Annals of the University of Craiova, Mathematics and Computer Science Series
Volume 45(2), 2018, Pages 275–282
ISSN: 1223-6934

From rotations of conics to a class of Riemann-Finslerian flows

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Dedicated to Academician Radu Miron on the occasion of his 91'th birthday

Abstract. The aim of this paper is to produce new examples of (semi-) Riemannian and Finsler structures in dimension two having as model a scalar deformation of conics which generalizes the rotation with a right angle. It continues [6] and [8] from the point of view of relationship between quadratic polynomials (which provide equations of conics in dimension 2) and Finsler geometries. A type of two-dimensional Finslerian flow is introduced, based on the previous deformation and we completely solve the corresponding particular case of Riemannian flow.

2010 Mathematics Subject Classification. 53C60, 51M04.
Key words and phrases. conic, invariant, Finsler fundamental function, deformation, Riemann-Finslerian flow.

1. Introduction

Two recent papers [6] and [8], devoted to Finsler geometry, start with a deformation of a conic Γ obtained by deforming the gradient vector field for the quadratic form defining Γ. These deformation are inspired by the scaling (linear) transformation of Computer Graphics: $(x, y) \in \mathbb{R}^2 \to (\lambda_x \cdot x, \lambda_y \cdot y) \in \mathbb{R}^2$, following [13, p. 136]. The well-known invariants from the Euclidean geometry of conics are computed for these new conics which depend on two scalars denoted α and β.

In this following note we present another type of deformation based on the well-known rotation of the plane. More precisely, we consider the consider the linear transformation $(x, y) \to (-\alpha y, \beta x)$, which for $\alpha = \beta = 1$ is the trigonometric rotation with the right angles. We call (α, β)-*rotated* the new conic and the diagonal case $\alpha = \beta$ is particularly analyzed, with a special view towards the trigonometric case $\alpha = \beta = 1$. Moreover, we treat this deformation in terms of complex numbers.

In the next section we move to the Riemann-Finslerian framework of dimension two and consider the deformation inspired by the previous section. We finish this paper with a type of Finslerian flows which can be the starting point of future studies following the way opened by the famous Ricci flow of Riemannian geometry, [4]. Due to the complex form of Finslerian deformation even in the Randers case, we can solve completely only the corresponding particular case of Riemannian flows. The solution is a time-dependent metric and a case of decreasing area is pointed out. We remark that in dimension four some recent bi-metric approaches of spacetime geometries appear in [1]-[2] and [3] while a geometrical study in arbitrary dimension is the very old paper [10].

Received August 24, 2018. Accepted December 10, 2018.

2. The generalized rotation of conics

In the two-dimensional Euclidean space $\mathbb{R}^2$ let us consider the conic Γ implicitly defined by $f \in C^\infty\left(\mathbb{R}^2\right)$ as: $\Gamma = \{(x,y) \in \mathbb{R}^2 \mid f(x,y) = 0\}$ where f is a quadratic function of the form $f(x,y) = r_{11}x^2 + 2r_{12}xy + r_{22}y^2 + 2r_{10}x + 2r_{20}y + r_{00}$ with $r_{11}^2 + r_{12}^2 + r_{22}^2 \neq 0$.

Definition 2.1. Fix the scalars α, β with $\alpha\beta \neq 0$. The (α,β)-*rotation* of Γ is the conic:

$$\begin{cases} \Gamma^r = \Gamma^r_{\alpha,\beta} : f^r(x,y) := f(-\alpha y, \beta x) = 0, \\ f^r(x,y) = (\beta^2 r_{22})x^2 + 2(-\alpha\beta r_{12})xy + (\alpha^2 r_{11})y^2 + 2(\beta r_{20})x + 2(-\alpha r_{10})y + r_{00}. \end{cases} \tag{1.1}$$

Example 2.1. i) Fix other non-vanishing scalars a, b. The ellipse $E(a,b) : \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$ and the hyperbola $H(a,b) : \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0$ have the following (α,β)-rotation:

$$E^r_{\alpha,\beta} : \frac{\beta^2 x^2}{b^2} + \frac{\alpha^2 y^2}{a^2} - 1 = 0, \quad H^r_{\alpha,\beta} : \frac{\beta^2 x^2}{b^2} - \frac{\alpha^2 y^2}{a^2} + 1 = 0. \tag{1.2}$$

Hence E^r is also an ellipse and H^r is a hyperbola. The equilateral hyperbola $\Gamma : xy = C = constant$ has the (α,β)-rotation:

$$\Gamma^r : \alpha\beta xy = -C \tag{1.3}$$

which is also an equilateral hyperbola.
ii) For $p > 0$ let the parabola $P(p) : y^2 - 2px = 0$. Its (α,β)-rotation is:

$$P^r_{\alpha,\beta} : x^2 + 2\frac{\alpha p}{\beta^2} y = 0 \tag{1.4}$$

which is also a parabola.
iii) Consider again the ellipse $E(a,b)$ with $a > b > 0$. The family of all *confocal* conics with $E(a,b)$ is given by:

$$\Gamma_\lambda : \frac{x^2}{a-\lambda} + \frac{y^2}{b-\lambda} - 1 = 0 \tag{1.5}$$

for $\lambda \in \mathbb{R} \setminus \{a,b\}$. The (α,β)-rotation of Γ_λ is:

$$(\Gamma_\lambda)^r_{\alpha,\beta} : \frac{\beta^2 x^2}{b-\lambda} + \frac{\alpha^2 y^2}{a-\lambda} - 1 = 0. \tag{1.6}$$

In order to study the (α,β)-rotations we recall the algebraic invariants associated to Γ:

$$\Delta = \begin{vmatrix} r_{11} & r_{12} & r_{10} \\ r_{12} & r_{22} & r_{20} \\ r_{10} & r_{20} & r_{00} \end{vmatrix}, \quad D = \delta + I r_{00} - r_{10}^2 - r_{20}^2, \quad I = r_{11} + r_{22}, \quad \delta = r_{11}r_{22} - r_{12}^2. \tag{1.7}$$

More precisely, the main result of this Section follows directly:

Theorem 2.1. *The new conic $\Gamma^r_{\alpha,\beta}$ has the following invariants:*

$$I^r = \alpha^2 r_{11} + \beta^2 r_{22}, \delta^r = (\alpha\beta)^2\delta, D^r = (\alpha\beta)^2\delta + \alpha^2(r_{11}r_{00} - r_{10}^2) + \beta^2(r_{22}r_{00} - r_{20}^2),$$
$$\Delta^r = (\alpha\beta)^2\Delta. \tag{1.8}$$

Then the initial conic Γ and Γ^r have the same nature.

A special attention deserves the diagonal case $\alpha = \beta$ for which we have:

$$I^r = \alpha^2 I, \quad \delta^r = \alpha^4 \delta, \quad D^r = (\alpha^4 - \alpha^2)\delta + \alpha^2 D, \quad \Delta^r = \alpha^4 \Delta. \tag{1.9}$$

By performing a second rotation for this last case we obtain:

$$\left(\Gamma^r_{\alpha,\alpha}\right)^r_{\alpha,\alpha} : \alpha^4(r_{11}x^2 + 2r_{12}xy + r_{22}y^2) - \alpha^2(2r_{10}x + 2r_{20}y) + r_{00} = 0. \tag{1.10}$$

The trigonometric case $\alpha = \beta = 1$ gives for the last two equations:

$$I^r = I, \quad \delta^r = \delta, \quad D^r = D, \quad \Delta^r = \Delta,$$

$$\left(\Gamma^r_{1,1}\right)^r_{1,1} : r_{11}x^2 + 2r_{12}xy + r_{22}y^2 - 2r_{10}x - 2r_{20}y + r_{00} = 0. \tag{1.11}$$

Returning to the general case of α and β we treat the mixed deformation with complex numbers following the model of [7]; a classification of conics written in the complex plane appears in [9, p. 640]. More precisely, with the usual notation $z = x + iy \in \mathbb{C}$ we derive the complex expression of Γ:

$$\Gamma : F(z, \bar{z}) := Az^2 + Bz\bar{z} + \bar{A}\bar{z}^2 + Cz + \bar{C}\bar{z} + r_{00} = 0 \tag{1.12}$$

with:

$$A = \frac{r_{11} - r_{22}}{4} - \frac{r_{12}}{2} i \in \mathbb{C}, \quad 2B = r_{11} + r_{22} = I \in \mathbb{R}, \quad C = r_{10} - r_{20} i \in \mathbb{C}. \tag{1.13}$$

It follows that the usual rotation performed with the angle φ to eliminate the mixed term xy has the meaning to reduce/rotate A in the real line while the translation which eliminates the term y has a similar meaning with respect to C. The inverse relationship between f and F is:

$$r_{11} = B + 2\Re A, \quad r_{22} = B - 2\Re A, \quad r_{12} = -2\Im A, \quad r_{10} = \Re C, \quad r_{20} = -\Im C \tag{1.14}$$

with $\Re$ and $\Im$ respectively the real and imaginary part. Hence the angle φ is provided by the formula:

$$\tan 2\varphi := \frac{2r_{12}}{r_{11} - r_{22}} = -\frac{\Im A}{\Re A} = -\tan \arg A \to 2\varphi = -\arg A. \tag{1.15}$$

The expression of the invariants of Γ in terms of A, B, C is:

$$I = 2B, \quad \delta = B^2 - 4|A|^2, \quad D = \delta + 2r_{00}I - |C|^2 \tag{1.16$_1$}$$

$$\Delta = r_{00}(B^2 - 4|A|^2) - B|C|^2 + 2\Re C(\Re A \Re C + \Im A \Im C) + 2\Im C(\Re C \Im A - \Re A \Im C). \tag{1.16$_2$}$$

The transformation of the complex coefficients under the (α, β)-rotation is:

$$A^r = \frac{\beta^2 - \alpha^2}{4} B - \frac{\alpha^2 + \beta^2}{2} \Re A - \alpha\beta \Im A i, B^r = \frac{\alpha^2 + \beta^2}{2} B + (\alpha^2 - \beta^2)\Re A,$$

$$\tilde{C} = -\beta \Im C + \alpha \Re C i. \tag{1.17}$$

For the considered particular case $\alpha = \beta$ we obtain:

$$A^r = -\alpha^2 A, \quad B^r = \alpha^2 B, \quad \tilde{C} = -\alpha C i \tag{1.18}$$

while the trigonometric case $\alpha = \beta = 1$ yields:

$$A^r = -A, \quad B^r = B, \quad \tilde{C} = -Ci \tag{1.19}$$

Returning to the general complex formalism above, in the case of a non-degenerate Γ, which means $\Delta \neq 0$, we can also express *the eccentricity* e by:

$$e^2 := 2 - \frac{I}{\lambda} = 1 - \frac{\delta}{\lambda^2}, \quad \lambda^2 - I\lambda + \delta = 0. \tag{1.20}$$

It follows that μ and e are provided by:

$$\lambda_{\pm} := B \pm 2|A| \to e^2 = \frac{\pm 4|A|}{B \pm 2|A|} \tag{1.21}$$

and hence the eccentricity is preserved by a diagonal rotation $\alpha = \beta$ since we use (1.18).

We finish this section by discussing the commutation of a rotation with the previous two gradient deformations of conics:
1) the (α, β)-*deformation* of Γ is the conic, [6, p. 87]:

$$\tilde{\Gamma} = \Gamma_{\alpha,\beta} : \alpha \left[\frac{1}{2} f_x\right]^2 + \beta \left[\frac{1}{2} f_y\right]^2 = 0. \tag{1.22}$$

2) the (α, β)-*mixed deformation* of Γ is the conic, [8]:

$$\Gamma^m = \Gamma^m_{\alpha,\beta} : f^m(x, y) := g_{\alpha,\beta}(In(x, y), \frac{1}{2}\nabla f(x, y)) = \alpha y \left[\frac{1}{2} f_x\right] + \beta x \left[\frac{1}{2} f_y\right] = 0. \tag{1.23}$$

A straightforward computation gives:
1+rotation)

$$(\tilde{\Gamma})^r : \beta^2(\alpha r_{12}^2 + \beta r_{22}^2)x^2 - 2\alpha\beta r_{12}(\alpha r_{11} + \beta r_{22})xy + \alpha^2(\alpha r_{11}^2 + \beta r_{12}^2)y^2 + 2\beta(\alpha r_{12}r_{10} + \beta r_{22}r_{20})x - 2\alpha(\alpha r_{11}r_{10} + \beta r_{12}r_{20})y + \alpha r_{10}^2 + \beta r_{20}^2 = 0, \tag{1.24}$$

$$\widetilde{\Gamma^r} : \beta^2(\alpha r_{12}^2 + \beta r_{22}^2)x^2 - 2\alpha\beta r_{12}(\alpha r_{11} + \beta r_{22})xy + \alpha^2(\alpha r_{11}^2 + \beta r_{12}^2)y^2 + 2\beta(\alpha r_{12}r_{10} + \beta r_{22}r_{20})x - 2\alpha(\alpha r_{11}r_{10} + \beta r_{12}r_{20})y = 0, \tag{1.25}$$

and then $(\tilde{\Gamma})^r = \widetilde{\Gamma^r}$ if and only if $\alpha r_{10}^2 + \beta r_{20}^2 = 0$ which for the diagonal case means $r_{10} = r_{20} = 0$. Hence, in this diagonal case the new conic is:

$$(\tilde{\Gamma})^r = \widetilde{\Gamma^r} : (r_{12}^2 + r_{22}^2)x^2 - 2Ir_{12}xy + (r_{11}^2 + r_{12}^2)y^2 = 0. \tag{1.26}$$

2+rotation)

$$(\Gamma^m)^r : \beta r_{12}x^2 - 2(\alpha r_{11} + \beta r_{22})xy + \alpha r_{12}y^2 - 2r_{10}x - 2r_{20}y = 0, \tag{1.27}$$

$$(\Gamma^r)^m : \beta r_{12}x^2 - (\alpha r_{11} + \beta r_{22})xy + \alpha r_{12}y^2 + r_{10}x - r_{20}y = 0, \tag{1.28}$$

and then $(\Gamma^m)^r = (\Gamma^r)^m$ if and only if $r_{10} = r_{20} = \alpha r_{11} + \beta r_{22} = 0$ which means that the new conic is:

$$(\Gamma^m)^r = (\Gamma^r)^m : r_{12}(\beta x^2 + \alpha y^2) = 0. \tag{1.29}$$

Let us remark that the origin belongs to both conics (1.26) and (1.29).

For the completeness of the subject we include here the iterations of gradient transformations 1 and 2:

$$\widetilde{\Gamma^m} : [r_{12}^2 + \frac{(\alpha r_{11} + \beta r_{22})^2}{\alpha\beta}](\beta x^2 + \alpha y^2) + 4r_{12}(\alpha r_{11} + \beta r_{22})xy + 2[\beta r_{12}r_{20} + (\alpha r_{11} + \beta r_{22})r_{10}]x + 2[(\alpha r_{11} + \beta r_{22})r_{20} + \alpha r_{12}r_{10}]y + \alpha r_{10}^2 + \beta r_{20}^2 = 0, \tag{1.30}$$

$$\tilde{\Gamma}^m : (\alpha r_{11} + \beta r_{12})r_{12}(\beta x^2 + \alpha y^2) + (\alpha^2 r_{11}^2 + 2\alpha\beta r_{12}^2 + \beta r_{22}^2)xy + \beta(\alpha r_{12}r_{10} + \beta r_{22}r_{20})x + \alpha(\alpha r_{11}r_{10} + \beta r_{12}r_{20})y = 0. \tag{1.31}$$

In particular, the diagonal condition $\alpha = \beta$ gives:

$$\widetilde{\Gamma^m} : (r_{12}^2 + I^2)(x^2 + y^2) + 4r_{12}Ixy + 2(r_{12}r_{20} + r_{10}I)x + 2(r_{20}I + r_{12}r_{10})y + r_{10}^2 + r_{20}^2 = 0, \tag{1.32}$$

$$\tilde{\Gamma}^m : r_{12}I(x^2+y^2)+(r_{11}^2+2r_{12}^2+r_{22}^2)xy+(r_{12}r_{10}+r_{22}r_{20})x+(r_{11}r_{10}+r_{12}r_{20})y=0. \tag{1.33}$$

3. The rotation of two-dimensional Finsler structures

Let M be an open subset of $\mathbb{R}^m$ considered as a smooth m-dimensional manifold with $m \geq 2$ and $\pi : TM \to M$ its tangent bundle. Let $x = (x^i) = (x^1, ..., x^m)$ be the coordinates on M and $(x, y) = (x^i, y^i) = (x^1, ..., x^m, y^1, ..., y^m)$ the induced coordinates on TM. Denote by O the null-section of π.

Recall after [12] that a *Finsler fundamental function* on M is a map $F : TM \to \mathbb{R}_+$ with the following properties:
F1) F is smooth on the slit tangent bundle $T_0M := TM \setminus O$ and continuous on O,
F2) F is positive homogeneous of degree 1: $F(x, \lambda y) = \lambda F(x, y)$ for every $\lambda > 0$,
F3) the matrix $(g_{ij}) = \left(\frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}\right)$ is invertible and its associated quadratic form is of constant rank.
The tensor field $g = \{g_{ij}(x, y); 1 \leq i, j \leq m\}$ is called *the Finsler metric* and the homogeneity of F implies:

$$F^2(x, y) = g_{ij}y^iy^j = y_iy^i \tag{2.1}$$

where $y_i = g_{ij}y^j$. The pair (M, F) is called *Finsler manifold.* We point out the possibility of singular Finsler metrics as in [11].

Fix now the dimension $m = 2$ for which we change the notation: $(x^1, x^2) \to (x, y)$, $(y^1, y^2) \to (\dot{x}, \dot{y})$. Fix also the vector $\bar{\alpha} = (\alpha, \beta) \in \mathbb{R}^2_{+,+}$ with all strictly positive components although there are cases when some of them can be null or even negative. Inspired by the previous Section we introduce:

Definition 3.1. The *$\bar{\alpha}$-rotation* of F is $F^r = F^r_{\bar{\alpha}} : TM \to \mathbb{R}$ given by:

$$F^r(x, y, \dot{x}, \dot{y}) = F(x, y, -\alpha\dot{y}, \beta\dot{x}). \tag{2.2}$$

From (2.1) due to homogeneity it results a basic equation of Finsler geometry:

$$\frac{1}{2}(F^2)_{y^i} = g_{ij}y^j \tag{2.3}$$

This new Fislerian fundamental function yields a new Finslerian metric $g^r = g^{\bar{\alpha}}$ which we call *the $\bar{\alpha}$-rotation* of g. A straightforward computation yields:

$$g^r_{11} = \beta^2 g_{22}, \quad g^r_{22} = \alpha^2 g_{11}, \quad g^r_{12} = -\alpha\beta g_{12}. \tag{2.4}$$

Example 3.1. (Euclidean geometry) The Euclidean metric g_{can} is transformed into the Riemannian metric: $g^r_{can} = diag(\beta^2, \alpha^2)$. Applying a second rotation we get $(g^r_{can})^r = \alpha^2\beta^2 g_{can}$ which is a homothetical transformation. Hence, if $\alpha\beta = 1$ we get an involution on the positive cone of conformal Euclidean metrics $ConfEuclidean = \{\lambda g_{can}; \lambda \in (0, +\infty)\}$.

Example 3.2. (Randers geometry) Let F be a Randers fundamental function of Minkowski type:

$$F_b(x, y, \dot{x}, \dot{y}) = F_b(\dot{x}, \dot{y}) = \sqrt{\dot{x}^2 + \dot{y}^2} + b\dot{x} \tag{2.5}$$

with $0 < b < 1$. The corresponding Finsler metric is:

$$g^b_{11} = 1 + b^2 + b\frac{2\dot{x}^3 + 3\dot{x}\dot{y}^2}{(\dot{x}^2 + \dot{y}^2)^{\frac{3}{2}}}, \quad g^b_{12} = \frac{b\dot{y}^3}{(\dot{x}^2 + \dot{y}^2)^{\frac{3}{2}}}, \quad g^b_{22} = 1 + \frac{b\dot{x}^3}{(\dot{x}^2 + \dot{y}^2)^{\frac{3}{2}}}. \tag{2.6}$$

The new Finslerian metric with $\alpha = \beta = 1$ is:

$$g^r_{11} = g^b_{22}, \quad g^r_{12} = -g^b_{12}, \quad g^r_{22} = g^b_{11}. \tag{2.7}$$

This proves that the new Finslerian structure F^r defines a completely new Finsler geometry on M.

Example 3.3. (Spherically symmetric Finsler functions) Let $I \subseteq \mathbb{R}_+$ be an interval and $A, B : I \to \mathbb{R}$ two smooth functions. We define the orthogonally invariant Finsler function:

$$F(x, y, \dot{x}, \dot{y}) = \sqrt{A(x^2 + y^2)(\dot{x}^2 + \dot{y}^2) + B(x^2 + y^2) < (x, y), (\dot{x}, \dot{y}) >^2_{can}}. \tag{2.8}$$

Its Finsler metric is a non-diagonal Riemannian one:

$$g_{11} = A + Bx^2, \quad g_{12} = Bxy, \quad g_{22} = A + By^2. \tag{2.9}$$

The new Finslerian fundamental function is:

$$F^m(x, y, \dot{x}, \dot{y}) = \sqrt{A(\beta^2\dot{x}^2 + \alpha^2\dot{y}^2) + B(-2\alpha\beta xy\dot{x}\dot{y} + \beta^2y^2\dot{x}^2 + \alpha^2x^2\dot{y}^2)} \tag{2.10}$$

and hence the new Finslerian metric is again a non-diagonal Riemannian metric:

$$g^m_{11} = \beta^2(A + By^2), \quad g^m_{12} = -\alpha\beta Bxy, \quad g^m_{22} = \alpha^2(A + Bx^2). \tag{2.11}$$

4. Finslerian flows

For the given manifold M let $Finsler(M \times \mathbb{R})$ be the infinite space of all possible time-dependent Finslerian metrics on M as well as $T^s_2(TM \times \mathbb{R})$ the space of all time-dependent symmetric tensor fields of $(0, 2)$-type on TM. Following the theory of geometric (more precisely Riemannian) flows we introduce:

Definition 4.1. A *Finslerian flow* on M is a dynamical system modeled by the partial differential equations:

$$\partial_t g_t = \mathcal{F}(g_t) \tag{3.1}$$

where $\mathcal{F} : Finsler(M \times \mathbb{R}) \to T^s_2(TM \times \mathbb{R})$ is a given map and g_t is a family of Finslerian metrics depending on the parameter t belonging to the interval $I \subseteq \mathbb{R}$.

Example 4.1. i) (Special Riemannian flows) If we restrict the functional $\mathcal{F}$ to $Riemann(M \times \mathbb{R})$ to be the (-2)Ricci curvature then we obtain the famous Ricci flow provided the proof of two outstanding conjectures: Poincaré Conjecture and Thurston Geometrization Conjecture. For a relationship between Randers metrics and Ricci solitons via the Zermelo navigation problem see [5].
ii) Other famous Riemannian flows are: the Calabi flow and the Yamabe flow.
iii) Time-dependent Randers metrics are recently used in the study of causal relationships on space-time manifolds in [14].

Returning to the general Finslerian framework and vector $\bar{\alpha}$ of previous Section we consider:

Definition 4.2. The *Finslerian $\bar{\alpha}$-rotation flow* is that given by:

$$\mathcal{F}(g) = g^r = g^{\bar{\alpha}}. \tag{3.2}$$

Inspired by [6, p. 96] we introduce the corresponding *aria variation* as the smooth function
$A : TM \times \mathbb{R} \to TM \times \mathbb{R}$ given by:

$$\partial_t A(x, y, \dot{x}, \dot{y}, t) = \sum_{i,j=1}^{2} g^r_{ij} g^{ij} \tag{3.3}$$

where, as usual, g^{ij} are the components of inverse g^{-1}.

Example 4.2. (Riemannian $\bar{\alpha}$-flow) With the computations of (2.4) we have:

$$\partial_t a_{11} = \beta^2 a_{22}, \quad \partial_t a_{12} = -\alpha\beta a_{12}, \quad \partial_t a_{22} = \alpha^2 a_{11}. \tag{3.4}$$

Then $a_{12}(t) = e^{-\alpha\beta t}$ on $I = \mathbb{R}$, $a_{11}(t) = u(x, y)\cosh(\alpha\beta t) + v(x, y)\sinh(\alpha\beta t)$ and $a_{22}(t) = \frac{\alpha}{\beta}[v(x, y)\cosh(\alpha\beta t) + u(x, y)\sinh(\alpha\beta t)$.

We have immediately that:

$$\partial_t A(x, y, \dot{x}, \dot{y}, t) = \alpha^2 a^2_{11} + 2\alpha\beta a^2_{12} + \beta^2 a^2_{22} \tag{3.5}$$

and hence we have:

$$\partial_t A = \alpha^2[(u^2 + v^2)\cosh(2\alpha\beta t) + uv\sinh(2\alpha\beta t)] + 2\alpha\beta e^{-2\alpha\beta t}. \tag{3.6}$$

It results:

$$A = \frac{\alpha}{2\beta}[(u^2 + v^2)\sinh(2\alpha\beta t) + uv\cosh(2\alpha\beta t)] - e^{-2\alpha\beta t}. \tag{3.7}$$

For $\alpha\beta < 0$ and $uv > 0$ it results a negative A which means an area-decreasing flow.

We finish with the following remark: in the reference [15], from two Finsler functions F_+, F_-, it is obtained a *bi-metric*:

$$F = \sqrt{F_+ \cdot F_-}. \tag{3.8}$$

The negative result of [15] concerning the physical implications of this metric as well as the considerations of our Section 1 suggests other two deformations:

$$F_{2,\alpha,\beta} = \sqrt{\alpha F^2_+ + \beta F^2_-}, \quad F_{m,\alpha,\beta} = \sqrt[m]{\alpha F^m_+ + \beta F^m_-}, \quad m \in \mathbb{N}^* \tag{3.9}$$

which will be studied in a future work.

Acknowledgements

The author has been supported by COST Action CA15117 (CANTATA).

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