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A Tauberian condition under which convergence follows from the weighted mean summability of sequences of fuzzy number

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Abstract. Let (u_n) be a sequence of fuzzy numbers and (p_n) be a sequence of nonnegative numbers such that $p_0 > 0$ and

$$P_n := \sum_{k=0}^{n} p_k \to \infty \quad \text{as} \quad n \to \infty.$$

The weighted mean of (u_n) is defined by

$$t_n := \frac{1}{P_n} \sum_{k=0}^{n} p_k u_k \quad \text{for} \quad n = 0, 1, 2, ...$$

It is well known that convergence of (u_n) implies that of the sequence (t_n) of its weighted means. However, the converse of this implication is not true in general. In this paper, we investigate under which conditions convergence of (u_n) follows from its weighted mean summability. We prove a Tauberian theorem including condition of slow decrease with respect to the weighted mean summability method for sequences of fuzzy numbers.

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1. Introduction

In this section, we begin with some remarks about history of fuzzy set theory and its applications to the $(\overline{N}, p)$ summability method, that is about the history from almost fifty years ago until these days. After dwelling on study that encourages us to do this research, we complete this section summarizing theorem attained in this article.

Improved based upon the fuzzy sets and fuzzy set operations which was introduced by Zadeh [13], fuzzy set theory has increasingly received attention from researchers in a diverse range of disciplines in the last few years. Aspiring to apply concept of fuzziness to individual works with a broad viewpoint from theoretical to practical in almost all sciences and technology, researchers have reached numerous and varied applications of its in fields such as statistics, nuclear science, biomedicine, agriculture, geography, weather prediction, finance and stock market, engineering, computer science, artificial intelligence, pattern recognition, handwriting analysis, decision theory, robotics etc. In addition to these, one of areas which the concept of fuzziness was carried out is also pure mathematics and there have been several authors discussing many important properties and applications of fuzzy sets. Dubois and Prade [4] introduced

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the fuzzy numbers and defined basic operations of addition, subtraction, multiplication and division. In [6], Goetschel and Voxman presented a less restrictive definition of fuzzy numbers. Matloka [7] introduced bounded and convergent sequences of fuzzy numbers, studied some of their properties and showed that every convergent sequence of fuzzy numbers is bounded. Nanda [9] studied the spaces of bounded and convergent sequence of fuzzy numbers and proved that they are complete metric spaces.

In recent years, there has been an increasing interest on summability methods of sequences of fuzzy numbers. One of these summability methods which has attracted the attention of many researchers is the $(\overline{N}, p)$ summability method. Tripathy and Baruah [12] introduced the $(\overline{N}, p)$ summability method for sequences of fuzzy numbers and obtained fuzzy analogues of classical Tauberian theorems for this method. Çanak [3] investigated some conditions needed for the $(\overline{N}, p)$ summable sequences to be convergent. Later, Önder et al. [10] established a Tauberian condition controlling one-sided oscillatory behavior of a sequence of fuzzy numbers for the $(\overline{N}, p)$ summability method.

Besides the studies mentioned up to now, the study that encourages us to do this research is in fact that including some results obtained by Móricz [8] for the $(\overline{N}, p)$ summability method for sequences of real numbers. In [8], Móricz defined the classes of all upper and lower allowed sequences with respect to (p_n) and obtained necessary and sufficient conditions for the $(\overline{N}, p)$ summable sequences of real numbers by means of defined classes. Here, our aim extend the results presented by Móricz for the $(\overline{N}, p)$ summable sequences of real numbers to the $(\overline{N}, p)$ summable sequences of fuzzy numbers using these classes.

In this paper, we establish a Tauberian theorem which convergence follows from the $(\overline{N}, p)$ summability under condition of slow decrease with respect to the $(\overline{N}, p)$ method.

2. Preliminaries

In this section, we begin with basic definitions and notations with respect to fuzzy numbers that will be used throughout this paper. In the sequel, we mention its linear structure, set operations on the space of fuzzy numbers and some algebraic properties related to its. We recall metric on the space of fuzzy numbers and exhibit a list of fundamental properties of its. We end this section by giving some definitions concerning the sequences of fuzzy numbers. For the sake of completeness of the paper, we give our study in Section 3.

In [6], Goetschel and Voxman introduced concept of fuzzy numbers as follows:

Definition 2.1. Consider a fuzzy subset of real line $u : \mathbb{R} \to [0, 1]$. Then the mapping u is a fuzzy number if it satisfies following additional properties:

(i) u is normal; i.e., there exists a $t_0 \in \mathbb{R}$ such that $u(t_0) = 1$.

(ii) u is fuzzy convex; i.e., for any $t_0, t_1 \in \mathbb{R}$ and for any $\alpha \in [0, 1]$, $u(\alpha t_0 + (1 - \alpha)t_1) \geq \min\{u(t_0), u(t_1)\}$.

(iii) u is upper semicontinuous on $\mathbb{R}$.

(iv) The support of u, $[u]_0 := \overline{\{t \in \mathbb{R} : u(t) > 0\}}$ is compact, where $\overline{\{t \in \mathbb{R} : u(t) > 0\}}$ denotes the closure of the set $\{t \in \mathbb{R} : u(t) > 0\}$ in usual topology of $\mathbb{R}$.

The set of all fuzzy numbers on $\mathbb{R}$ is denoted by E^1.

We recall the linear structure of E^1 as follows. For $u \in E^1$, the α-level set of u is defined by

$$[u]_\alpha := \begin{cases} \{x \in \mathbb{R} : u(x) \geq \alpha\}, & 0 < \alpha \leq 1, \\ \overline{\{x \in \mathbb{R} : u(x) > \alpha\}}, & \alpha = 0. \end{cases}$$

Then, it is easily established (see [5]) that u is a fuzzy number if and only if $[u]_\alpha$ is a closed, bounded and nonempty interval for each $\alpha \in [0,1]$ with $[u]_\beta \subseteq [u]_\alpha$ if $0 \leq \alpha \leq \beta \leq 1$. From this characterization of fuzzy numbers, it follows that a fuzzy number u is completely determined by the end points of the intervals $[u]_\alpha = [u^-(\alpha), u^+(\alpha)]$ where $u^-(\alpha) \leq u^+(\alpha)$ and $u^-(\alpha), u^+(\alpha) \in \mathbb{R}$ for each $\alpha \in [0,1]$.

In the sequel, Goetschel and Voxman [6] presented another representation of a fuzzy number as a pair of functions that satisfy some properties.

Theorem 2.1. [6] *Let $u \in E^1$ and $[u]_\alpha = [u^-(\alpha), u^+(\alpha)]$. Then the functions $u^-, u^+ : [0,1] \to \mathbb{R}$, defining the endpoints of the α-level sets, satisfy following conditions:*

(i) *$u^-(\alpha) \in \mathbb{R}$ is a bounded, non-decreasing and left continuous function on $(0,1]$.*

(ii) *$u^+(\alpha) \in \mathbb{R}$ is a bounded, non-increasing and left continuous function on $(0,1]$.*

(iii) *The functions $u^-(\alpha)$ and $u^+(\alpha)$ are right continuous at $\alpha = 0$.*

(iv) *$u^-(1) \leq u^+(1)$.*

Conversely, if the pair of functions f and g satisfies the above conditions (i)-(iv), then there exists a unique fuzzy number u such that $[u]_\alpha := [f(\alpha), g(\alpha)]$ for each $\alpha \in [0,1]$ and $u(x) := \sup_{\alpha \in [0,1]} \{\alpha : f(\alpha) \leq x \leq g(\alpha)\}$.

Suppose that $u, v \in E^1$ are represented by $[u^-(\alpha), u^+(\alpha)]$ and $[v^-(\alpha), v^+(\alpha)]$ for each $\alpha \in [0,1]$, respectively. Then, the operations addition, subtraction and scalar multiplication on the set of fuzzy numbers are defined as follows:

$$[u+v]_\alpha := \left[u^-(\alpha) + v^-(\alpha), u^+(\alpha) + v^+(\alpha)\right],$$

$$[u-v]_\alpha := \left[u^-(\alpha) - v^+(\alpha), u^+(\alpha) - v^-(\alpha)\right],$$

$$[ku]_\alpha = k[u]_\alpha := \begin{cases} [ku^-(\alpha), ku^+(\alpha)], & k \geq 0, \\ [ku^+(\alpha), ku^-(\alpha)], & k < 0. \end{cases}$$

The set of all real numbers can be embedded in E^1. For $r \in \mathbb{R}$, $\bar{r} \in E^1$ is defined by

$$\bar{r}(x) := \begin{cases} 1, & x = r, \\ 0, & x \neq r. \end{cases}$$

The following lemma deals with the algebraic properties of fuzzy numbers.

Lemma 2.2. [2] *On the set of fuzzy numbers there are two binary operations, denoted by $+$, . and called addition, scaler multiplication, respectively. These operations satisfy following properties:*

(i) *The addition of fuzzy numbers is associative and commutative, i.e., $u + v = v + u$ and $u + (v + w) = (u + v) + w$, for any $u, v, w \in E^1$.*

(ii) *$\bar{0} \in E^1$ is neutral element with respect to $+$, i.e., $u + \bar{0} = \bar{0} + u = u$, for any $u \in E^1$.*

(iii) *With respect to $+$, none of $u \in E^1 \setminus \mathbb{R}$ has opposite in E^1.*

(iv) *$\bar{1} \in E^1$ is neutral element with respect to ., i.e., $u\bar{1} = \bar{1}u = u$, for any $u \in E^1$.*

(v) *For any $a, b \in \mathbb{R}$ with $ab \geq 0$ and any $u \in E^1$, we have $(a+b)u = au + bu$. For general $a, b \in \mathbb{R}$, this property does not hold.*

(vi) *For any $a \in \mathbb{R}$ and $u, v \in E^1$, we have $a(u + v) = au + av$.*

(vii) *For any* $a, b \in \mathbb{R}$ *and* $u \in E^1$, *we have* $(ab)u = a(bu)$.

As a conclusion, we attain by Lemma 2.2 that the space of fuzzy numbers is not a linear space.

Concept of metric space may be defined as an arbitrary fuzzy set which a distance between all elements of the set are described. It is possible to define several different metrics on the space of fuzzy numbers; however, the most well known and preferential metric among these metrics is the Hausdorff distance for fuzzy numbers based on the classical Hausdorff distance between compact convex subsets of $\mathbb{R}^n$. Let W denote the set of all closed and bounded intervals. For the case when $A = [a^-, a^+]$, $B = [b^-, b^+]$ are the two intervals, the Hausdorff distance on W is defined by

$$d(A, B) := \max\left\{\left|a^- - b^-\right|, \left|a^+ - b^+\right|\right\}.$$

It can be noted that W is a complete separable metric space on the basis of the Haussdorff distance (cf. Nanda [9]). Now, we may define the metric D on the space of fuzzy numbers with the help of the Hausdorff metric d.

Definition 2.2. ([2]) Let $D : E^1 \times E^1 \to \mathbb{R}_+$ and let $u, v \in E^1$ represented respectively by $[u^-(\alpha), u^+(\alpha)]$ and $[v^-(\alpha), v^+(\alpha)]$ for each $\alpha \in [0, 1]$

$$D(u, v) = \sup_{\alpha \in [0,1]} d([u]_\alpha, [v]_\alpha).$$

Then D is called the Hausdorff distance between fuzzy numbers u and v.

It is easy to see that

$$D(u, \bar{0}) = \sup_{\alpha \in [0,1]} \max\left\{\left|u^-(\alpha)\right|, \left|u^+(\alpha)\right|\right\} = \max\left\{\left|u^-(0)\right|, \left|u^+(0)\right|\right\}.$$

The following proposition presents some fundamental properties of the Hausdorff distance between fuzzy numbers.

Proposition 2.3. [2] *Let* $u, v, w, z \in E^1$ *and* $k \in \mathbb{R}$. *Then following statements hold true.*

(i) (E^1, D) *is a complete metric space.*
(ii) $D(u + w, v + w) = D(u, v)$*; i.e.,* D *is translation invariant.*
(iii) $D(ku, kv) = |k|\, D(u, v)$.
(iv) $D(u + v, w + z) \leq D(u, w) + D(v, z)$.
(v) $|D(u, \bar{0}) - D(v, \bar{0})| \leq D(u, v) \leq D(u, \bar{0}) + D(v, \bar{0})$.

For $u, v \in E^1$, partial ordering relation on E^1 is defined as follows:([1])

- $u \preceq v$ if and only if $[u]_\alpha \preceq [v]_\alpha$, i.e. $u^-(\alpha) \leq v^-(\alpha)$ and $u^+(\alpha) \leq v^+(\alpha)$ for any $\alpha \in [0, 1]$.
- We say that $u \prec v$ if $u \preceq v$ and there exists $\alpha_0 \in [0, 1]$ such that $u^-(\alpha_0) < v^-(\alpha_0)$ or $u^+(\alpha_0) < v^+(\alpha_0)$.
- We say that $u, v \in E^1$ are incomparable if neither $u \preceq v$ nor $v \preceq u$.

Lemma 2.4. [1] *Given two fuzzy numbers* u *and* v *the following statements are equivalent:*

(i) $D(u, v) \leq \epsilon$,
(ii) $u - \bar{\epsilon} \preceq v \preceq u + \bar{\epsilon}$,

where $\epsilon > 0$.

Lemma 2.5. [11] *Let* $u, v, w \in E^1$. *If* $u + w \preceq v + w$, *then* $u \preceq v$.

We now refer following definitions concerning sequences of fuzzy numbers which will be needed in the sequel.

Definition 2.3. [7] A sequence $u = (u_n)$ of fuzzy numbers is a function u from the set $\mathbb{N} = \{0, 1, 2, ...\}$ into the set E^1. The fuzzy number u_n denotes the value of the function at a point $n \in \mathbb{N}$ and is called the *n-th* term of the sequence. The set of all sequences of fuzzy numbers is denoted by $\omega(F)$.

Definition 2.4. [7] A sequence $u = (u_n)$ of fuzzy numbers is said to be convergent to the fuzzy number μ_0, written as $\lim_{n\to\infty} u_n = \mu_0$, if for every $\epsilon > 0$ there exists a positive integer $n_0 = n_0(\epsilon)$ such that

$$D(u_n, \mu_0) < \epsilon \quad \text{whenever} \quad n \geq n_0. \tag{1}$$

The number μ_0 is called the limit of (u_n). The set of all convergent sequences of fuzzy numbers is denoted by $c(F)$.

Definition 2.5. [12] Let $u = (u_n)$ be a sequence of fuzzy numbers and $p = (p_n)$ be a sequence of nonnegative numbers such that $p_0 > 0$ and

$$P_n := \sum_{k=0}^{n} p_k \to \infty \quad \text{as} \quad n \to \infty. \tag{2}$$

The weighted means of $(u_n) \in \omega(F)$ is defined by

$$t_n := \frac{1}{P_n} \sum_{k=0}^{n} p_k u_k \quad \text{for} \quad n \in \mathbb{N}.$$

The sequence (u_n) is said to be summable by weighted mean method determined by the sequence (p_n) to the fuzzy number μ_0 if for every $\epsilon > 0$ there exists a positive integer $n_0 = n_0(\epsilon)$ such that

$$D(t_n, \mu_0) < \epsilon \quad \text{whenever} \quad n \geq n_0.$$

The weighted mean methods are also called Riesz methods or the $(\overline{N}, p)$ methods in the literature. The $(\overline{N}, p)$ summability method is regular if and only if condition (2) is satisfied. In other words, every convergent sequence of fuzzy numbers is also $(\overline{N}, p)$ summable to the same number under condition (2). However, converse of this statement is not true in general. Truth of that is possible under some suitable condition which is so-called a Tauberian condition on the sequence. Any theorem stating that convergence of a sequence follows from its $(\overline{N}, p)$ summability and some Tauberian condition is said to be a Tauberian theorem for the $(\overline{N}, p)$ summability method.

At present, we remind the classes of all upper and lower allowed sequences with respect to (p_n) and their natural subclasses given by Móricz [8].

Let (ρ_n) be a strictly increasing sequence of positive integers such that $\rho_n \to \infty$ as $n \to \infty$. The sequence (ρ_n) is an upper allowed sequence with respect to (p_n) if condition

$$\liminf_{n\to\infty} \frac{P_{\rho_n}}{P_n} > 1 \tag{3}$$

is satisfied. Similarly, the sequence (ρ_n) is a lower allowed sequence with respect to (p_n) if condition

$$\liminf_{n\to\infty} \frac{P_n}{P_{\rho_n}} > 1 \tag{4}$$

is satisfied. In this case, the classes of all upper and lower allowed sequence with respect to (p_n) is denoted by Λ_u and Λ_ℓ, respectively. The natural subclasses of these classes are constructed as follows. Define

$$\rho_n^u(\lambda) := \min\left\{ m > n : \sum_{k=n+1}^{m} \frac{p_k}{P_k} \geq \lambda - 1 \right\} \quad \text{for} \quad \lambda > 1,$$

then we have

$$P_{\rho_n^u(\lambda)} = P_n + \sum_{k=n+1}^{\rho_n^u(\lambda)} \frac{p_k}{P_k} P_k \geq P_n + P_n \sum_{k=n+1}^{\rho_n^u(\lambda)} \frac{p_k}{P_k} \geq \lambda P_n.$$

In this case, it may be considered $\widetilde{\Lambda}_u := \{(\rho_n^u(\lambda))_n,\ \lambda > 1\}$ instead of Λ_u. Similarly, we define

$$\rho_n^\ell(\lambda) := \max\left\{ m < n : \sum_{k=m+1}^{n} \frac{p_k}{P_k} \geq \lambda - 1 \right\} \quad \text{for} \quad \lambda > 1,$$

then we have

$$P_n = P_{\rho_n^\ell(\lambda)} + \sum_{k=\rho_n^\ell(\lambda)+1}^{n} \frac{p_k}{P_k} P_k \geq P_{\rho_n^\ell(\lambda)} + P_{\rho_n^\ell(\lambda)} \sum_{k=\rho_n^\ell(\lambda)+1}^{n} \frac{p_k}{P_k} \geq \lambda P_{\rho_n^\ell(\lambda)}.$$

In this case, it may be considered $\widetilde{\Lambda}_\ell := \left\{(\rho_n^\ell(\lambda))_n,\ \lambda > 1\right\}$ instead of Λ_ℓ.

At present, we define a fuzzy analogue of concept of slow decrease with respect to the $(\overline{N}, p)$ method by means of defined classes.

Definition 2.6. A sequence $u = (u_n)$ of fuzzy numbers is said to be slowly decreasing with respect to the $(\overline{N}, p)$ method if for every $\epsilon > 0$ there exist $n_0 = n_0(\epsilon)$ and $\lambda > 1$ such that for all $n \geq n_0$

$$u_k \succeq u_n - \bar{\epsilon} \quad \text{whenever} \quad n < k \leq \rho_n^u(\lambda). \tag{5}$$

Equivalently, it can be said that if a sequence (u_n) of fuzzy numbers is slowly decreasing with respect to the $(\overline{N}, p)$ method if for every $\epsilon > 0$ there exist $n_0 = n_0(\epsilon)$ and $\lambda > 1$ such that for all $n \geq n_0$

$$u_n \succeq u_k - \bar{\epsilon} \quad \text{whenever} \quad \rho_n^\ell(\lambda) < k \leq n. \tag{6}$$

As a matter of fact, if we assume that (u_n) is slowly decreasing with respect to the $(\overline{N}, p)$ method, then there would exist an $\epsilon_1 > 0$ such that for all $\lambda > 1$ and $n_1 \in \mathbb{N}$, we have for $n \geq n_1$,

$$u_n \not\succeq u_k - \bar{\epsilon_1} \quad \text{whenever} \quad \rho_n^\ell(\lambda) < k \leq n.$$

Therefore, there exists $\alpha_0 \in [0, 1]$ such that

$$u_n^-(\alpha_0) < u_k^-(\alpha_0) - \epsilon_1 \quad \text{or} \quad u_n^+(\alpha_0) < u_k^+(\alpha_0) - \epsilon_1.$$

If we consider the case $u_n^-(\alpha_0) < u_k^-(\alpha_0) - \epsilon_1$, then we can see that condition of slow decrease with respect to the $(\overline{N}, p)$ method given for sequences of real numbers

$$\lim_{\lambda \to 1^+} \liminf_{n\to\infty} \min_{\rho_n^\ell(\lambda) < k \leq n} (u_n - u_k) \geq 0$$

is not satisfied for the sequence $(u_n^-(\alpha_0))$ of real numbers. In other words, $(u_n^-(\alpha_0))$ is not slowly decreasing with respect to the $(\overline{N}, p)$ method. Similarly, we can say that $(u_n^+(\alpha_0))$ is not slowly decreasing with respect to the $(\overline{N}, p)$ method. This contradicts the hypothesis that (u_n) is slowly decreasing with respect to the $(\overline{N}, p)$ method.

3. Main Result

In this section, we establish a Tauberian theorem which convergence follows from the $(\overline{N}, p)$ summability under condition of slow decrease with respect to the $(\overline{N}, p)$ method.

Theorem 3.1. *Let (u_n) be a sequence of fuzzy numbers. If (u_n) is $(\overline{N}, p)$ summable to $\mu_0 \in E^1$ and slowly decreasing with respect to the $(\overline{N}, p)$ method, then (u_n) converges to μ_0.*

Proof. Assume that $(u_n) \in \omega(F)$ is $(\overline{N}, p)$ summable to $\mu_0 \in E^1$ and slowly decreasing with respect to the $(\overline{N}, p)$ method. For each n such that $(\rho_n) \in \Lambda_u$,

$$\begin{aligned}
\frac{P_{\rho_n}}{P_{\rho_n} - P_n} t_{\rho_n} + t_n &= \frac{P_{\rho_n}}{P_{\rho_n} - P_n}\left(\frac{1}{P_{\rho_n}}\sum_{k=0}^{\rho_n} p_k u_k\right) + \frac{1}{P_n}\sum_{k=0}^{n} p_k u_k \\
&= \frac{1}{P_{\rho_n} - P_n}\left(\sum_{k=0}^{n} p_k u_k + \sum_{k=n+1}^{\rho_n} p_k u_k\right) + \frac{1}{P_n}\sum_{k=0}^{n} p_k u_k \\
&= \left(\frac{1}{P_{\rho_n} - P_n} + \frac{1}{P_n}\right)\sum_{k=0}^{n} p_k u_k + \frac{1}{P_{\rho_n} - P_n}\sum_{k=n+1}^{\rho_n} p_k u_k \\
&= \frac{P_{\rho_n}}{P_{\rho_n} - P_n}\left(\frac{1}{P_n}\sum_{k=0}^{n} p_k u_k\right) + \frac{1}{P_{\rho_n} - P_n}\sum_{k=n+1}^{\rho_n} p_k u_k \\
&= \frac{P_{\rho_n}}{P_{\rho_n} - P_n} t_n + \frac{1}{P_{\rho_n} - P_n}\sum_{k=n+1}^{\rho_n} p_k u_k. \qquad (7)
\end{aligned}$$

By using (3), we have

$$\begin{aligned}
\limsup_{n\to\infty} \frac{P_{\rho_n}}{P_{\rho_n} - P_n} &= \limsup_{n\to\infty} \frac{1}{1 - \dfrac{P_n}{P_{\rho_n}}} = \left\{\liminf_{n\to\infty}\left(1 - \frac{P_n}{P_{\rho_n}}\right)\right\}^{-1} \\
&= \left\{1 - \limsup_{n\to\infty}\frac{P_n}{P_{\rho_n}}\right\}^{-1} = \left\{1 - \frac{1}{\liminf\limits_{n\to\infty}\dfrac{P_{\rho_n}}{P_n}}\right\}^{-1} < \infty. \qquad (8)
\end{aligned}$$

Then by taking into account (8) and the assumed summability $(\overline{N},p)$ of (u_n) to μ_0, we obtain

$$\begin{aligned}\limsup_{n\to\infty} D\left(\frac{P_{\rho_n}}{P_{\rho_n}-P_n}t_{\rho_n}, \frac{P_{\rho_n}}{P_{\rho_n}-P_n}t_n\right) &= \limsup_{n\to\infty} \frac{P_{\rho_n}}{P_{\rho_n}-P_n}D(t_{\rho_n},t_n)\\ &= \limsup_{n\to\infty} \frac{P_{\rho_n}}{P_{\rho_n}-P_n}D(t_{\rho_n}+\mu_0,t_n+\mu_0)\\ &\le \limsup_{n\to\infty} \frac{P_{\rho_n}}{P_{\rho_n}-P_n}\left[D(t_{\rho_n},\mu_0)+D(t_n,\mu_0)\right]\\ &\le \limsup_{n\to\infty} \frac{P_{\rho_n}}{P_{\rho_n}-P_n}\limsup_{n\to\infty}\left[D(t_{\rho_n},\mu_0)+D(t_n,\mu_0)\right]\\ &\le \limsup_{n\to\infty} \frac{P_{\rho_n}}{P_{\rho_n}-P_n}\left[\limsup_{n\to\infty} D(t_{\rho_n},\mu_0)+\limsup_{n\to\infty} D(t_n,\mu_0)\right]\\ &\le \limsup_{n\to\infty} \frac{P_{\rho_n}}{P_{\rho_n}-P_n}\left[\lim_{n\to\infty} D(t_{\rho_n},\mu_0)+\lim_{n\to\infty} D(t_n,\mu_0)\right]=0.\end{aligned}$$

From this point of view, for large enough n and $\epsilon>0$ we can write

$$D\left(\frac{P_{\rho_n}}{P_{\rho_n}-P_n}t_{\rho_n}, \frac{P_{\rho_n}}{P_{\rho_n}-P_n}t_n\right)<\frac{\epsilon}{3}.$$

Thus, by Lemma 2.4, we obtain that

$$\begin{aligned}\frac{P_{\rho_n}}{P_{\rho_n}-P_n}t_n-\frac{\overline{\epsilon}}{3} &\preceq \frac{P_{\rho_n}}{P_{\rho_n}-P_n}t_{\rho_n}\\ &\preceq \frac{P_{\rho_n}}{P_{\rho_n}-P_n}t_n+\frac{\overline{\epsilon}}{3}.\end{aligned}\tag{9}$$

Since (u_n) is $(\overline{N},p)$ summable to μ_0, we have $\lim_{n\to\infty} D(t_n,\mu_0)=0$. Then, for each $\epsilon>0$, $D(t_n,\mu_0)<\dfrac{\epsilon}{3}$. Hence, by Lemma (2.4), we obtain

$$\begin{aligned}\mu_0-\frac{\overline{\epsilon}}{3} &\preceq t_n\\ &\preceq \mu_0+\frac{\overline{\epsilon}}{3}.\end{aligned}\tag{10}$$

In addition, since (u_n) is slowly decreasing with respect to the $(\overline{N},p)$ method, $(\rho_n)\in\Lambda_u$ and we may consider $\widetilde{\Lambda}_u$ instead of Λ_u by the mentioned construction, we get

$$\begin{aligned}\frac{1}{P_{\rho_n}-P_n}\sum_{k=n+1}^{\rho_n}p_k u_k &\succeq \frac{1}{P_{\rho_n}-P_n}\sum_{k=n+1}^{\rho_n}p_k(u_n-\frac{\overline{\epsilon}}{3})\\ &= u_n-\frac{\overline{\epsilon}}{3}.\end{aligned}\tag{11}$$

By using the identity (7) and combining (9), (10) and (11) we obtain for each $\epsilon > 0$ that

$$\begin{aligned}\frac{P_{\rho_n}}{P_{\rho_n}-P_n}t_n+\frac{2\overline{\epsilon}}{3}+\mu_0 \succeq \frac{P_{\rho_n}}{P_{\rho_n}-P_n}t_{\rho_n}+t_n &= \frac{P_{\rho_n}}{P_{\rho_n}-P_n}t_n+\frac{1}{P_{\rho_n}-P_n}\sum_{k=n+1}^{\rho_n}p_k u_k\\ &\succeq \frac{P_{\rho_n}}{P_{\rho_n}-P_n}t_n+u_n-\frac{\overline{\epsilon}}{3}.\end{aligned}$$

Therefore, by Lemma 2.5 we get

$$\overline{\epsilon}+\mu_0 \succeq u_n. \tag{12}$$

On the other hand, for each n such that $(\rho_n)\in\Lambda_\ell$,

$$\begin{aligned}\frac{P_{\rho_n}}{P_n-P_{\rho_n}}t_{\rho_n}+\frac{1}{P_n-P_{\rho_n}}\sum_{k=\rho_n+1}^{n}p_k u_k &= \frac{P_{\rho_n}}{P_n-P_{\rho_n}}\left(\frac{1}{P_{\rho_n}}\sum_{k=0}^{\rho_n}p_k u_k\right)\\ &\quad+\frac{1}{P_n-P_{\rho_n}}\sum_{k=\rho_n+1}^{n}p_k u_k\\ &=\frac{1}{P_n-P_{\rho_n}}\left(\sum_{k=0}^{\rho_n}p_k u_k+\sum_{k=\rho_n+1}^{n}p_k u_k\right)=\frac{1}{P_n-P_{\rho_n}}\sum_{k=0}^{n}p_k u_k\\ &=\frac{P_n}{P_n-P_{\rho_n}}t_n=\left(1+\frac{P_{\rho_n}}{P_n-P_{\rho_n}}\right)t_n\\ &=\frac{P_{\rho_n}}{P_n-P_{\rho_n}}t_n+t_n.\end{aligned} \tag{13}$$

By using (4), we get

$$\limsup_{n\to\infty}\frac{P_{\rho_n}}{P_n-P_{\rho_n}}=\limsup_{n\to\infty}\frac{1}{\dfrac{P_n}{P_{\lambda_n}}-1}=\left\{\liminf_{n\to\infty}\frac{P_n}{P_{\lambda_n}}-1\right\}^{-1}<\infty \tag{14}$$

Then going through the similar process above and by taking into account (14), we get

$$\lim_{n\to\infty}D\left(\frac{P_{\rho_n}}{P_n-P_{\rho_n}}t_{\rho_n},\frac{P_{\rho_n}}{P_n-P_{\rho_n}}t_n\right)=0.$$

Hence, for sufficiently large n and $\epsilon>0$, we can write

$$D\left(\frac{P_{\rho_n}}{P_n-P_{\rho_n}}t_{\rho_n},\frac{P_{\rho_n}}{P_n-P_{\rho_n}}t_n\right)<\frac{\epsilon}{3}.$$

So, By Lemma (2.4), we obtain that

$$\frac{P_{\rho_n}}{P_n-P_{\rho_n}}t_{\rho_n}-\frac{\overline{\epsilon}}{3}\preceq\frac{P_{\rho_n}}{P_n-P_{\rho_n}}t_n\preceq\frac{P_{\rho_n}}{P_n-P_{\rho_n}}t_{\rho_n}+\frac{\overline{\epsilon}}{3}. \tag{15}$$

Furthermore, since (u_n) is slowly decreasing with respect to the $(\overline{N},p)$ method, $(\rho_n)\in\Lambda_\ell$ and we may consider $\widetilde{\Lambda}_\ell$ instead of Λ_ℓ by the mentioned construction, we have

$$\frac{1}{P_n-P_{\rho_n}}\sum_{k=\rho_n+1}^{n}p_k u_k\preceq\frac{1}{P_n-P_{\rho_n}}\sum_{k=\rho_n+1}^{n}p_k(u_n+\frac{\overline{\epsilon}}{3})=u_n+\frac{\overline{\epsilon}}{3}. \tag{16}$$

Combining (10),(15) and (16), we obtain by the identity (13) that, for each $\epsilon > 0$

$$\begin{aligned}\frac{P_{\rho_n}}{P_n - P_{\rho_n}} t_{\rho_n} - \frac{2\overline{\epsilon}}{3} + \mu_0 \preceq \frac{P_{\rho_n}}{P_n - P_{\rho_n}} t_n + t_n &= \frac{P_{\rho_n}}{P_n - P_{\rho_n}} t_{\rho_n} + \frac{1}{P_n - P_{\rho_n}} \sum_{k=\rho_n+1}^{n} p_k u_k \\ &\preceq \frac{P_{\rho_n}}{P_n - P_{\rho_n}} t_{\rho_n} + u_n + \frac{\overline{\epsilon}}{3}.\end{aligned}$$

Therefore, by Lemma 2.5, we get

$$\mu_0 - \overline{\epsilon} \preceq u_n. \tag{17}$$

Combining (12) and (17), for each $\epsilon > 0$ we obtain

$$\mu_0 - \overline{\epsilon} \preceq u_n \preceq \overline{\epsilon} + \mu_0.$$

By Lemma 2.4, we conclude that

$$D(u_n, \mu_0) \leq \epsilon.$$

This implies that $\lim_{n\to\infty} u_n = \mu_0$. □

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