ then the blocking occurs, *i.e.* (9) holds.

In the inhomogeneous case it is only a rough sufficient condition. Indeed the following statement gives a more accurate condition for (9).

We define

$$\mathcal{H} = \{\boldsymbol{\tau} \in L^2(\mathcal{D})^{3\times 3}; \quad \tau_{ij} = \tau_{ji}, \quad \text{trace}(\boldsymbol{\tau}) = 0 \text{ in } \mathcal{D}\}$$

which stands for the deviatoric subspace of $L^2(\mathcal{D})^{3\times 3}_S$, and we consider

$$\mathcal{A}_{\boldsymbol{b}} = \big\{\boldsymbol{\tau} \in \mathcal{H}\,;\; \exists p \in L^2(\mathcal{D}), \quad \mathbf{div}\, \boldsymbol{\tau} - \nabla p = -\boldsymbol{b} \text{ in } \mathcal{D}, \quad (\boldsymbol{\tau} - pI)\boldsymbol{n} = 0 \text{ on } \partial_1 \mathcal{D}\big\},$$

where $(\boldsymbol{\tau} - pI)\boldsymbol{n} = 0$ lies in $H^{-1/2}(\partial\mathcal{D})^3$. Using the characterization of the gradient of a distribution (see for instance [13], p.14) we obtain another characterization of the set $\mathcal{A}_{\boldsymbol{b}}$:

$$\mathcal{A}_{\boldsymbol{b}} = \left\{\boldsymbol{\tau} \in \mathcal{H}\,; \quad \int_{\mathcal{D}} \boldsymbol{\tau} : D(\boldsymbol{v}) = \int_{\mathcal{D}} \boldsymbol{b} \cdot \boldsymbol{v},\ \forall \boldsymbol{v} \in \mathcal{V}\right\}.$$

In [6] it is proved the following result.

Theorem 3.1. *The Bingham fluid is blocked,* i.e., *(9) holds, if and only if there exists a function* $\boldsymbol{\sigma} \in \mathcal{A}_{\boldsymbol{b}}$ *such that* $g(x) \geq |\boldsymbol{\sigma}(x)|$*, a.e.* $x \in \mathcal{D}$*.*

4. The stationary anti-plane flow

We consider in this section the particular case of the stationary anti-plane flow. Therefore, $\mathcal{D} = \Omega \times \mathbb{R}$ where Ω is a bounded domain in $\mathbb{R}^2$. The boundary of Ω, denoted by Γ, is divided into two parts $\Gamma = \Gamma_0 \cup \Gamma_1$, such that $\partial_0\mathcal{D} = \Gamma_0 \times \mathbb{R}$, $\partial_1\mathcal{D} = \Gamma_1 \times \mathbb{R}$. We are looking for a flow in the Ox_3 direction, *i.e.* $\boldsymbol{u} = (0, 0, u)$, which does not depend on x_3 and t so that $\rho = \rho(x_1, x_2)$ and $u = u(x_1, x_2)$. Note that under these assumptions the equations (1-2) are satisfied, hence the density ρ represents now a parameter of the inhomogeneous problem and we cannot talk about a density dependent model anymore. Indeed the density is implied only in the spatial distribution of inhomogeneous parameters g, η and the body forces f are defined as follows

$$\eta(x_1, x_2) = \eta(\rho(x_1, x_2)), \quad g(x_1, x_2) = g(\rho(x_1, x_2)), \quad f(x_1, x_2) = \rho(x_1, x_2) f_3(x_1, x_2),$$

where f_3 denotes the component of the forces in the Ox_3 direction. We suppose in the following that

$$f, g, \eta \in L^\infty(\Omega), \quad g \geq 0, \quad \eta(x) \geq \eta_0 > 0, \text{ a.e. } x \in \Omega.$$

If we define

$$V = \{v \in H^1(\Omega); \quad v = 0 \quad \text{on} \quad \Gamma_0\}$$

then the variational formulation (7) for the anti-plane flow becomes

$$u \in V, \qquad \forall v \in V, \qquad \int_\Omega \eta(x)\nabla u(x) \cdot \nabla(v(x) - u(x))\, dx$$
$$+ \int_\Omega g(x)|\nabla v(x)|\, dx - \int_\Omega g(x)|\nabla u(x)|\, dx \geq \int_\Omega f(x)(v(x) - u(x))\, dx. \qquad (11)$$

The above problem is a standard variational inequality. If $\text{meas}(\Gamma_0) > 0$ then it has a unique solution u. If $\Gamma_0 = \emptyset$ and $\int_\Omega f(x)\, dx = 0$ then a solution exists and it is unique up to an additive constant. In the following we will always assume that one of these cases holds.

In order to give the variational formulation in terms of stresses for (11) we define $H = (L^2(\Omega))^2$ and

$$A_f = \{\boldsymbol{\tau} \in H; \quad \operatorname{div} \boldsymbol{\tau} = -f \quad \text{in} \ \ \Omega, \quad \boldsymbol{\tau} \cdot \boldsymbol{n} = 0 \quad \text{on}\ \Gamma_1\}, \qquad (12)$$

where $\boldsymbol{\tau} \cdot \boldsymbol{n}$ is considered in $H^{-\frac{1}{2}}(\Gamma)$. Let $J : H \to \mathbb{R}$ be the following functional

$$J(\boldsymbol{\tau}) = \int_\Omega \frac{1}{2\eta(x)}[|\boldsymbol{\tau}(x)| - g(x)]_+^2\, dx. \qquad (13)$$

The variation formulation in terms of stresses is given in the next proposition,proved in [6].

Proposition 4.1. *There exists at least a* $\boldsymbol{\sigma} \in A_f$ *minimizing* J *on* A_f, i.e. $J(\boldsymbol{\sigma}) \leq J(\boldsymbol{\tau})$, *for all* $\tau \in A_f$, *which is characterized by*

$$\boldsymbol{\sigma} \in A_f \quad \textit{and} \quad \int_\Omega \frac{[|\boldsymbol{\sigma}(x)| - g(x)]_+}{\eta(x)|\boldsymbol{\sigma}(x)|}\boldsymbol{\sigma}(x) \cdot \boldsymbol{\tau}(x)\, dx = 0, \quad \forall\, \boldsymbol{\tau} \in A_0 \qquad (14)$$

where A_0 *is* A_f *with* $f = 0$.

The following theorem, proved in[6], gives the connection between (11) and (14).

Theorem 4.1. *Let* u *be the solution of (11) and let* $\boldsymbol{\sigma}$ *be a solution of (14). Then we have*

$$\nabla u(x) = \frac{[|\boldsymbol{\sigma}(x)| - g(x)]_+}{\eta(x)|\boldsymbol{\sigma}(x)|}\boldsymbol{\sigma}(x), \quad \textit{a.e.} \quad x \in \Omega. \qquad (15)$$

The above theorem gives the opportunity to describe the rigid zones Ω_r and the shearing zones Ω_s defined by

$$\Omega_r = \{x \in \Omega; \quad |\nabla u(x)| = 0\}, \quad \Omega_s = \{x \in \Omega; \quad |\nabla u(x)| > 0\}.$$

Indeed, from (15) we have the following result.

Corollary 4.1. *The solution* $\boldsymbol{\sigma}$ *of (14) is unique in* Ω_s, i.e., *if* $\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2$ *are two solutions of (14) then* $\boldsymbol{\sigma}_1(x) = \boldsymbol{\sigma}_2(x)$ *a.e.* $x \in \Omega_s$. *For any* $\boldsymbol{\sigma}$ *solution of (14) we have*

$$\Omega_r = \{x \in \Omega; \quad |\boldsymbol{\sigma}(x)| \leq g(x)\}, \quad \Omega_s = \{x \in \Omega; \quad |\boldsymbol{\sigma}(x)| > g(x)\}. \qquad (16)$$

The previous description of the rigid zones can be used to study the blocking property, *i.e.*, when the whole Ω is a rigid zone ($\Omega = \Omega_r$). In this case $u \equiv 0$ is the solution of (11) characterized by the following problem:
Find the link between f and g such that

$$\int_\Omega g(x)|\nabla v(x)|\, dx \geq \int_\Omega f(x)v(x)\, dx, \qquad \forall v \in V. \tag{17}$$

As in the threedimensional case, the blocking always occurs for large enough yield distribution. Indeed, there exists an homogeneous yield limit $g^*_{\text{hom}} > 0$ given by

$$g^*_{\text{hom}} = \sup_{v\in V,\, v\neq \text{const}} \frac{\displaystyle\int_\Omega f(x)v(x)\, dx}{\displaystyle\int_\Omega |\nabla v(x)|\, dx}$$

such that if $g(x) \geq g^*_{\text{hom}}$, a.e. $x \in \Omega$ then the blocking occurs, *i.e.* (17) holds. Moreover we have the the following complete characterization of the blocking property.

Proposition 4.2. *The Bingham fluid is blocked if and only if there exists $\boldsymbol{\sigma} \in A_f$ such that $|\boldsymbol{\sigma}(x)| \leq g(x)$ a.e. $x \in \Omega$.*

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