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Common fixed point of set valued graph A_φ-contraction pair and generalized φ-weak G-contraction on metric space endowed with a graph

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Abstract. In this paper, we define the notion of graph A_φ-contraction pair and generalized φ-weak G contraction on subsets of a metric space involving a graph. Using such contractions, the existence as well as uniqueness of common fixed point for set valued mappings with set valued domain involving a directed graph have been examined. Suitable examples are presented to validate the non-triviality the results. We particularly generalize and extend the results due to Zhang and Song [Fixed point theory for generalized φ-weak contractions. Appl. Math. Lett., 22:75-78, 2009].

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1. Introduction

Boyd and Wong [6] investigated fixed points of nonlinear contractions. Ran and Reurings [19] combined the Banach's and Kanster-Tarski's fixed point theorems for continuous functions. Motivated by the work of Ran and Reurings, Neito & Rodríguez-López [16, 17] proved the uniqueness of fixed point without monotonicity and continuity properties. Jachymski [11] introduced the structure of graph on a metric space (MS, in short) by replacing the order structure. A few relevant work in this context are [5, 7, 14, 23].

Nadler [13] established the set valued version of Banach's theorem in a complete MS. Study of common fixed point (CFP, in short) has attracted researchers over the years [9, 10, 12, 15, 18, 22].

Generalized φ-weak contractions were introduced by Zhang and Song [24] to prove some CFP results for single valued maps in a complete MS. Akram et. al [3] established a characterization for metric completeness with the help of A-contractions. In the current paper, we define the notions of graph A_φ-contraction pair and generalized φ-weak G contraction on bounded and closed subsets of a MS involving the directed graph. Using such contractions, we obtain some new CFP results in a complete MS.

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2. Preliminaries

Let (X, d) be a MS and $W(X)$ be the collection of all nonempty closed and bounded subsets of X. If

$$H(E, F) = \max\{\sup_{v \in F} d(v, E), \sup_{u \in E} d(u, F)\}, E, F \in W(X)$$

where $d(u, F) = \inf_{v \in F} d(u, v)$. Then H is called Hausdorff metric induced by d and $(W(X), H)$ forms a metric space.

A directed graph G is an ordered pair $(V(G), E(G))$, where $V(G)$ is the set of vertices and $E(G)$ is the set of edges. We consider G as $V(G) = X$ and the set $E(G)$ of its edges also contains all its loops. Also, we assume that G does not contain parallel edges.

The conversion of a graph G is termed as G^{-1} and can be found from G by considering the reverse direction of edges of G. Also, the undirected graph is termed as $\tilde{G}$ and it can be found from G by ignoring the directions. If we consider a directed graph G such that the set of edges is symmetric, then we get

$$E(\tilde{G}) = E(G) \cup E(G^{-1}).$$

If a, b are two vertices of G, then a path in G from a to b is a finite sequence $\{a_i\}_{i=0}^{n}$ of $n + 1$ vertices such that $a_0 = a$, $a_n = b$ and $(a_{i-1}, a_i) \in E(G)$ for $i = 1, 2, \ldots, n$.

The graph G is said to be connected if there is at least one path between every pair of vertices in G. Further, G is weakly connected if $\tilde{G}$ is connected.

For more fixed point results in the similar setting we refer to [1, 2, 8] .

Suppose $P, Q \subset X (P, Q \neq \phi)$. Then, by $(P, Q) \subset E(G)$, we mean that 'there is an edge between P and Q' , i.e., there is an edge between some $p \in P$ and $q \in Q$. Moreover, by, 'there is a path between P and Q' , we mean that there is a path between some $p \in P$ and $q \in Q$.

For $S, R : W(X) \to W(X)$, the set X_{SR} is defined as below:

$$X_{SR} = \{P \in W(X) : (P, S(P)) \subset E(G) \text{ and } (S(P), RS(P)) \subset E(G)\}.$$

Definition 2.1. [3] Suppose A is the collection of all functions $\alpha : \mathbb{R}_+^3 \to \mathbb{R}_+$ satisfying

(i) α is continuous on $\mathbb{R}_+^3$.

(ii) $p \leq kq$ for some $0 \leq k \in 1$ whenever $p \leq \alpha(p, q, q)$ or $p \leq \alpha(q, p, q)$ or $p \leq \alpha(q, q, p)$ for each p, q.

Definition 2.2. [3] Suppose that (X, d) is a MS and R is a self map on X. R is called an A-contraction if

$$d(Ra_0, Rb_0) \leq \alpha(d(a_0, b_0), d(a_0, Ra_0), d(b_0, Rb_0))$$

for each $a_0, b_0 \in X$ and some $\alpha \in A$.

Definition 2.3. [21] Consider the class of functions $\Phi = \{\varphi | \varphi : \mathbb{R}_+ \to \mathbb{R}_+\}$, which satisfies the following assertions:

(i) $u_1 \leq u_2$ implies $\varphi(u_1) \leq \varphi(u_2)$;

(ii) $(\varphi^n(u))_{n \in N} \to 0$ for each $u > 0$;

(iii) $\sum \varphi^n(u)$ converges for each $t > 0$;

When (i-ii) are true, then φ is said to be a comparison function (CF). If (iii) is true as well, then φ is called a strong CF.

Definition 2.4. [4] A self mapping $R : X \to X$ on an MS (X, d) is called be a φ-weak contraction if there exists a map $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ with $\varphi(0) = 0$ and $\varphi(u) > 0$ for each $u > 0$ satisfying

$$d(Ra, Rb) \leq d(a, b) - \varphi(d(a, b)), \text{ for each } a, b \in X.$$

Rhoades [20] generalized of Banach's principle as follows.

Theorem 2.1. [20] *Suppose that (X, d) is an MS and R is a self-map on X satisfying*

$$d(Ra, Rb) \leq d(a, b) - \varphi(d(a, b)), \textit{ for each } a, b \in X$$

where $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous and non decreasing function with $\varphi(0) = 0$ and $\varphi(u) > 0$ for each $u > 0$. Then R has a unique fixed point.

Definition 2.5. [24] Suppose that (X, d) is an MS. Two self maps S, R on X are said to be generalized φ-weak contractions if there exists a map $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ with $\varphi(0) = 0$ and $\varphi(u) > 0$ for each $u > 0$ satisfying

$$d(Ra_0, Sb_0) \leq N(a_0, b_0) - \varphi(N(a_0, b_0)); \text{ for each } a_0, b_0 \in X$$

where $N(a_0, b_0) = \max\{d(a_0, b_0), d(a_0, Ra_0), d(b_0, Sb_0), \frac{1}{2}(d(a_0, Sb_0) + d(b_0, Ra_0))\}$.

The following was proved by Zhang and Song [24].

Theorem 2.2. *Suppose that (X, d) is an MS and S, R are two self maps on X such that for all $a, b \in X$*

$$d(Ra, Sb) \leq N(a, b) - \varphi(N(a, b));$$

where $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a lower semi continuous function with $\varphi(0) = 0$ and $\varphi(u) > 0$ for each $u > 0$. Then R and S have a unique CFP.

3. Common fixed point of set valued graph A_φ-contraction pair

In this section, we prove a CFP theorem by defining graph A_φ-contraction pair.

Definition 3.1. Let A_φ be the collection of all functions $\alpha : \mathbb{R}_+^3 \to \mathbb{R}_+$ which satisfy the following conditions:
- (i) α is continuous on $\mathbb{R}_+^3$.
- (ii) for each $p, q \in \mathbb{R}_+$, $p \leq \alpha(p, q, q)$ or $p \leq \alpha(q, p, q)$ or $p \leq \alpha(q, q, p)$, then $p \leq \varphi(q)$, where φ is a strong CF.

In this definition, if we take $\varphi(u) = ku$ as $0 \leq k < 1$ for each $u > 0$, then we obtain $\alpha \in A$.

Definition 3.2. Suppose $S, R : W(X) \to W(X)$ are two set valued maps on $W(X)$. The pair (S, R) of maps is called graph A_φ-contraction pair if the assertions hold which are given below:
- (i) for each $P_0 \in W(X), (P_0, S(P_0)) \subset E(G)$ and $(S(P_0), RS(P_0)) \subset E(G)$;
- (ii) there exists some $\alpha \in A_\varphi$ and $(P_0, Q_0) \subset E(G)$ such that

$$H(S(P_0), R(Q_0)) \leq \alpha(H(P_0, Q_0), H(P_0, S(P_0)), H(Q_0, R(Q_0))).$$

Remark 3.1. If a pair (S, R) of maps on *W(X)* is graph A_φ-contraction for graph G, then the pair is also graph A_φ-contraction for the graph G^{-1} and $\widetilde{G}$.

Definition 3.3. Let S, R be set valued maps on $W(X)$. We say that G is a μ-graph whenever for each sequence $\{A_k\}_{k\geq 0}$ in X with $A_k \to A$ and $(A_{2k}, A_{2k+1}) \subset E(G)$ for each $k \geq 0$, there is a subsequence $\{A_{2k_r}\}$ of $\{A_{2k}\}$ such that either R is continuous and $(A, A_{2k_r+1}) \subset E(G)$ for each $r \geq 0$ or S is continuous and $(A_{2k_r}, A) \subset E(G)$ for each $r \geq 0$.

Theorem 3.1. *Suppose $(W(X), H)$ is a complete MS involving a directed graph G and S, R set valued maps on $W(X)$. Suppose that:*

(i) *G is a μ-graph;*

(ii) *there is a sequence $\{A_k\}_{k\in\mathbb{N}}$ in X such that*

$$(A_{2k}, S(A_{2k})) \subset E(G) \Rightarrow (A_{2k+2}, S(A_{2k+2})) \subset E(G)$$

and

$$(A_{2k+1}, R(A_{2k+1})) \subset E(G) \Rightarrow (A_{2k+3}, R(A_{2k+3})) \subset E(G);$$

(iii) *the pair (S, R) is graph A_φ-contraction.*

Then S, R have a CFP. Moreover, if for any two CFPs P_0, Q_0 of S and R respectively, there exists $W \in W(X)$ such that $(P_0, W) \subset E(G)$ and $(W, Q_0) \subset E(G)$, then S, R have a unique CFP.

Proof. Let E_0 be an arbitrary element in $W(X)$. So from assumption $(E_0, S(E_0)) \subset E(G)$ and $(S(E_0), RS(E_0)) \subset E(G)$. These imply that there exists some $x_0 \in E_0$ such that there is an edge between x_0 and some $x_1 \in S(E_0)$.

Let $E_1 = S(E_0)$, then the inclusion $(E_1, R(E_1)) \subset E(G)$ gives the existence of an edge between x_1 and $x_2 \in R(E_1)$.

Next assume that $E_2 = R(E_1)$. Continuing this way, we take $E_1 = S(E_0), E_2 = R(E_1), \ldots, E_{2k+1} = S(E_{2k}), E_{2k+2} = R(E_{2k+1})$, for $k \in \mathbb{N}$. Since $(E_0, S(E_0)) \subset E(G)$ and $(E_1, R(E_1)) \subset E(G)$ for $E_0, E_1 \in W(X)$. Then from the assumption for $E_2, E_3 \in W(X)$, we get $(E_2, S(E_2)) \subset E(G)$ and $(E_3, R(E_3)) \subset E(G)$. Continuing this way, we have $(E_{2k}, S(E_{2k})) \subset E(G)$ and $(E_{2k+1}, R(E_{2k+1})) \subset E(G)$ for each $k \in \mathbb{N}$. Thus $(E_{2k}, E_{2k+1}) \subset E(G)$ and $(E_{2k+1}, E_{2k+2}) \subset E(G)$, for each $k \in \mathbb{N}$.

Now from (iii), we have

$$\begin{aligned} H(E_{2k+1}, E_{2k+2}) &= H(S(E_{2k}), R(E_{2k+1})) \\ &\leq \alpha(H(E_{2k}, E_{2k+1}), H(E_{2k}, S(E_{2k})), H(E_{2k+1}, R(E_{2k+1}))) \\ &= \alpha(H(E_{2k}, E_{2k+1}), H(E_{2k}, E_{2k+1}), H(E_{2k+1}, E_{2k+2})). \end{aligned}$$

From the definition of α,

$$H(E_{2k+1}, E_{2k+2}) \leq \varphi(H(E_{2k}, E_{2k+1})), \text{ for all } k \in \mathbb{N}.$$

Similarly, from (ii) $(E_{2k}, S(E_{2k})) \subset E(G) \Rightarrow (E_{2k+2}, S(E_{2k+2})) \subset E(G)$. i.e., $(E_{2k+2}, E_{2k+3}) \subset E(G)$. Thus by using (iii)

$$\begin{aligned} H(E_{2k+2}, E_{2k+3}) &= H(R(E_{2k+1}), S(E_{2k+2})) \\ &= H(S(E_{2k+2}), R(E_{2k+1})) \\ &\leq \alpha(H(E_{2k+2}, E_{2k+1}), H(E_{2k+2}, S(E_{2k+2}), H(E_{2k+1}, R(E_{2k+1}))) \\ &= \alpha(H(E_{2k+1}, E_{k+2}), H(E_{2k+2}, E_{2k+3}), H(E_{2k+1}, E_{2k+2})). \end{aligned}$$

From the definition of α,

$$H(E_{2k+2}, E_{2k+3}) \leq \varphi(H(E_{2k+1}, E_{2k+2})), \text{ for all } k \in \mathbb{N}.$$

Continuing this way, we get,

$$\begin{aligned} H(E_{2k+2}, E_{2k+3}) &\leq \varphi(H(E_{2k+1}, E_{2k+2})) \\ &\leq \varphi(\varphi(H(E_{2k}, E_{2k+1}))) \\ &= \varphi^2(H(E_{2k}, E_{2k+1})) \\ &. \\ &. \\ &. \\ &\leq \varphi^{2k+2}(H(E_0, E_1)). \end{aligned}$$

Thus

$$H(E_k, E_{k+1}) \leq \varphi^k(H(E_0, E_1)), \text{ for all } k \in \mathbb{N}.$$

Since $H(E_0, E_1) \geq 0$. So, from the Definition 2.3 (ii), we get $\lim_{k\to\infty} \varphi^k(H(E_0, E_1)) = 0$.

Now for any $\varepsilon > 0$, there is a $k_0 \in \mathbb{N}$ such that for each $k \geq k_0$

$$\varphi^k(H(E_0, E_1)) < \varphi - \varphi(\varepsilon).$$

Hence

$$H(E_k, E_{k+1}) < \varphi - \varphi(\varepsilon), \text{ for each } k \geq k_0 \ . \tag{3.1}$$

Also for any positive integer $m, k \in \mathbb{N}$ with $m > k > k_0$, we prove that

$$H(E_k, E_m) < \varepsilon. \tag{3.2}$$

We prove the inequality 3.2 by using mathematical induction on m. The inequality 3.2 holds for $m = k+1$ by using 3.1. Assume that 3.2 is true for $m = l$. i.e., $H(E_k, E_l) < \varepsilon$. So that $m = l + 1$, we have

$$\begin{aligned} H(E_k, E_m) &\leq H(E_k, E_{k+1}) + H(E_{k+1}, E_{l+1}) \\ &< \varepsilon - \varphi(\varepsilon) + H(E_{k+1}, E_{l+1}) \\ &< \varepsilon - \varphi(\varepsilon) + \varphi(H(E_k, E_l)) \\ &< \varepsilon - \varphi(\varepsilon) + \varphi(\varepsilon) \\ &= \varepsilon. \end{aligned}$$

Hence using mathematical induction on m, we see that 3.2 holds for $m > k \geq k_0$. Thus $\{E_k\}$ is a Cauchy sequence in $W(X)$. As $(W(X), H)$ is complete, we get $E_k \to P_0$, for some $P_0 \in W(X)$.

Next, we assert that P_0 is a CFP of S and R. As $E_k \to P_0$ and for $E_{2k} \in W(X)$, we have $(E_{2k}, E_{2k+1}) \subset E(G)$ for each $k \in \mathbb{N}$. Because G is a μ-graph, there exists a subsequence $\{E_{2k_p}\}$ of $\{E_{2k}\}$ such that either R is continuous and $(P_0, E_{2k_p+1}) \subset E(G)$ or S is continuous and $(E_{2k_p}, P_0) \subset E(G)$.

Assume that R is continuous and $(P_0, E_{2k_p+1}) \subset E(G)$. Since every subsequence of a convergent sequence is convergent and has the same limit. Therefore,

$$P_0 = \lim_{p\to\infty} (E_{2k_p+1}) \quad \Rightarrow \quad R(P_0) = \lim_{p\to\infty} R(E_{2k_p+1}) = \lim_{p\to\infty} (E_{2k_p+2}) = P_0.$$

That is $R(P_0) = P_0$. Moreover, from (iii) we get,

$$H(S(P_0), R(E_{2k_p+1})) \leq \alpha(H(P_0, E_{2k_p+1}), H(P_0, S(P_0)), H(E_{2k_p+1}, R(E_{2k_p+1}))).$$

Taking $p \to \infty$, we obtain

$$H(S(P_0), P_0) \leq \alpha(0, H(P_0, S(P_0)), 0) \quad \Rightarrow \quad H(S(P_0), P_0) \leq \varphi(0) = 0.$$

That is $S(P_0) = P_0$. Thus P_0 is a CFP of S and R. Similarly, assume that S is continuous and $(E_{2k_p}, P_0) \subset E(G)$. Thus

$$P_0 = \lim_{p\to\infty} E_{2k_p+2} \quad \Rightarrow \quad S(P_0) = \lim_{p\to\infty} S(E_{2k_p+2}) = \lim_{p\to\infty} E_{2k_p+3} = P_0.$$

That is $S(P_0) = P_0$. Again, from (iii) we have,

$$H(S(E_{2k_p}, R(P_0)) \leq \alpha(H(E_{2k_p}, P_0), H(E_{2k_p}, S(E_{2k_p})), H(P_0, R(P_0))).$$

Taking $p \to \infty$, we obtain

$$H(P_0, R(P_0)) \leq \alpha(0, 0, H(P_0, R(P_0))) \quad \Rightarrow \quad H(P_0, R(P_0)) \leq \varphi(0) = 0.$$

That is $R(P_0) = P_0$. Thus P_0 is a CFP of S and R.

Finally, we prove that P_0 is unique. Suppose V is another fixed point of S and R. Then $(P_0, W) \subset E(G)$ and $(W, Q_0) \subset E(G)$. Being G a directed graph, we get $(P_0, Q_0) \subset E(G)$. Now,

$$\begin{aligned}
H(P_0, Q_0) &= H(S(P_0), R(Q_0)) \\
&\leq \alpha(H(P_0, Q_0), H(P_0, S(P_0)), H(Q_0, R(Q_0))) \\
&\leq \alpha(H(P_0, Q_0), 0, 0) \\
&\leq \varphi(0) \\
&= 0.
\end{aligned}$$

Thus $P_0 = Q_0$. Hence P_0 is the unique CFP of S and R. □

Corollary 3.2. *Let $(W(X), H)$ be a complete MS endowed with a directed graph G and $S, R : W(X) \to W(X)$ be set valued mappings satisfying:*

(i) *G is a μ-graph ;*

(ii) *there is a sequence $\{A_k\}_{k\in N}$ in X such that*

$$(A_{2k}, S(A_{2k})) \subset E(G) \Rightarrow (A_{2k+2}, S(A_{2k+2})) \subset E(G)$$

and

$$(A_{2k+1}, R(A_{2k+1})) \subset E(G) \Rightarrow (A_{2k+3}, R(A_{2k+3})) \subset E(G);$$

(1) *there exists some $\alpha \in A$ such that*

$$H(S(P_0), R(Q_0)) \leq \alpha(H(P_0, Q_0), H(P_0, S(P_0)), H(Q_0, R(Q_0))),$$

for each $(P_0, Q_0) \subset E(G)$;

(2) *X_{SR} is nonempty.*

Then S, R have a CFP.

Following example demonstrates the conditions of Theorem 3.1.

Example 3.1. Suppose $X = \{1, 2, 3, 4\} = V(G)$ and

$$E(G) = \{(1, 3), (1, 4), (3, 2), (2, 4), (3, 3), (4, 3), (4, 4)\}.$$

Assume that $V(G)$ is endowed with metric d which is defined as

$$d(3, 3) = d(4, 4) = 0,$$

$$d(4, 3) = \frac{1}{k+1},$$

$$d(1,3) = d(1,4) = d(2,3) = d(2,4) = \frac{k+1}{k+2}.$$

Define the Hausdorff metric as follows

$$H(P_0, Q_0) = \begin{cases} \frac{1}{k+1}, & \text{if } P_0, Q_0 \subseteq \{3,4\} \text{ with } P_0 \neq Q_0 \\ \frac{k+1}{k+2}, & \text{if } P_0 \text{ or } Q_0 \nsubseteq \{3,4\} \text{ with } P_0 \neq Q_0 \\ 0, & \text{if } P_0 = Q_0. \end{cases}$$

The mappings $S, R : W(X) \to W(X)$ are defined as:

$$S(P_0) = \begin{cases} \{3\}, & \text{if } P_0 \subseteq \{3,4\} \\ \{4\}, & \text{if } P_0 \nsubseteq \{3,4\}. \end{cases}$$

$$R(P_0) = \begin{cases} \{3\}, & \text{if } P_0 \subseteq \{3,4\} \\ \{3,4\}, & \text{if } P_0 \nsubseteq \{3,4\}. \end{cases}$$

Now for each $P_0, Q_0 \in W(X)$, consider the cases given below:

(1) If $P_0, Q_0 \subseteq \{3,4\}$, $H(S(P_0), R(Q_0)) = H(\{3\}, \{3\}) = 0$

(2) If $P_0 \nsubseteq \{3,4\}$ and $Q_0 \subseteq \{3,4\}$, we get

$$H(S(P_0), R(Q_0)) = H(\{4\}, \{3\}) = \frac{1}{k+1}.$$

Since

$$\frac{1}{k+1} \leq \alpha(\frac{k+1}{k+2}, \frac{k+1}{k+2}, \frac{1}{k+1})$$
$$\Rightarrow H(S(P_0), R(Q_0)) \leq \alpha(H(P_0, Q_0), H(P_0, S(P_0)), H(Q_0, R(Q_0))).$$

Hence all the conditions of Theorem 3.1 are satisfied, where $\varphi(t) = \frac{4t}{5}$. Moreover, $\{3\}$ is the unique CFP of S and R.

4. Generalized φ-weak G-contraction

In this section, we establish another CFP theorem by defining generalized φ-weak G contraction.

Definition 4.1. Suppose that $S, R : W(X) \to W(X)$ are two set valued maps. The pair (S, R) is called a generalized ϕ weak G-contraction if the following assertions hold:

(i) for each $P_0 \in W(X)$, $(P_0, S(P_0)) \subset E(G)$ and $(S(P_0), RS(P_0)) \subset E(G)$;

(ii) there is a lower semi continuous function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ with $\phi(t) > 0$ for $t \in (0, \infty)$ and $\phi(0) = 0$ such that for each $(P_0, Q_0) \subset E(G)$

$$H(S(P_0), R(Q_0)) \leq M_{S,R}(P_0, Q_0) - \phi(M_{S,R}(P_0, Q_0)) \tag{4.1}$$

where

$$M_{S,R} = \max\{H(P_0, Q_0), H(P_0, S(P_0)), H(Q_0, R(Q_0)), \frac{H(Q_0, S(P_0)) + H(P_0, R(Q_0))}{2}\}$$

Theorem 4.1. *Suppose $(W(X), H)$ is a complete MS with a directed graph G and S, R set valued maps on $W(X)$. If*

(i) *G is a μ-graph;*

(ii) *the pair (S, R) is generalized ϕ weak G-contraction.*

Then S and R have a CFP.

Proof. Let E_0 be an arbitrary element in $W(X)$. So from assumption $(E_0, S(E_0)) \subset E(G)$ and $(S(E_0), RS(E_0)) \subset E(G)$. These imply that there exists some $x_0 \in E_0$ such that there is an edge between x_0 and some $x_1 \in S(E_0)$.

Let $E_1 = S(E_0)$, then the inclusion $(E_1, R(E_1)) \subset E(G)$ gives the existence of an edge between x_1 and $x_2 \in R(E_1)$.
Let $E_2 = R(E_1)$. Continuing this way, we take $E_1 = S(E_0), E_2 = R(E_1), \ldots, E_{2k+1} = S(E_{2k}), E_{2k+2} = R(E_{2k+1})$, for $k \in \mathbb{N}$. Since $(E_0, S(E_0)) \subset E(G)$ and $(E_1, R(E_1)) \subset E(G)$ for $E_0, E_1 \in W(X)$. Then from the assumption for $E_2, E_3 \in W(X)$, we get $(E_2, S(E_2)) \subset E(G)$ and $(E_3, R(E_3)) \subset E(G)$. Preceding in this way, we get $(E_{2k}, S(E_{2k})) \subset E(G)$ and $(E_{2k+1}, R(E_{2k+1})) \subset E(G)$, for all $k \in \mathbb{N}$.

Thus $(E_{2k}, E_{2k+1}) \subset E(G)$ and $(E_{2k+1}, E_{2k+2}) \subset E(G)$, for all $k \in \mathbb{N}$.

Now from (4.1) we have

$$H(E_{2k+1}, E_{2k+2}) = H(S(E_{2k}), R(E_{2k+1})) \leq M_{S,R}(E_{2k}, E_{2k+1}) - \varphi(M_{S,R}(E_{2k}, E_{2k+1})) \tag{4.2}$$

where

$$\begin{aligned} M_{S,R}(E_{2k}, E_{2k+1}) =& \max\{H(E_{2k}, E_{2k+1}), H(E_{2k}, S(E_{2k})), H(E_{2k+1}, R(E_{2k+1})), \\ & \frac{H(E_{2k+1}, S(E_{2k})) + H(E_{2k}, R(E_{2k+1}))}{2}\} \\ =& \max\{H(E_{2k}, E_{2k+1}), H(E_{2k}, E_{2k+1}), H(E_{2k+1}, E_{2k+2}), \\ & \frac{H(E_{2k+1}, E_{2k+1}) + H(E_{2k}, E_{2k+2})}{2}\} \\ =& \max\{H(E_{2k}, E_{2k+1}), H(E_{2k+1}, E_{2k+2}), \frac{H(E_{2k}, E_{2k+2})}{2}\} \\ \leq& \max\{H(E_{2k}, E_{2k+1}), H(E_{2k+1}, E_{2k+2}), \\ & \frac{H(E_{2k}, E_{2k+1}) + H(E_{2k+1}, E_{2k+2})}{2}\} \\ =& \max\{H(E_{2k}, E_{2k+1}), H(E_{2k+1}, E_{2k+2})\}. \end{aligned}$$

Thus (4.2) becomes

$$\begin{aligned} H(E_{2k+1}, E_{2k+2}) \leq& \max\{H(E_{2k}, E_{2k+1}), H(E_{2k+1}, E_{2k+2})\} \\ & - \varphi[\max\{H(E_{2k}, E_{2k+1}), H(E_{2k+1}, E_{2k+2})\}] \\ =& H(E_{2k}, E_{2k+1}). \end{aligned}$$

That is

$$H(E_{2k+1}, E_{2k+2}) \leq H(E_{2k}, E_{2k+1}).$$

Similarly,

$$\begin{aligned} H(E_{2k+2}, E_{2k+3}) &= H(R(E_{2k+1}), S(E_{2k+2})) \\ &= H(S(E_{2k+2}), R(E_{2k+1})) \\ &\leq M_{S,R}(E_{2k+2}, E_{2k+1}) - \varphi(M_{S,R}(E_{2k+2}, E_{2k+1})) \end{aligned}$$

where

$$\begin{aligned}
M_{S,R}(E_{2k+2},E_{2k+1}) =& \max\{H(E_{2k+2},E_{2k+1}),H(E_{2k+2},S(E_{2k+2})),H(E_{2k+1},R(E_{2k+1})),\\
&\frac{H(E_{2k+1},S(E_{2k+2}))+H(E_{2k+2},R(E_{2k+1}))}{2}\}\\
=&\max\{H(E_{2k+2},E_{2k+1}),H(E_{2k+2},E_{2k+3}),H(E_{2k+1},E_{2k+2}),\\
&\frac{H(E_{2k+1},E_{2k+3})+H(E_{2k+2},E_{2k+2})}{2}\}\\
=&\max\{H(E_{2k+2},E_{2k+1}),H(E_{2k+2},E_{2k+3}),\frac{H(E_{2k+1},E_{2k+3})}{2}\}\\
\leq&\max\{H(E_{2k+2},E_{2k+1}),H(E_{2k+2},E_{2k+3}),\\
&\frac{H(E_{2k+1},E_{2k+2})+H(E_{2k+2},E_{2k+3})}{2}\}\\
&=\max\{H(E_{2k+2},E_{2k+1}),H(E_{2k+2},E_{2k+3})\}.
\end{aligned}$$

Thus from (4.2), we have

$$H(E_{2k+2},E_{2k+3})\leq H(E_{2k+1},E_{2k+2}),\ \text{for all } k\in\mathbb{N}.$$

Hence

$$H(E_n,E_{n+1})\leq H(E_{n-1},E_n),\ \text{for all } k\in\mathbb{N}.$$

Thus $\{H(E_k,E_{k+1})\}$ is a decreasing sequence of non negative real numbers. So it is convergent to some $b\geq 0$. i.e., $\lim\limits_{k\to\infty}H(E_k,E_{k+1})=b$. We claim that $b=0$.

Also, $\lim\limits_{k\to\infty}H(E_k,E_{k+1})=\lim\limits_{k\to\infty}M_{S,R}(E_{k-1},E_k)=b$.

Now, by lower semi continuity of φ, we have

$$\varphi(b)\leq\lim_{k\to\infty}inf\varphi(M_{S,R}(E_{k-1},E_k)).$$

Taking limit as $k\to\infty$ in the following inequality

$$H(E_k,E_{k+1})\leq M_{S,R}(E_{k-1},E_k)-\varphi(M_{S,R}(E_{k-1},E_k))$$

we get

$$b\leq b-\varphi(b)\Rightarrow\varphi(b)\leq 0.$$

Thus $\varphi(b)=0$, by the property of the function φ. Hence $\lim\limits_{k\to\infty}H(E_k,E_{k+1})=b=0$.

Next, we show that $\{E_k\}$ is a Cauchy sequence. If $\{E_k\}$ is not a Cauchy sequence, then there exists $\varepsilon>0$ and subsequences $\{k_r\}$ and $\{m_r\}$ of positive integers such that

$$k_r>m_r>r,H(E_{m_r},E_{k_r-1})<\varepsilon,H(E_{m_r},E_{k_r})\geq\varepsilon,$$

for all $r\in\mathbb{N}$.

Then

$$\varepsilon\leq H(E_{m_r},E_{k_r})\leq H(E_{m_r},E_{k_r-1})+H(E_{k_r-1},E_{k_r}).\tag{4.3}$$

From (4.3) it follows that $H(E_{m_r},E_{k_r})\to\varepsilon^+$ as $k\to\infty$. If we take $E_{2k+1}=E_{m_r},E_{2k+2}=E_{k_r}$ in 4.2, we get the next relation

$$H(E_{m_r},E_{k_r})\leq M_{S,R}(E_{k_r-2},E_{m_r})-\varphi(M_{S,R}(E_{k_r-2},E_{m_r}))\tag{4.4}$$

where

$$\begin{aligned}M_{S,R}(E_{k_r-2},E_{m_r}) =& \max\{H(E_{k_r-2},E_{m_r}),H(E_{k_r-2},S(E_{k_r-2})),H(E_{m_r},R(E_{m_r})),\\&\frac{H(E_{m_r},S(E_{k_r-2}))+H(E_{k_r-2},R(E_{m_r}))}{2}\}\\=&\max\{H(E_{k_r-2},E_{m_r}),H(E_{k_r-2},E_{k_r-1}),H(E_{m_r},E_{m_r+1}),\\&\frac{H(E_{m_r},E_{k_r-1})+H(E_{k_r-2},E_{m_r+1})}{2}\}.\end{aligned}$$

Now we consider the following cases:

If $M_{S,R}(E_{k_r-2},E_{m_r}) = H(E_{k_r-2},E_{m_r})$, then taking limit as $r\to\infty$ in 4.4, we get

$$\varepsilon \le \varepsilon - \varphi(\varepsilon) \Rightarrow \varphi(\varepsilon) = 0.$$

By our assumption about φ, we have $\varepsilon = 0$, which is a contradiction.

When $M_{S,R}(E_{k_r-2},E_{m_r}) = H(E_{k_r-2},E_{k_r-1})$, then taking limit as $r\to\infty$ in 4.4, we get

$$\varepsilon \le 0 - \varphi(0), \quad \text{gives a contradiction.}$$

If $M_{S,R}(E_{k_r-2},E_{m_r}) = H(E_{m_r},E_{m_r+1})$, then taking limit as $r\to\infty$ in 4.4, we get

$$\varepsilon \le 0 - \varphi(0), \quad \text{gives a contradiction.}$$

Finally, if $M_{S,R}(E_{k_r-2},E_{m_r}) = \frac{H(E_{m_r},E_{k_r-1})+H(E_{k_r-2},E_{m_r+1})}{2}$, then taking limit as $r\to\infty$ in 4.4, we have

$$\varepsilon \le 1/2(\varepsilon+\varepsilon) - \varphi(1/2(\varepsilon+\varepsilon)),$$

which is a contradiction. Hence $\{E_k\}$ must be a Cauchy sequence in $W(X)$. As $(W(X),H)$ is complete, we get $E_k \to P$, for some $P \in W(X)$.

Next we prove that P is a CFP of S and R. Since $E_k \to P$ and for $E_{2k} \in W(X)$, we have $(E_{2k},E_{2k+1}) \subset E(G)$, for each $k \in \mathbb{N}$. Because G is a μ- graph, there exists a subsequence $\{E_{2k_p}\}$ of $\{E_{2k}\}$ such that either R is continuous and $(P,E_{2k_p+1}) \subset E(G)$ or S is continuous and $(E_{2k_p},P) \subset E(G)$.

Assume that R is continuous and $(P,E_{2k_p+1}) \subset E(G)$. Since

$$P = \lim_{p\to\infty}(E_{2k_p+1}) \quad\Rightarrow\quad R(P) = \lim_{p\to\infty} R(E_{2k_p+1}) = \lim_{p\to\infty}(E_{2k_p+2}) = P.$$

That is $R(P) = P$. Now from 4.1, we get

$$H(S(P),R(E_{2k_p+1})) \le M_{S,R}(P,E_{2k_p+1}) - \phi(M_{S,R}(P,E_{2k_p+1})) \tag{4.5}$$

where

$$\begin{aligned}M_{S,R}(P,E_{2k_p+1}) =& \max\{H(P,E_{2k_p+1}),H(P,S(P)),H(E_{2k_p+1},R(E_{2k_p+1})),\\&\frac{H(E_{2k_p+1},S(P))+H(P,R(E_{2k_p+1}))}{2}\}\\=&\max\{H(P,E_{2k_p+1}),H(P,S(P)),H(E_{2k_p+1},E_{2k_p+2}),\\&\frac{H(E_{2k_p+1},S(P))+H(P,E_{2k_p+2})}{2}\}.\end{aligned}$$

Taking limit as $p\to\infty$ in 4.5 and apply the same procedure as in 4.4, we get $H(S(P),P) = 0$. Thus $P = S(P)$. Hence P is a CFP of S and R.

Similarly, assume that S is continuous and $(E_{2k_p},P) \subset E(G)$. Using the same procedure we get P is a CFP of S and R. □

Corollary 4.2. *Suppose $(W(X), H)$ is a complete MS involving a directed graph G and R a set valued map on $W(X)$. If*
(i) *G is a μ-graph;*
(ii) *R is generalized φ- weak G- contraction.*
Then R has a fixed point.

5. Conclusion

We put forward the notions of graph A_φ-contraction pair and generalized φ-weak G contraction on bounded and closed subsets of a metric space and established some CFP results. We assumed certain conditions such as the underlying graph G is a μ-graph. Obtaining the results by relaxing that condition is a suggested future work.

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