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Infinitely many weak solutions for a fourth-order Kirchhoff type elliptic equation

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Abstract. In this article, by using critical point theory, we prove the existence of infinitely many weak solutions for a fourth-order Kirchhoff type elliptic equation involving multi-singular inverse square potentials. Precisely this work is devoted to consider a fourth order elliptic equation involving multi-singular inverse square potentials on the smooth bounded domain $\Omega \subset \mathbb{R}^N (N \geq 5)$

$$\begin{cases} \Delta^2 u - M\left(\int_\Omega \left(|\nabla\ u|^2 - \sum_{i=1}^{k} \frac{\mu_i}{|x-a_i|^2}|u|^2\right) dx\right) \left(\Delta\ u - \sum_{i=1}^{k} \frac{\mu_i}{|x-a_i|^2} u\right) \\ \quad = \lambda f(x,u) + u|u|^{2^{**}-2} & \text{for } x \in \Omega \backslash \{a_1, ..., a_k\} \\ u = \Delta u = 0 & \text{for } x \in \partial\Omega, \end{cases}$$

where Δ^2 is the biharmonic operator and 2^{**} is the critical Sobolev exponent, $a_i \in \Omega, i = 1, 2, ..., k$, for $k \geq 1$ are different points, $0 \leq \mu_i \in \mathbb{R}$ and $\sum_{i=1}^{k} \mu_i < \bar{\mu} := \left(\frac{N-2}{2}\right)^2$ where $\bar{\mu}$ is the best constant in the Hardy inequality, $f : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is L^1-Carathéodory function and $M \in C^1([0, +\infty[, \mathbb{R})$.

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1. Introduction

In this work we consider a fourth order elliptic equation involving multi-singular inverse square potentials on the smooth bounded domain $\Omega \subset \mathbb{R}^N (N \geq 5)$

$$\begin{aligned} \Delta^2 u \quad & -M\left(\int_\Omega \left(|\nabla\ u|^2 - \sum_{i=1}^{k} \frac{\mu_i}{|x-a_i|^2}|u|^2\right) dx\right) \left(\Delta\ u - \sum_{i=1}^{k} \frac{\mu_i}{|x-a_i|^2} u\right) \\ & = \lambda f(x,u) + u|u|^{2^{**}-2}, \end{aligned} \tag{1.1}$$

for $x \in \Omega \backslash \{a_1, ..., a_k\}$, coupled with boundary condition

$$u(x) = \Delta u(x) = 0 \text{ for } x \in \partial\Omega,$$

where Δ^2 is the biharmonic operator; that is,

$$\Delta^2 u = \sum_{i=1}^{N} \frac{\partial^4}{\partial x_i^4} u + \sum_{i \neq j}^{N} \frac{\partial^4}{\partial x_i^2 x_j^2} u,$$

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and $2^{**} := \frac{2N}{N-4}$ is the critical Sobolev exponent, $a_i \in \Omega, i = 1, 2, ..., k$, for $k \geq 1$ are different points, $0 \leq \mu_i \in \mathbb{R}$ and $\sum_{i=1}^{k} \mu_i < \bar{\mu} := \left(\frac{N-2}{2}\right)^2$ where $\bar{\mu}$ is the best constant in the Hardy inequality, $f : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is L^1-Carathéodory function and $M \in C^1([0, +\infty[, \mathbb{R})$.

In 1883 Kirchhoff [24] extended the classical D'Alembert's wave equation first by introducing the equation

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L |\frac{\partial u}{\partial x}| dx\right)\frac{\partial^2 u}{\partial x^2} = 0.$$

After that, many authors studied the following nonlocal elliptic boundary value problem

$$\begin{cases} -M\left(\int_\Omega |\nabla u|^2 dx\right)\Delta u(x) = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

which is called the Kirchhoff type problem. Some interesting results for the problems of Kirchhoff type can be found in the papers [2, 13, 19, 20, 25, 26, 28, 30]. Along with it, the problem (1.1) is a modified version of time-independent Schrödinger-Poisson equation appearing in an interesting physical context. According to a classical model, the interaction of a change particle with an electro-magnetic field can be described by coupling the nonlinear Schrödinger-Poisson equations. (We refer the readers to [4].)

In recent years, there has been a growing interest in the study of Schrödinger-Poisson systems. For example the authors in [27] consider a modified version of the Schrödinger-Poisson systems in $\mathbb{R}^3$, which describes the interaction of a charge particle with an electro-magnetic field. Particularly in 2014 using variational methods and critical points theory, Ferrara, Khademloo and Heidarkhani [11] established the multiplicity results of nontrivial and nonnegative solutions for the following perturbed fourth-order Kirchhoff type elliptic problem

$$\begin{cases} \Delta_p^2 u - \left[M\left(\int_\Omega |\nabla u|^p\right)\right]^{p-1}\Delta_p u + \rho|u|^{p-2}u = \lambda f(x, u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $p > \max\left\{1, \frac{N}{2}\right\}, \Delta_p^2 u = \Delta\left(|\Delta u|^{p-2}\Delta u\right)$ is an operator of fourth order, the so-called p-biharmonic operator. $\lambda > 0$ is a real number, $\Omega \in \mathbb{R}^N$ $(N \geq 1)$ is a bounded smooth domain, $\rho > 0$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is an L^1- Carathéodory function.

As far as we know, the literature does not contain any result on the existence of nontrivial weak solutions to the problems involving singular points in domain.

The problem of finding such types of solutions is a very classical problem. It has been studied in [15, 30] for example. Ferrara and Molica Bisci [12] studied the existence of solutions for the elliptic problem with Hardy potential

$$\begin{cases} \Delta_p u = \mu\frac{|u|^{p-2}u}{|x|^p} + \lambda f(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Huang and Liu [22] studied the sign-changing solutions for p-biharmonic equations with Hardy potential

$$\begin{cases} \Delta_p^2 u - \dfrac{a}{|x|^{2p}}|u|^{p-2}u = f(x,u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega. \end{cases}$$

Xu and Bai in [30] studied the existence of infinitely many weak solutions for a fourth-order Kirchhoff type elliptic problems with Hardy potential

$$\begin{cases} M\left(\int_\Omega |\Delta u|^p dx\right)\Delta_p^2 u - \dfrac{a}{|x|^{2p}}|u|^{p-2}u = \lambda f(x,u) + \mu g(x,u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega. \end{cases}$$

Many authors have considered problems involving multi-singular inverse square potentials. They have discussed on existence, multiplicity or behavior of the solutions. For example see [8, 21, 29].

In this article a fourth-order Kirchhoff type elliptic equation is investigated which involves multi-singular inverse square potential. Motivated by this large interest presented, for example in [22, 30], we establish the existence of an interval Λ such that, for each $\lambda \in \Lambda$ the problem (1.1) admits a sequence of pairwise distinct solutions. The plan of the paper is as follows: In section 2, some necessary preliminary facts and the main result are presented. In section 3, we give the main existence results.

2. Preliminaries and main results

Before giving preliminaries, we introduce some notations and assumptions.

Definition 2.1. A function $f : \Omega\times\mathbb{R} \to \mathbb{R}$ is said to be an L^1-Carathéodory function if it satisfies
(i) for each $x \in \Omega$ the function $f(x,.) : \mathbb{R} \to \mathbb{R}$ is continuous;
(ii) for each $u \in \mathbb{R}$ the function $f(.,u) : \Omega \to \mathbb{R}$ is measurable;
(iii) for every positive integer k there exists $h_k \in L^1(\Omega, \mathbb{R}_+)$ such that

$$|f(x,u)| \le h_k(u) \quad \text{for all } |u| \le k \text{ and almost every } x \in \Omega.$$

$\mathcal{H}_1$) $M : [0,+\infty[\to \mathbb{R}$ is a continuous function such that there are two positive constants m_0, m_1 such that

$$0 < m_0 \le 1 \le M(t) \le m_1, \quad \text{for all } t \ge 0, \tag{2.1}$$

and

$$\tilde{M}(t) \ge M(t)t, \tag{2.2}$$

where

$$\tilde{M}(t) = \int_0^t M(s)ds \quad \text{for } t \ge 0.$$

$\mathcal{H}_2$) $0 \le \mu_1 \le \mu_2 \le ... \le \mu_k < \bar{\mu}$ and $\sum_{i=1}^k \mu_i < \bar{\mu}$.
$\mathcal{H}_3$) $f : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is an L^1-Carathéodory function and $F(x,t) \ge 0$ for every $(x,t) \in \Omega \times [0,+\infty[$, where

$$F(x,t) = \int_0^t f(x,s)\,ds, \quad (x,t) \in \Omega \times \mathbb{R}.$$

$\mathcal{H}_4$) Choose $s > 0$ such that $B(0,s) \subset \Omega$, where $B(0,s)$ denotes the ball with center at 0 and radius of s. Put

$$\theta_1 = \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \int_{\frac{s}{2}}^{s} \left|\frac{12\,(N+1)}{s^3} r - \frac{24N}{s^2} + \frac{9\,(N-1)}{s}\frac{1}{r}\right|^2 r^{N-1} dr,$$

$$\theta_2 = \int_{B(0,s)\setminus B\left(0,\frac{s}{2}\right)} \sum_{i=1}^{N} \left(\frac{12x_i}{s^3} - \frac{24x_i}{s^2} + \frac{9x_i}{sr}\right)^2 dx,$$

where Γ denotes the Gamma function, and

$$\Theta = \theta_1 + \theta_2. \tag{2.3}$$

Assume that λ_1 be the positive first eigenvalue of the second order problem

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Then $\lambda_1(\lambda_1 - c)$ will be the positive first eigenvalue of the fourth-order problem

$$\begin{cases} \Delta^2 u + c\Delta u = \lambda u & \text{in } \Omega, \\ \Delta u = u = 0 & \text{on } \partial\Omega. \end{cases}$$

where $c < \lambda_1$. From Poincaré inequality, one has

$$\frac{\|\Delta u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)}}{\|u\|^2_{L^2(\Omega)}} \geq \lambda_1(\lambda_1 - c) \quad \text{for all} \quad u \in H^2(\Omega) \cap H^1_0(\Omega), \tag{2.4}$$

(see [3, 17, 28]). For $\mu_i = 0$ using the well known Sobolev-Hardy inequalities, we get

$$\left(\int_\Omega \frac{|u|^p}{|x-a|^s} dx\right)^{\frac{2}{p}} \leq C_{p,s} \int_\Omega |\nabla u|^2 dx, \qquad \text{for all } u \in H^1_0(\Omega),$$

where $a \in \Omega$, $2 \leq p \leq 2^{**}$ and $0 \leq s < 2$. In the case $p = s = 2$, it is the same as the well known Hardy inequality

$$\int_\Omega \frac{u^2}{|x-a|^2} dx \leq \frac{1}{\bar{\mu}} \int_\Omega |\nabla u|^2 dx, \qquad \text{for all } u \in H^1_0(\Omega), \tag{2.5}$$

(see [7, 9, 21]).
Let $X = H^2(\Omega) \cap H^1_0(\Omega)$ be the Hilbert space equipped with the inner product

$$(u,v)_1 = \int_\Omega \left(\Delta u \Delta v + \nabla u \nabla v - \sum_{i=1}^{k} \frac{\mu_i}{|x-a_i|^2} uv\right) dx,$$

and the deduced norm

$$\|u\|_1^2 = \int_\Omega \left(|\Delta u|^2 + |\nabla u|^2 - \sum_{i=1}^{k} \frac{\mu_i}{|x-a_i|^2} |u|^2\right) dx.$$

This norm is equivalent to

$$\|u\|^2 = \int_\Omega (|\Delta u|^2 + |\nabla u|^2) dx,$$

which it corresponds to the following inner product

$$(u,v) = \int_\Omega (\Delta u \Delta v + \nabla u \nabla v) dx,$$

in X. Having two equivalent norms $\|.\|$ and $\|.\|_1$, one will always be able to find the constants C_1 and C_2 such that the following inequality holds:

$$C_1\|u\|_1^2 \leq \|u\|^2 \leq C_2\|u\|_1^2, \tag{2.6}$$

for every $u \in X$. More precisely, by $\mathcal{H}_2$,

$$\sum_{i=1}^{k} \frac{\mu_i}{|x-a_i|^2} > 0.$$

Thus

$$\|u\|_1^2 = \int_\Omega \Big(|\Delta u|^2 + |\nabla u|^2 - \sum_{i=1}^{k} \frac{\mu_i}{|x-a_i|^2}|u|^2\Big)dx < \int_\Omega \left(|\Delta u|^2 + |\nabla u|^2\right) dx = \|u\|^2,$$

and from $\mathcal{H}_2$ and (2.5) one has

$$\begin{aligned}\|u\|_1^2 &= \int_\Omega \Big(|\Delta u|^2 + |\nabla u|^2 - \sum_{i=1}^{k} \frac{\mu_i}{|x-a_i|^2}|u|^2\Big)dx \\ &\geq \int_\Omega |\Delta u|^2 dx + \left(1 - \frac{\sum_{i=1}^{k}\mu_i}{\bar{\mu}}\right)\int_\Omega |\nabla u|^2 dx \\ &\geq \left(1 - \frac{\sum_{i=1}^{k}\mu_i}{\bar{\mu}}\right)\|u\|^2.\end{aligned}$$

So for the constants $C_1 = 1$ and $C_2 = \left(1 - \frac{\sum_{i=1}^{k}\mu_i}{\bar{\mu}}\right)^{-1}$ the inequality (2.6) holds.

Definition 2.2. $u \in X$ is said to be a weak solution of the problem (1.1) if

$$\int_\Omega \Delta u \Delta v dx + M\left(\int_\Omega \left(|\nabla u|^2 - \sum_{i=1}^{k} \frac{\mu_i}{|x-a_i|^2}|u|^2\right)dx\right)$$
$$\left(\int_\Omega \nabla u \nabla v - \sum_{i=1}^{k} \frac{\mu_i}{|x-a_i|^2} uvdx\right) = \lambda\left(\int_\Omega f(x,u)\, vdx + \frac{1}{\lambda}\int_\Omega |u|^{2^*-2}uvdx\right),$$

for every $u,\ v \in X$.

The corresponding energy functional of problem (1.1) is defined by

$$I_\lambda(u) = \Phi(u) - \lambda\Psi(u),$$

where

$$\begin{aligned}\Phi(u) &= \frac{1}{2}\int_\Omega |\Delta u|^2 dx + \frac{1}{2}\tilde{M}\left(\int_\Omega \left(|\nabla u|^2 - \sum_{i=1}^{k} \frac{\mu_i}{|x-a_i|^2}|u|^2\right)dx\right), \\ \Psi(u) &= \int_\Omega \left(F(x,u) + \frac{1}{\lambda 2^{**}}|u|^{2^{**}}\right)dx.\end{aligned} \tag{2.7}$$

The functionals Φ and Ψ are well defined and continuously Gâteaux differentiable. Their derivatives are

$$\Phi'(u)\, v =$$

$$\int_{\Omega}\Delta u\Delta v dx+M\left(\int_{\Omega}\left((|\nabla u|^2-\sum_{i=1}^{k}\frac{\mu_i}{|x-a_i|^2}|u|^2\right)dx\right)\left(\int_{\Omega}\nabla u\nabla v-\sum_{i=1}^{k}\frac{\mu_i}{|x-a_i|^2}uvdx\right), \tag{2.8}$$

and

$$\Psi^{'}(u)(v)=\int_{\Omega}f(x,u)\,vdx+\frac{1}{\lambda}\int_{\Omega}|u|^{2^{**}-2}uvdx, \tag{2.9}$$

for every $u,v\in X$.
It is standard to show that the nontrivial solutions of problem (1.1) are equivalent to the nonzero critical points of $I_\lambda(u)$. For more details, we refer the readers to [6] and [23].
Our main results are the following:

Theorem 2.1. *Suppose that $\mathcal{H}_1-\mathcal{H}_4$ hold. Let there exists $s>0$ as considered in $\mathcal{H}_4$ such that if we put*

$$\alpha:=\liminf_{t\to+\infty}\frac{\sup_{\|\xi\|_{L^2(\Omega)}\le t}\int_{\Omega}F(x,\xi)\,dx}{t^2},\quad \beta:=\limsup_{t\to+\infty}\frac{\int_{B\left(0,\frac{s}{2}\right)}F(x,t/h)\,dx}{t^2}, \tag{2.10}$$

one has

$$\alpha<L\beta,$$

where

$$L=\frac{\lambda_1(\lambda_1-c)\,m_0h^2}{m_1\Theta C_2},$$

in which $h>1$, Θ as in (2.3) and C_2 as in (2.6). Then for every

$$\lambda\in\Lambda:=\frac{\lambda_1(\lambda_1-c)\,m_0}{2C_2}\left]\frac{1}{L\beta},\frac{1}{\alpha}\right[,$$

the problem (1.1) possesses an unbounded sequence of weak solutions in X.

Theorem 2.2. *Suppose that $\mathcal{H}_1$ and $\mathcal{H}_2$ hold. Also assume*
($\mathcal{H}_3{}'$) $f:\Omega\times\mathbb{R}\longrightarrow\mathbb{R}$ is an L^1-Carathéodory function and there exists $c>0$ such that

$$F(x,t)\ge 0 \text{ for every } (x,t)\in\Omega\times[0,c]\,.$$

($\mathcal{H}_4{}'$) There exists $s>0$ as considered in $\mathcal{H}_4$ such that, if we put

$$\tilde{\alpha}:=\liminf_{t\to 0^+}\frac{\sup_{\|\xi\|_{L^2(\Omega)}\le t}\int_{\Omega}F(x,\xi)\,dx}{t^2},\quad \tilde{\beta}:=\limsup_{t\to 0^+}\frac{\int_{B\left(0,\frac{s}{2}\right)}F(x,t/h)\,dx}{t^2},$$

one has

$$\tilde{\alpha}<L\tilde{\beta},$$

where

$$L=\frac{\lambda_1(\lambda_1-c)\,m_0h^2}{m_1\Theta C_2},$$

in which $h>1$, Θ as in (2.3) and C_2 as in (2.6).
Then, for every

$$\lambda\in\tilde{\Lambda}:=\frac{\lambda_1(\lambda_1-c)\,m_0}{2C_2}\left]\frac{1}{L\tilde{\beta}},\frac{1}{\tilde{\alpha}}\right[,$$

the problem (1.1) admits a sequence $\{u_n\}$ of weak solutions such that $u_n\to 0$ strongly in X.

Remark 2.1. The conditions $\mathcal{H}_3, \mathcal{H}_4, \mathcal{H}_3{}'$ and $\mathcal{H}_4{}'$ guarantee that problem (1.1) has infinitely many solutions. Precisely $\mathcal{H}_3$ and $\mathcal{H}_4$ ensure that the primitive F of f satisfies a suitable oscillatory behavior at infinity for obtaining unbounded solutions and $\mathcal{H}_3{}'$ and $\mathcal{H}_4{}'$ ensure that the primitive F of f satisfies a suitable oscillatory behavior near the origin for finding arbitrarily small solutions. (See [18]).

Our main tool to prove the above-mentioned theorems is the following infinitely many critical points theorem [6].

Theorem 2.3. *Let X be a reflexive real Banach space; $\Phi, \Psi : X \to \mathbb{R}$ be two Gâteaux differentiable functionals such that Φ is sequentially weakly lower semicontinuous, strongly continuous, and coercive and Ψ is sequentially weakly upper semicontinuous. For every $r > \inf_X \Phi$, let us put*

$$\varphi(r) = \inf_{u\in\Phi^{-1}(]-\infty,r[)} \frac{\sup_{v\in\Phi^{-1}(]-\infty,r[)} \Psi(v) - \Psi(u)}{r - \Phi(u)}$$

$$\gamma = \liminf_{r\to+\infty} \varphi(r), \qquad \delta = \liminf_{r\to(\inf_X \Phi)^+} \varphi(r).$$

Then, one has
(i) If $\gamma < +\infty$ then, for each $\lambda \in \left]0, \frac{1}{\gamma}\right[$, the following alternative holds: either the functional $\Phi - \lambda\Psi$ has a global minimum, or there exists a sequence $\{u_n\}$ of critical points (local minima) of $\Phi - \lambda\Psi$ such that $\lim_{n\to+\infty} \Phi(u_n) = +\infty$.
(ii) If $\delta < +\infty$ then, for each $\lambda \in \left]0, \frac{1}{\delta}\right[$, the following alternative holds: either there exists a global minimum of Φ which is a local minimum of $\Phi - \lambda\Psi$, or there exists a sequence $\{u_n\}$ of pairwise distinct critical points (local minima) of $\Phi - \lambda\Psi$, with $\lim_{n\to+\infty} \Phi(u_n) = \inf_X \Phi$, which weakly converges to a global minimum of Φ.

3. Proof of main Theorems

In this section our main results will be proved. We apply part (i) of Theorem 2.3 to prove Theorem 2.1. In fact it will be indicated that the functional I_λ does not have a global minimum.

Proof of Theorem 2.1. Let Φ, Ψ be the functionals defined in (2.7). Since

$$\tilde{M}\left(\int_\Omega \left(|\nabla\ u|^2 - \sum_{i=1}^k \frac{\mu_i}{|x-a_i|^2}|u|^2\right) dx\right) = \int_0^{\int_\Omega \left(|\nabla\ u|^2 - \sum_{i=1}^k \frac{\mu_i}{|x-a_i|^2}|u|^2\right) dx} M\left(s\right) ds,$$

thus from (2.5) and (2.6) we get

$$\Phi\left(u\right) \geq \begin{cases} \dfrac{1}{2C_2}\|u\|^2 & \text{if} \quad m_0 \geq 1, \\ \dfrac{m_0}{2C_2}\|u\|^2 & \text{if} \quad m_0 < 1, \end{cases} \tag{3.1}$$

and

$$\Phi\left(u\right) \leq \begin{cases} \dfrac{1}{2}\|u\|^2 & \text{if} \quad m_1 \leq 1, \\ \dfrac{m_1}{2}\|u\|^2 & \text{if} \quad m_1 > 1. \end{cases}$$

The above inequalities and $\mathcal{H}_1$ conclude that

$$\frac{m_0}{2C_2}\|u\|^2 \leq \Phi(u) \leq \frac{m_1}{2}\|u\|^2, \tag{3.2}$$

where $C_2 = \left(1 - \frac{\sum_{i=1}^{k}\mu_i}{\bar{\mu}}\right)^{-1}$. Consequently Φ is coercive. From the weakly lower semicontinuity of norm, the monotonicity and continuity of $\tilde{M}$, it is clear that Φ is sequentially weakly lower semicontinuous. Moreover, Φ is continuously Gâteaux differentiable and its Gâteaux derivative admits a continuous inverse.

For fixed $u \in X$, let $u_n \rightharpoonup u$ weakly in X as $n \to \infty$. Then u_n converges uniformly to u on Ω as $n \to \infty$. (See [31]). Since $g(x,u) = f(x,u) + u|u|^{2^{**}-2}$ is continuous in $\mathbb{R}$ for every $x \in \Omega$, so $g(x,u_n) \to g(x,u)$ for every $x \in \Omega$ as $n \to \infty$. Hence $\Psi'(u_n) \to \Psi'(u)$ as $n \to \infty$, which means that Ψ is strongly continuous on X. This implies that Ψ' is a compact operator by Proposition 26.2 of [31]. Thus it is sequentially weakly upper semicontinuous.

Now, we wish to prove that

$$\gamma < \infty.$$

By (2.4) and (3.2), one has

$$\begin{aligned} \Phi^{-1}(]-\infty, r[) &= \{u \in X, \Phi(u) < r\} \\ &\subset \left\{u \in X : \frac{m_0}{2C_2}\|u\|^2 < r\right\} \\ &\subset \left\{u \in X : \|u\|^2 < \frac{2rC_2}{m_0}\right\} \\ &\subset \left\{u \in X : \|u\|^2_{L^2(\Omega)} < \frac{2rC_2}{\lambda_1(\lambda_1 - c)m_0}\right\}. \end{aligned} \tag{3.3}$$

Note that $\Phi(0) = 0$ and $\Psi(0) = 0$. For every $r > 0$, it is obtained by (3.3) that

$$\begin{aligned} \varphi(r) &= \inf_{u\in\Phi^{-1}(]-\infty,r[)} \frac{\sup_{u\in\Phi^{-1}(]-\infty,r[)}\Psi(v) - \Psi(u)}{r - \Phi(u)} \\ &\leq \frac{\sup_{u\in\Phi^{-1}(]-\infty,r[)}\Psi(v)}{r} \\ &\leq \frac{\sup_{\|\xi\|_{L^2(\Omega)}\leq d}\int_\Omega F(x,\xi)\,dx}{r} + \frac{1}{\lambda 2^{**}}\frac{\sup_{\|\xi\|_{L^2(\Omega)}\leq d}\int_\Omega |u|^{2^{*}*}dx}{r}, \end{aligned}$$

where

$$d = \frac{2rC_2}{\lambda_1(\lambda_1 - c)m_0}.$$

Let $\{d_n\}$ be a sequence of positive numbers such that $d_n \to +\infty$ and

$$\lim_{n\to+\infty} \frac{\sup_{\|\xi\|_{L^2(\Omega)}\leq d_n}\int_\Omega F(x,\xi)\,dx}{d_n^2} = \liminf_{t\to+\infty}\frac{\sup_{\|\xi\|_{L^2(\Omega)}\leq t}\int_\Omega F(x,\xi)\,dx}{t^2}. \tag{3.4}$$

Let

$$r_n = \frac{\lambda_1(\lambda_1 - c)m_0}{2C_2}d_n^2 \quad \text{for all} \quad n \in \mathbb{N}.$$

From (2.10) and (3.4) we get

$$\begin{aligned}\gamma &= \liminf_{r\to+\infty}\varphi(r) \le \liminf_{n\to+\infty}\varphi(r_n)\\ &\le \frac{2C_2}{\lambda_1(\lambda_1-c)m_0}\lim_{n\to+\infty}\frac{\sup_{\|\xi\|_{L^2(\Omega)}\le d_n}\int_\Omega F(x,\xi)\,dx}{d_n^2}\\ &\quad+\frac{1}{2^{**}\lambda}\frac{2C_2}{\lambda_1(\lambda_1-c)m_0}\limsup_{n\to+\infty}\frac{\|u\|_{L^{2^{**}}(\Omega)}^{2^{**}}}{d_n^2}\\ &\le \frac{1}{2^{**}\lambda}\frac{2C_2}{\lambda_1(\lambda_1-c)m_0}\alpha<+\infty.\end{aligned} \tag{3.5}$$

By (3.5) it gives that

$$\gamma<\frac{2C_2}{\lambda_1(\lambda_1-c)m_0}\alpha<\frac{1}{\lambda}. \tag{3.6}$$

Since $\alpha<L\beta$, we have $\Lambda\subset\left]0,\frac{1}{\gamma}\right[$.

Fix $\lambda\in\Lambda$, the previous inequality assures that the conclusion (i) of Theorem 2.3 can be used and either I_λ has a global minimum or there exists a sequence $\{u_n\}$ of solutions of problem (1.1) such that $\lim_{n\to\infty}\|u\|=+\infty$.

The other step is to verify that the functional $\Phi-\lambda\Psi$ for $\lambda\in\Lambda$ is unbounded from below. Since

$$\frac{1}{\lambda}<\frac{2C_2}{\lambda_1(\lambda_1-c)m_0}L\beta=\frac{2h^2}{m_1\Theta}\beta,$$

there exists a sequence $\{a_n\}$ of positive numbers and $\zeta>0$ such that $a_n\to+\infty$ and

$$\frac{1}{\lambda}<\zeta<\frac{2h^2}{m_1\Theta}\frac{\int_{B\left(0,\frac{s}{2}\right)}F(x,a_n/h)\,dx}{a_n^2}, \tag{3.7}$$

for n large enough. Consider a sequence $\{w_n\}$ in X defined by setting

$$w_n(x)=\begin{cases}0 & x\in\bar{\Omega}\setminus B(0,s),\\ \frac{a_n}{h}\left(\frac{4}{s^3}\rho^3-\frac{12}{s^2}\rho^2+\frac{9}{s}\rho-1\right) & x\in B(0,s)\setminus B\left(0,\frac{s}{2}\right),\\ \frac{a_n}{h} & x\in B\left(0,\frac{s}{2}\right),\end{cases} \tag{3.8}$$

where $\rho=dist\,(x,0)=\sqrt{\sum_{i=1}^N x_i^2}$. Obviously $w_n\in X$. A direct calculation shows

$$\frac{\partial w_n(x)}{\partial x_i}=\begin{cases}0 & x\in\left(\bar{\Omega}\setminus B(0,s)\right)\cap B\left(0,\frac{s}{2}\right),\\ \frac{a_n}{h}\left(\frac{12\rho x_i}{s^3}-\frac{24x_i}{s^2}+\frac{9x_i}{s\rho}\right) & x\in B(0,s)\setminus B\left(0,\frac{s}{2}\right),\end{cases}$$

and

$$\frac{\partial^2 w_n(x)}{\partial x_i^2}=\begin{cases}0 & x\in\left(\bar{\Omega}\setminus B(0,s)\right)\cap B\left(0,\frac{s}{2}\right),\\ \frac{a_n}{h}\left(\frac{12\left(x_i^2+\rho^2\right)}{s^3\rho}-\frac{24}{s^2}+\frac{9\left(\rho^2-x_i^2\right)}{s\rho^3}\right) & x\in B(0,s)\setminus B\left(0,\frac{s}{2}\right).\end{cases} \tag{3.9}$$

By (2.3) and (3.9) we have

$$\sum_{i=1}^N\frac{\partial^2 w_n(x)}{\partial x_i^2}=\begin{cases}0 & x\in\left(\bar{\Omega}\setminus B(0,s)\right)\cap B\left(0,\frac{s}{2}\right),\\ \frac{a_n}{h}\left(\frac{12\rho(N+1)}{s^3}-\frac{24N}{s^2}+\frac{9(N-1)}{s\rho}\right) & x\in B(0,s)\setminus B\left(0,\frac{s}{2}\right),\end{cases} \tag{3.10}$$

and

$$\begin{aligned}\int_{\Omega}|\Delta w_n(x)|^2dx &= \left(\frac{a_n}{h}\right)^2\frac{2\pi^{N/2}}{\Gamma\left(\frac{N}{2}\right)}\int_{\frac{s}{2}}^{s}\left|\frac{12(N+1)}{s^3}-\frac{24N}{s^2}+\frac{9(N-1)}{s}\frac{1}{r}\right|^2r^{N-1}dr\\ &= \frac{\theta_1}{h^2}a_n^2, \qquad (3.11)\end{aligned}$$

and

$$\begin{aligned}\int_{\Omega}|\nabla w_n(x)|^2dx &= \int_{B(0,s)/B\left(0,\frac{s}{2}\right)}\sum_{i=1}^{N}\left(\frac{a_n}{h}\right)^2\left(\frac{12(N+1)x_i}{s^3}-\frac{24x_i}{s^2}+\frac{9}{s}\frac{x_i}{\rho}\right)^2dx\\ &= \left(\frac{a_n}{h}\right)^2\int_{B(0,s)/B\left(0,\frac{s}{2}\right)}\sum_{i=1}^{N}\left(\frac{a_n}{h}\right)^2\left(\frac{12\rho x_i}{s^3}-\frac{24x_i}{s^2}+\frac{9}{s}\frac{x_i}{\rho}\right)^2dx\\ &= \left(\frac{a_n}{h}\right)^2\theta_2. \qquad (3.12)\end{aligned}$$

Thus, by (2.1), (3.11) and (3.12) one gets that

$$\begin{aligned}\Phi(w_n) &\leq \frac{m_1}{2}\|w_n\|^2 = \frac{m_1}{2}\int_{\Omega}|\left(\Delta w_n|^2+|\nabla w_n|^2\right)dx\\ &= \frac{m_1}{2}\left(\frac{\theta_1}{h^2}a_n^2+\frac{\theta_2}{h^2}a_n^2\right) = \frac{m_1\Theta}{2h^2}a_n^2. \qquad (3.13)\end{aligned}$$

On the other hand, by (2.10) and (3.7), one has

$$\begin{aligned}\Psi(w_n) &= \int_{\Omega}\left(F(x,w_n(x))+\frac{1}{\lambda 2^{**}}|w_n|^{2^{**}}\right)dx\\ &\geq \int_{B(0,s/2)}F(x,a_n/h)\,dx\\ &\geq \frac{m_1\Theta\zeta a_n^2}{2h^2}. \qquad (3.14)\end{aligned}$$

Hence, it follows from (3.13), (3.14) and (3.7) that

$$\begin{aligned}\Phi(w_n)-\lambda\Psi(w_n) &\leq \frac{m_1\Theta\zeta a_n^2}{2h^2}-\lambda\int_{B(0,s/2)}F(x,a_n/h)\,dx\\ &\leq \frac{m_1\Theta}{2h^2}(1-\lambda\zeta)a_n^2, \qquad (3.15)\end{aligned}$$

for every $n\in\mathbb{N}$ large enough. Thus

$$\lim_{n\to+\infty}(\Phi(w_n)-\lambda\Psi(w_n)) = -\infty.$$

From the Theorem 2.3 case (i) there exists an unbounded sequence $\{u_n\}$ of critical points of the functional $\Phi-\lambda\Psi$. The proof is completed in terms of the relation between the critical points of $\Phi-\lambda\Psi$ and the weak solutions of problem (1.1).

In the following, by using part (ii) of Theorem 2.3, we prove Theorem 2.1. Precisely we establish the existence of infinitely many solutions of the problem (1.1) converging at zero. □

Proof of Theorem 2.2. Let $\{d_n\}$ be a sequence of positive numbers such that $d_n \to 0^+$, and

$$\begin{aligned}\delta :=& \liminf_{r\to 0^+} \varphi(r) \le \liminf_{n\to+\infty} \varphi(r_n) \\ \le& \frac{2C_2}{\lambda_1(\lambda_1 - c)\, m_0} \lim_{n\to+\infty} \frac{\sup_{\|\xi\|_{L^2(\Omega)}\le d_n} \int_\Omega F(x,\xi)\,dx}{d_n^2} \\ &+ \frac{1}{2^{**}\lambda} \frac{2C_2}{\lambda_1(\lambda_1 - c)\, m_0} \limsup_{n\to+\infty} \frac{\|\xi\|_{L^{2^{**}}(\Omega)}^{2^{**}}}{d_n^2} \\ \le& \frac{1}{2^{**}\lambda} \frac{2C_2}{\lambda_1(\lambda_1 - c)\, m_0} \tilde{\alpha} < +\infty, \end{aligned} \tag{3.16}$$

so $\tilde{\Lambda} \subseteq \left]0, \frac{1}{\delta}\right[$.

Let Φ and Ψ be as in (2.7) and $\{a_n\}$ a sequence of positive numbers such that $a_n \to 0^+$ and $\zeta > 0$

$$\frac{1}{\lambda} < \zeta < \frac{2h^2}{m_1\Theta} \frac{\int_{B\left(0,\frac{s}{2}\right)} F(x, a_n/h)\,dx}{a_n^2},$$

for n large enough. From the fact that $\min_X \Phi = \Phi(0) = 0$, (3.13), (3.14), and (3.7) one gets

$$\begin{aligned}\Phi(w_n) - \lambda\Psi(w_n) \quad &\le \frac{m_1\Theta\zeta a_n^2}{2h^2} - \lambda \int_{B(0,s/2)} F(x, a_n/h)\,dx \\ &\le \frac{m_1\Theta}{2h^2}(1 - \lambda\zeta)\, a_n^2 < 0 = \Phi(0) - \lambda\Psi(0),\end{aligned}$$

for every $n \in \mathbb{N}$ large enough, where $\{w_n\}$ is the sequence in X defined in (3.8). Taking into account $\|w_n\| \to 0$, we derive $\Phi - \lambda\Psi$ does not have a local minimum at zero. The conclusion is obtained from case (ii) of Theorem (2.3).

Remark 3.1. We point out that the authors in [16], using the same variational framework but different technical arguments, ensured the existence of infinitely many solutions for the problem (1.1) where $k = 1$ and $f(x,u) = |u|^{q-2}u, 1 < q < 2$ or $2 < q < 2^{**}$.

Remark 3.2. By using the strong maximum principle and quite standard arguments, it is verifiable that such solutions which are achieved in Theorem 2.1 do not change sign, so we can assume that they are positive. The authors in [1] obtained a maximum principle for fourth order operators with singular terms. They proved that in the case of Navier boundary condition $u = \Delta u = 0$ on $\partial\Omega$, maximum principle holds and if the minimizers exist, then one can expect a non-negative solution. (See [5, 10, 14]).

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