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On solutions of functional equations with linear translations

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Abstract. In this paper we study the polynomial functional equations of the form $af(a_1x + a_0) + bf(b_1x + b_0) = g(x)$, where $g(x)$ is a polynomial of the degree $n \geq 0$. Theorem 2.3 affirms that the given equation has a unique polynomial solution provided if $aa_1^i + bb_1^i \neq 0$ for each integer $i \geq 0$. Other non-polynomial solution depends on solutions of the homogeneous equation $af(a_1x + a_0) + bf(b_1x + b_0) = 0$.

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1. Introduction

Theory of functional equations is a large and important domain of mathematics [1, 4, 5, 7]. Formally, a functional equation is a relation between concrete variables where some variables are functions or functions with their derivatives. Some properties of solutions of a given differential equation may be determined without finding their exact form [8]. We study the functional equations without derivatives of functions. One of the general forms of a functional equation is the following:

$F(x_1, x_2, ..., x_n, f(g_1(x_1, x_2, ..., x_n)), ..., f(g_m(x_1, x_2, ..., x_n))) = 0$,

where $g_1, g_2, ..., g_m$ are given functions. A solution of this equation is a function $f(x)$, which satisfies the functional equation.

Our aim is to analyze the functional equations with linear translations of the form:

$af(a_1x + a_0) + bf(b_1x + b_0) = g(x)$,

where $g(x)$ is a given function and a, b, a_1, b_1, a_0, b_0 are given numbers. The equations of that form are subject of interest for mathematical competitions of diverse level. There are many books which analyze concrete functional equations, but propose only outline of solving ways [3, 5, 6].

We present a general approach of solving functional equations with linear translations. We study the case when the function $g(x)$ is a polynomial. This approach is important from the didactic point of view. The process of composing problems has the goal of forming the capability to analyze notions and their properties, of consolidating knowledge, of creating premises for their application, of developing school students' mathematical creativity etc. The notions of equation and function are fundamental in the course of elementary mathematics and encapsulate a rich potential for solving problems of an inter- and trans-disciplinary character. The general approach allows the structuring of the algorithm of composing concrete equations, whose "spicy" character arises from the way of selecting the coefficients a, a_1, b, b_1 and the

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degrees of polynomial $g(x)$. Some problems of this type sometimes are declared as "trick problems". The polynomials are considered on the field of reals $\mathbb{R}$. But the main results are true for the field of complex number too and, more general, for topological commutative fields of characteristic zero.

2. Equations with linear translations

Fix an equation

$$af(a_1x + a_0) + bf(b_1x + b_0) = g(x), \tag{1}$$

where $g(x)$ is a polynomial.

The problem consists of finding the solutions $f(t)$ of equation (1). The equation (1) determines the homogeneous equation

$$af(a_1x + a_0) + bf(b_1x + b_0) = 0. \tag{2}$$

Let S be the set of solutions of the equation (1) and S_0 the set of solutions of the equation (2). Obviously, $S_0 \neq \emptyset$, since $0 \in S_0$.

The following elementary fact establishes the relation between S and S_0.

Proposition 2.1. *If $S \neq \emptyset$ and $f_0 \in S$, then $S = \{f_0 + h : h \in S_0\}$.*

Corollary 2.2. *Either $S = \emptyset$, or $|S| = |S_0|$.*

If $b = 0$ and $aa_1 \neq 0$, then $f(x) = a^{-1}g(a_1^{-1}(x - a_0))$ is the unique solution of the equation (1).

Assume that $aba_1 \neq 0$ and $b_1 = 0$. In this case, equation (1) has the form $af(a_1x + a_0) + bf(b_0) = g(x)$. Putting $t = a_1x + a_0$, we obtain $af(t) + bf(b_0) = g(a_1^{-1}(t - a_0))$ and $(a + b)f(b_0) = g(a_1^{-1}(b_0 - a_0))$. If $a + b = 0$ and $g(a_1^{-1}(b_0 - a_0)) \neq 0$, then the equation (1) doesn't have any solution. If $a + b = 0$ and $g(a_1^{-1}(b_0 - a_0)) = 0$, then $f(t) = a^{-1}g(a_1^{-1}(t - a_0)) + c$, where c is an arbitrary constant, is the general solution and the equation (1) has an infinite number of solutions. If $a + b \neq 0$, then $f(b_0)$ $= (a + b)^{-1}g(a_1^{-1}(b_0 - a_0))$ and $f(t) = a^{-1}g(a_1^{-1}(t - a_0)) - a^{-1}bf(b_0)$ is the unique solution of the equation (1).

Hence, is important the case $aba_1b_1 \neq 0$ and $a + b \neq 0$.

Let $\Delta_n = aa_1^n + bb_1^n$, $n \in \mathbb{N} = \{0, 1, 2, ...\}$.

One of the main results is the following theorem.

Theorem 2.3. *Let $g(x) = c_nx^n + c_{n-1}x^{n-1} + ... + c_1x + c_0$ ($c_n \neq 0$) be a polynomial of degree $\deg(g) = n \geq 0$ and $m \geq n$. If $\Delta_i \neq 0$ for any $i \leq m$ then in the class $P(m)$ of all polynomials of the degree $\leq m$ there exists a unique polynomial solution $f(x)$ of the equation (1). The polynomial degree $\deg(f) = n$.*

Proof. Assume that the polynomial $f(x) = e_px^p + e_{p-1}x^{p-1} + ... + e_1x + e_0$ is a solution of the equation (1) and $p \leq m$. Then $af(a_1x + a_0) = ae_pa_1^px^p + d_{p-1}x^{p-1} + ... + d_1x + d_0$ and $bf(b_1x + b_0) = be_pb_1^px^p + l_{p-1}x^{p-1} + ... + l_1x + l_0$.

If $p > n$, then $e_p(aa_1^n + bb_1^n) = 0$ and $e_p = 0$.

Thus, we can assume that $p = n$. In this case, $e_n = (aa_1^n + bb_1^n)^{-1}c_n$. Newton's binomial formulas permit to calculate in a unique way all coefficients e_j for $j \leq n$. The proof is complete.

If $\Delta_i = 0$ for some $i \leq n$, then the existence of the polynomial solutions depends on the case. The method from the proof of the theorem allows to obtain all polynomial solutions of the equation (1). $\square$

Further, we propose concrete examples in which the polynomial $g(x)$ has different degrees.

Example 2.1. Consider the equation $4f(x-3) - f(2x+1) = x$, where $\deg(g) = 1$. We find that $\Delta_0 = 4\cdot 1^0 - 1\cdot 2^0 = 3 \neq 0$, $\Delta_1 = 4\cdot 1^1 - 1\cdot 2^1 = 2 \neq 0$, $\Delta_2 = 4\cdot 1^2 - 1\cdot 2^2 = 0$ and $\Delta_i = 4\cdot 1^i - 1\cdot 2^i = 4 - 2^i < 0$, for all $i \geq 3$. Assume that the polynomial $f(x) = e_2x^2 + e_1x + e_0$ is a solution of the given equation. It follows that $4f(x-3) - f(2x+1)$ $= 4e_2x^2 + (-24e_2 + 4e_1)x + 36e_2 - 12e_1 + 4e_0 - (4e_2x^2 + (4e_2 + 2e_1)x + e_2 + e_1 + e_0)$ and $0e_2x^2 + (-28e_2 + 2e_1)x + (35e_2 - 13e_1 + 3e_0) = x$. Fix $e_2 = c$. Then $e_1 = \frac{1}{2}(28c + 1)$ and $e_0 = 49c + \frac{13}{6}$. In this case, the equation has an infinite number of polynomial solutions of the degree ≤ 2: $S(2) = \{cx^2 + (14c + \frac{1}{2})x + 49c + \frac{13}{6} \mid c \in R\}$. The function $f(x) = \frac{1}{2}x + \frac{13}{6}$ is the unique solution of the degree 1. The equation has not polynomials solutions of the degree ≥ 3.

Example 2.2. Consider an equation where $\deg(g) = 2$, for instance $4f(x-3) - f(2x+1) = x^2 + x - 2$. In this case, $\Delta_0 = 3 \neq 0$, $\Delta_1 = 2 \neq 0$, $\Delta_2 = 4\cdot 1^2 - 1\cdot 2^2 = 0$ and $\Delta_i = 4\cdot 1^i - 1\cdot 2^i = 4 - 2^i < 0$ for all $i \geq 3$. If $f(x) = e_2x^2 + e_1x + e_0$ is a solution, we have $4f(x-3) - f(2x+1) = 40e_2x^2 + (-28e_2 + 2e_1)x + (35e_2 - 13e_1 + 3e_0) = x^2 + x - 2$. We obtain the contradiction $0 \cdot e_2 = 1$. Since $\Delta_i \neq 0$ for $n \geq 3$, then $e_n\Delta_n = 0$ if and only only if $e_n = 0$ and the equation has not polynomials solutions.

Example 2.3. Consider the equation $4f(x) - f(2x) = g(x)$. In this case, we obtain: $\Delta_0 = 3 \neq 0$, $\Delta_1 = 2 \neq 0$, $\Delta_2 = 0$ and $\Delta_i = 4 - 2^i < 0$ for any $i \geq 3$. Fix some polynomial solution $f(x) = e_3x^3 + e_2x^2 + e_1x + e_0$ with $\deg(f) = 3$. If $g(x) = c_3x^3 + c_2x^2 + c_1x + c_0$, then $4f(x) - f(2x) = -4e_3x^3 + 0e_2x^2 + 2e_1x + 3e_0$. Therefore $e_3 = -\frac{1}{4}c_3$, $0 \cdot e_2 = c_2$, $e_1 = \frac{1}{2}c_1$, $e_0 = \frac{1}{3}c_0$.

For this example we have the following conclusions.

Conclusion 2.1. *For $c_2 = 0$ there exists an infinite number of solutions of degree 3:*

$$f(x) = -\frac{1}{4}e_3x^3 + e_2x^2 + \frac{1}{2}e_1x + \frac{1}{3}e_0.$$

Conclusion 2.2. *For $c_2 \neq 0$ the equation has no polynomial solutions.*

Conclusion 2.3. *Any polynomial solution $f(x)$ of the given equation has* $\deg(f) \leq 3$.

Remark 2.1. We have a similar situation in the case when $g(x)$ is a polynomial of degree ≥ 4.

Example 2.4. Consider the equation $4f(x-1) - f(2x) = g(x)$. In this case, we obtain: $\Delta_0 \neq 0$, $\Delta_1 \neq 0$, $\Delta_2 = 0$ and $\Delta_i = 4 - 2^i < 0$ for any for all $i \geq 3$. Any polynomial solution is of the order $\leqslant 3$.

Let $g(x) = c_3x^3 + c_2x^2 + c_1x + c_0$. Fix some polynomial solution $f(x) = e_3x^3 + e_2x^2 + e_1x + e_0$ with $\deg(f) = 3$. We obtain $f(2x) = 8c_3x^3 + 4c_2x^2 + 2e_1x + e_0$ and $4f(x-1) - f(2x) = -4e_3x^3 - 12e_2x^2 + (2e_1 - 8e_2 + 12e_3)x + (3e_0 - 4e_1 + 4e_2 - 4e_3)$. Therefor $-4e_3 = c_3$, $0 \cdot e_2 - 12e_3 = c_2$, $2e_1 - 8e_2 + 12e_3 = c_1$, $3e_0 - 4e_1 + 4e_2 - 4e_3$ $= c_0$.

If $c_2 = 3c_3$, then the equation has an infinite number of polynomial solutions of degree 3 with $e_3 = -\frac{1}{4}c_3$, e_2 - arbitrary, $e_1 = \frac{1}{2}(c_1 + 8e_2 - 12e_3)$, $e_0 = \frac{1}{3}(c_0 + 4e_1 - 4e_2 + 4e_3)$.

If $c_2 \neq 3c_3$, then the equation has no solutions.

Example 2.5. Consider the equation $2f(x+1)+f(-2x) = g(x)$. We obtain: $\Delta_0 \neq 0$, $\Delta_1 = 2 \cdot 1^1 + 1 \cdot (-2)^1 = 0$ and $\Delta_i = 2 \cdot 1^i - 1 \cdot (-2)^i = 2 + (-2)^i \neq 0$, $i \geq 2$.

If $g(x) = c_4x^4 + c_3x^3 + c_2x^2 + c_1x + c_0$, then any polynomial solution is of the order $\leqslant 4$.

Let $f(x) = e_4x^4 + e_3x^3 + e_2x^2 + e_1x + e_0$ be some polynomial solution. We obtain $2f(x+1) = 2e_4x^4 + (8e_4 + 2e_3)x^3 + (12e_4 + 6e_3 + 2e_2)x^2 + (8e_4 + 6e_3 + 4e_2 + 2e_1)x + 2(e_4 + e_3 + e_2 + e_1 + e_0)$, $f(2x) = 16e_4x^4 - 8e_3x^3 + 4e_2x^2 - 2e_1x + e_0$ and $2f(x+1) + f(-2x) = 18c_4x^4 + (8e_4 - 6e_3)x^3 + (12e_4 + 6e_3 - 2e_2)x^2 + (8e_4 + 6e_3 + 4e_2 + 0 \cdot e_1)x + (2e_4 + 2e_3 + 2e_2 + 2e_1 + 3e_0)$.

Hence, we have the relations: $18e_4 = c_4$, $8e_4 - 6e_3 = c_3$, $12e_4 + 6e_3 - 2e_2 = c_2$, $8e_4 + 6e_3 + 4e_2 + 0 \cdot e_1 = c_1$, $2e_4 + 2e_3 + 2e_2 + 2e_1 + 3e_0 = c_0$.

Therefore $e_4 = \frac{1}{18}c_4$, $e_3 = \frac{2}{27}c_4 - \frac{1}{6}c_3$, $e_2 = \frac{5}{9}c_4 - \frac{1}{2}c_3 - \frac{1}{2}c_2$, $c_1 = \frac{28}{9}c_4 - 3c_3 - 2c_2 + 0 \cdot e_1$, e_1 - is arbitrary and $e_0 = -\frac{37}{81}c_4 + \frac{4}{9}c_3 + \frac{1}{3}c_2 + \frac{2}{3}e_1 - \frac{1}{3}c_0$.

If $c_1 = \frac{28}{9}c_4 - 3c_3 - 2c_2$, then the equation has an infinite number of polynomial solutions of degree 4.

If $c_1 \neq \frac{28}{9}c_4 - 3c_3 - 2c_2$, then the equation has no solution.

Remark 2.2. We have similar situations in the case when $g(x)$ is a polynomial of the degree > 4.

3. Periodicity and non-polynomial solutions

Consider the functional equation (1) where $g(x)$ is a polynomial and $|a_1| = |b_1|$. If $\Delta_0 = a + b \neq 0$ and $\Delta_1 = aa_1 + bb_1 \neq 0$, then $\Delta_i \neq 0$ for each $i \in \mathbb{N}$ and the functional equation has a unique polynomial solution. In other cases, the functional equation either has an infinite number of polynomial solutions or has no polynomial solution.

Let $f_0(x)$ be a given polynomial solution of the functional equation (1).

Consider the homogeneous functional equation (2). After the substitution $t = a_1x + a_0$ we obtain the equation $af(t) + bf(\frac{b_1}{a_1}t + b_0 - \frac{a_0b_1}{a_1}) = 0$.

We set $c = \frac{b_1}{a_1}$, $k = -\frac{b}{a}$ and $d = b_0 - ka_0$.

In this case, the equation (2) has the form

$$f(t) = kf(ct + d) \tag{3}$$

If in the equation (3) we have $c = \pm 1$, we can distinguish few cases:

1. $c = 1$ and $d = 0$;
2. $c = -1$, $d = 0$;
3. $c = 1$ and $d \neq 0$;
4. $c = -1$, $d \neq 0$ and $k = 1$;
5. $c = -1$, $d \neq 0$ and $k = -1$;
6. $c = -1$, $d \neq 0$ and $k \notin \{-1, 1\}$;
7. $c = 1$, $d \neq 0$ and $k \notin \{-1, 1\}$.

Case 1. $c = 1$ and $d = 0$.

If $k = 1$ the equation (3) has the form $f(t) = f(t)$ and has an infinite number of solutions.

If $c = 1$, $k \neq 1$ and $d = 0$, the function $f(x) = 0$ is the unique solution of the functional equation (1).

Case 2. $c = -1$, $d = 0$. In this case equation (3) has the form $f(t) = kf(-t)$.

If $k = 1$, any even (symmetric) function is a solution of equation (3).

We denote by F_1 the class of all functions with the property $f(t) = f(-t)$.

Conclusion 3.1. *If $k = -1$, $c = -1$ and $d = 0$, then $F = \{f_0 + f : f \in F_1\}$ is the class of all solutions of the functional equation* (1).

For $k = -1$ we obtain $f(-t) = -f(t)$ that any odd function is a solution of equation (3).

We denote by F_{-1} the class of all functions with the property $f(-t) = -f(t)$.

Conclusion 3.2. *If $k = -1$, $c = -1$ and $d = 0$, then $F = \{f_0 + f : f \in F_{-1}\}$ is the class of all the solutions of the functional equation* (1).

Conclusion 3.3. *For $k \notin \{-1, 1\}$, $d = 0$, $c \notin \{-1, 1\}$ the function $f = 0$ is the unique solution of the functional equation (2) and the functional equation* (1) *has a unique polynomial solution f_0.*

Case 3. $c = 1$ and $d \neq 0$.

In this case, equation (3) has the form $f(t) = kf(t + d)$.

On the semi-interval $I_0 = [0, d)$ fix some function h_0. Let $I_n = [nd, (n + 1)d)$, $n \in Z = \{0, \pm 1, \pm 2, ...\}$. On I_n we construct the function $h_n(x) = k^{-n} f_0(x - nd)$ for any $x \in I_n$. We say that such function h is (d, k)-periodic.

Let $F_{(k,d)} = \{f : f(t) \text{ be } (d, k)\text{-periodic}\}$.

If $k = 1$, then the $(d, 1)$-periodic function is periodic with period d.

Conclusion 3.4. *If $c = 1$, $d \neq 0$, then $F = \{f_0 + f : f \in F_{(k,d)}\}$ is the class of all solutions of the functional equation* (1).

Case 4. $c = -1$, $d \neq 0$ and $k = 1$.

In this case, equation (3) has the form $f(t) = f(-t + d)$.

Consider the intervals $I_n = (nd, (n + 1)d)$, where $n \in Z$.

Let $h(x)$ be a function on the space of reals $\mathbb{R}$ with the properties: $h(0) = h(d)$ and $h((n + 1)d) = h(-nd)$ for each $n \in \mathbb{Z} = \{0, \pm 1, \pm 2, ...\}$; if $x \in (0, d) = I_0$, then $h(x) = h(d - x)$; if $n \in \mathbb{N}$, then h is constructed on $(nd, (n + 1)d) = I_n$ in an arbitrary way; if $n \in \mathbb{N}$ and $x \in I_{-n-1} = ((-n - 1)d, -nd)$, then $h(x) = h(-x + d) \in I_n$.

Since $-(-x + d) + d = x$, the function h is defined correctly as the solution of the equation $f(t) = f(-t+d)$. Let F_d be the class of such functions. Then F_d is the class of all solutions of the equation (2).

Conclusion 3.5. *If $c = -1$, $d \neq 0$ and $k = 1$, then $F = \{f_0 + h : h \in F_d\}$ is the class of all solutions of the equation* (1).

Case 5. $c = -1$, $d \neq 0$ and $k = -1$.

In this case, $f(t) = -f(-t + d)$. As in the case 4, we denote $I_n = (nd, (n + 1)d)$, where $n \in Z$.

On the space of reals $\mathbb{R}$ consider the function h defined as follows:

- $h(nd) = -h(-nd + d)$ for any $n \in \mathbb{Z}$;
- if $t \in I_0$, then $h(d - t) = -h(t)$ and $h(\frac{d}{2}) = 0$;
- if $n \in N$, then the restriction $h \mid I_n$ is any arbitrary function on I_n;
- if $t \in I_{-n-1} = ((-n - 1)d, -nd)$, then $h(t) = -h(-t + d)$.

The function h is correctly defined. The class F_d of such functions is the class of all solutions of the equation (2), and $F = \{f_0 + h : h \in F_d\}$ is the class of all solutions of the equation (1).

Case 6. $c = -1$, $k \notin \{-1, 1\}$ and $d \neq 0$.

In this case, $f(t) = kf(-t + d)$.

Fix a number $t \in \mathbb{R}$. We put $\lambda(t) = -t + d$. We observe that $\lambda(\lambda(t)) = t$. The equation (3) has the form $f(t) = kf(\lambda(t))$. Since $\lambda(\lambda(t)) = t$ we observe that $f(t) = kf(\lambda(t)) = k \cdot kf(\lambda(\lambda(t))) = k^2 f(t)$. Since $k \notin \{-1, 1\}$, then $f(t) = 0$. Hence, in this case $h = 0$ is the unique solution of the equation (2) and the equation (1) has only polynomial solutions.

Example 3.1. Consider the equation $f(2x+2)-f(2x) = 24x^2+16x+2$. In this case, we obtain: $\Delta_n = 1 \cdot 2^n - 1 \cdot 2^n = 0$ for each $n \geq 0$. After the substitution $t = 2x$ we obtain the equation $f(t+2)-f(t) = 6t^2+8t+2$. Let $f(t) = pt^3+qt^2+rt+s$ be a solution of the equation. Since $f(t+2) = pt^3+(6p+q)t^2+(12p+4q+r)t+(8p+4q+2r+s)$ we obtain $f(t+2)-f(t)$ $=6pt^2+(12p+2q)t+(8p+4q+2r)$. Hence $p = 1$, $q = -1$, $r = -1$ and $f(x) = x^3 - x^2 - x + s$ is a polynomial solution of the degree 3. The equation has not solutions of the degree ≤ 2. Assume that the function $g(x)$ is a solution of the given equation. Then $h(x) = g(x) - f(x)$ is a solution of the homogeneous equation $h(x + 2) - h(x) = 0$ and $h(x + 2) = h(x)$. Hence $h(x)$ is a periodic function of the period 2. If $h(x)$ is given on the semi-interval $[0, 2)$, then $h(x)$ and $g(x)$ are determined uniquely.

Example 3.2. Consider the equation $f(2x+2)+f(2x) = 6x^2+4x+2$. In this case, we obtain: $\Delta_n = 1 \cdot 2^n + 1 \cdot 2^n = 2^{2+1} > 0$ for each $n \geq 0$. After the substitution $t = 2x$ we obtain the equation $f(t + 2) - f(t) = t^2 + 2t + 2$. Let $f(t) = pt^2 + qt + r$ be a solution of the equation. Since $f(t + 2) = pt^2 + (4p + q)t + (4p + 2q + r)$ we obtain $f(t + 2) + f(t)$ $=2pt^2 + (4p + 2q)t + (4p + 2q + 2r)$, $p = 3$, $q = -4$, $r = -1$. Hence $f(x) = 3x^2 - 4x - 1$ is the unique polynomial solution of the given equation. If $g(x)$ is a given solution of the equation, then $h(x) = g(x) - f(x)$ is a solution of the homogeneous equation $h(x + 2) + h(x) = 0$ and $h(x + 2) = -h(x)$. Hence $h(x)$ is a periodic function of the period 4. On the interval $[-2, 2]$ we have $h(x) = -h(x + 2)$. If h_0 is constructed on $[-2, 0)$ then for $x \in [0, 2)$ we put $h_0(x) = -h(x - 2)$. Then from $h(x + 4) = h(x)$ we determine the whole function $h(x)$. For instance, if $h_0(x) = x(x + 2)$ for any $x \in [-2, 0)$, then $h_0(x) = -x(x - 2)$. Then from $h(x) = h(x + 4)$ it follows that $h(x)$ and $g(x)$ are determined uniquely. In this case the function $h(x)$ and $g(x)$ are continuous and $g(x) = 2x^2 - 2x - 1$ for any $x \in [0, 2)$ and $g(x) = 4x^2 - 6x - 1$ for $x \in [-2, 0)$.

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