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Some numerical results with a selective eddy-viscosity model for Large-Eddy Simulations

Delia Jiroveanu, Georges-Henri Cottet, and Bertrand Michaux

Abstract. We present some numerical simulations with a selective version of the original Smagorinsky model on the 3D vortex tubes reconnection problem and the decaying 3D incompressible isotropic turbulence case. The selective procedure used here follows an idea proposed by the mecanicists [7] that we interpret through a mathematical criterium suggested by Constantin & Fefferman [5] concerning the regularity of the vorticity field direction. Numerical results show the efficiency of the selective model over the original model.

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1. Introduction

The understanding of turbulence, and implicitly, its prediction and control represent one of the most challenging problem in science and engineering.

It is generally accepted that the Navier-Stokes equations describe accurately the motion of every incompressible viscous fluids. However, the mathematical theory of these equations is still incomplete, the existence for all time of strong solutions or the uniqueness of weak solutions in the three dimensional case, beeing classical examples of open problems.

One natural computational method for solving these equations is to procede to a direct numerical simulation (DNS), in which all the scales of motion including the energy dissipation scales are explicitly computed. The main difficulty with this method is that its "computational cost" for typical engineering applications is too high for the available computing ressources. Indeed, since the total number of degrees of freedom necessary to represent a turbulent flow is of the order of $Re^{9/4}$ in three-dimension, where Re is the turbulent Reynolds number, it is clear that the direct simulation of all scales of high Reynolds number flows is actually impractical[1]. However, DNS is a very useful tool for studying simple flows at low and moderate Reynolds numbers, and also in turbulence research, where it is used especially for the validation of turbulence closure models.

A popular alternative to DNS is the technique of Large Eddy Simulation (LES), which proposes to compute only the large scales of the motion, larger than some cutoff wavenumber, and to model the effect of the small eddies, which are not very dependent on the geometry of the considered flow. The basic idea of LES is to define a large scale field through a filtering operation on the flow variables. Thus, the LES equations are

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[1]see [18] for a useful discussion about computational requirements in DNS

formaly obtained by filtering the Navier- Stokes equations. The decomposition of the flow variables into a space-averaged mean flow (large scales) and a perturbation field (small scales), induces in the Navier-Stokes equations a closure problem, caused by the nonlinear interactions between the large scales and the small ones. The main goal of LES is to accurately model the effect of small scales (subgrid scales) on the dynamics of large scales. Many LES models have been proposed along the time. Particularily in recent years, there was a significant progress toward the development of turbulence models which *should* guaranty the accuracy of the most physically relevant quantities for real applications. An overview of the classical LES models, with their practical capabilities and weaknesses, can be found in [6]. Since the LES models are supposed to assure the closure of the filtred Navier-Stokes equations, it is natural to think that they must ensure a certain regularity of the solution for these PDE and hence they should solve the uniqueness problem of the weak solution. For almost all LES models, no satisfactorily mathematical theory has yet been found. In [6] we show in a simple manner that a LES model of p-laplacian type (the Smagorinsky model, which will be presented in the following section) represents a good regularization for the Navier-Stokes equations, in the sense that it "solves" the uniqueness question. For further works on the mathematical properties of this model we refer to [11], [8]. On the other hand, it is known that in practice, the Smagorinsky model turns out to be too dissipative, especially close to walls. One possibility to remediate this drawback is to compute "dynamically" the constant of the model during the simulation in terms of the resolved velocity. It is the idea of dynamical models which gives generally good results [9]. Another approach consists in shrinking the support of the eddy viscosity of the original Smagorinsky model, by adding "dissipation" only in regions of the flow characterized by important values of the vorticity magnitude and a strong variability of the vorticity field direction. This idea, suggested by Bartello (1993) was firstly implemented in the case of the selective structure model (see [7]). We implemented it in the case of the Smagorinsky model, giving also a mathematical justification of the proposed selective version.

The aim of this paper is to present numerical simulations with the selective version of the Smagorinsky model, showing its efficiency by comparisons with the classical Smagorinsky model and reference DNS results.

The outline of this paper is as follows. In section 2 we recall the LES equations and we briefly present the concept of subgrid scale modeling. A selective variant of the original Smagorinsky model is presented and justified through a mathematical criterium suggested by Constantin and Fefferman as hypothesis which assures the regularity, and hence existence of strong solutions for 3D Navier-Stokes equations [5]. Numerical results with the proposed model on two test cases: the 3D vortex tubes reconnection and the decaying grid turbulence case are presented in section 3. Some concluding remarks are finally proposed.

2. Subgrid Scale Modeling. Eddy-Viscosity Models

As we previously said, the idea of Large-Eddy Simulation is to use a spatial filter in order to separate the large and small eddies which coexist in a turbulent flow. Usually, the filtred variables of the flow (filtred velocity and pressure) are defined by convolution with a smoothing kernel. Assuming that the considered filter commutes with the differentiation operators, and applying it to the Navier-Stokes equations, we obtain a set of PDE which describes the motion of large scales in a turbulent flow (LES equations) in a spatial domain $\Omega \subset \mathbf{R}^3$ and for a time $t > 0$

$$\partial_t \overline{u}_i + \partial_j(\overline{u}_i \overline{u}_j) + \partial_i p = \partial_j(\nu \partial_j \overline{u}_i + T_{ij}) \tag{1}$$

$$\partial_i \overline{u}_i = 0. \tag{2}$$

with as unknows the resolved (filtred) velocity $\overline{u}$ and pressure $\overline{p}$. ν is the kinematic viscosity (we consider here the density of the fluid $\rho = 1$) and (T_{ij}) the subgrid scale tensor which "comes" from the filtering of the convection term in the Navier-Stokes equations

$$T_{ij} = \overline{u}_i \overline{u}_j - \overline{u_i u_j}.$$

Usually, this tensor is splitted into a trace free component

$$\tau_{ij} = T_{ij} - T_{kk}\delta_{ij}/3 \text{ where } \delta \text{ is the Kronecker symbol}$$

and an isotropic one that is absorbed into a modified pressure P

$$P = \overline{p} - T_{kk}/3.$$

The LES equations become in this case

$$\partial_t \overline{u}_i + \partial_j(\overline{u}_i \overline{u}_j) + \partial_i P = \partial_j(\nu \partial_j \overline{u}_i + \tau_{ij}) \tag{3}$$

$$\partial_i \overline{u}_i = 0 \tag{4}$$

in which the modified subgrid stress τ has to be modeled. An important class of subgrid scales models (SGS) are the eddy viscosity models based on the Boussinesq's hypothesis. This assumes that the SGS stress tensor is proportional to the filtered strain-rate tensor $\overline{S}$. Then, τ is represented as

$$\tau_{ij} = 2\nu_t \overline{S}_{ij}$$

where ν_t is the eddy-viscosity, which is not a constant and must be specified.

One of the most popular models for Large Eddy Simulation is the Smagorinsky model [20] for which the eddy viscosity is assumed to be proportional to the filtred strain rate $|\overline{S}|$ $(= \sqrt{2\overline{S}_{ij}\overline{S}_{ij}})$

$$\nu_t^{sm} = (C_{sm}\Delta)^2 |\overline{S}|. \tag{5}$$

Δ is a characteristic length scale and C_{sm} the model constant which depends on the characteristics of the flow. It is usually evaluated so that the model reproduces the Kolmogorov spectrum in simulations of the homogeneous isotropic turbulence. We refer to [17] for a discussion on the evaluation of this parameter.

It is known that in this form, the classical Smagorinsky model performs rather well for isotropic turbulence, but it does not predict satisfactorily the transition to turbulence. To remedy its excessive dissipation, dynamical models were proposed with a model constant computed as function of time and space, in terms of the resolved scales. These models have been succesfully applied to various kinds of flows, but in order to provide strict dissipation it requires, in general, to assume flow homogeneity in at least one direction and thus does not readily apply to general geometries.

Another possibility to improve the qualities of this model, consists in applying it only in regions of high values of the magnitude of the vorticity where the vorticity field direction is not "regular enough". This can be done by multiplying the eddy viscosity with a filter able to detect the "turbulent regions" into a flow (we explain later how we construct this filter). This procedure was already implemented in the case of the structure function model (see [7]). One can understand this practical idea in the light of a sharp result obtained by Constantin and Fefferman [5] which gives a necessary condition for the loss of regularity in the 3D Navier- Stokes equations. It states that "if in regions where the magnitude of the vorticity at locations x and y (at time t) exceeds a certain value $\tilde{\Omega}$, the sinus of the angle formed by the unit vectors of direction of vorticity at x and y is small enough, then the solution of the initial value problem for the Navier-Stokes equations is strong and hence smooth on the time interval $[0, T]$".

Therefore, the idea of the selective variant of Smagorinsky's model is to switch off the eddy viscosity in regions where *the vorticity field is coherent enough* [2].

In practice we used the following algorithm for defining a filter "detector of turbulence at small scales": we compute the angle between the vorticity $\omega(x,t)$ at a given grid point and the average vorticity at the six closest neighbouring points $\tilde{\omega}(x,t)$ by

$$\beta(x,t) = arcsin \frac{\omega(x,t) \times \tilde{\omega}(x,t)}{|\omega(x,t)||\tilde{\omega}(x,t)|}. \tag{6}$$

And we define the function filter as

$$\Psi(\beta) = \chi(\mathcal{C}\{\beta / \sin\beta < \sin\beta_0\}) \tag{7}$$

where β_0 is a threshold angle such that $\sin\beta_0$ is small enough. For practical computations we choose β_0 to be the most probable value according to DNS of isotropic turbulence simulations at a resolution of 128^3 which give a value arround $15°$. (χ denotes the characteristic function of a considered set and $\mathcal{C}A$ the complementary of a given set A).

Finally, we define the eddy viscosity of the selective model by

$$\nu^{select} = \Psi(\beta)\nu_t^{sm}.$$

Thus, the effect of multiplying the eddy viscosity of the Smagorinsky model with this filter function is the following: if this angle is between a certain value β_0 and $\pi - \beta_0$, the eddy-viscosity is turned on (this correspond to the case of "non-alignment of the vorticity field direction"). Otherwise, only the molecular dissipation is active.

3. Numerical Results

This section is concerned with some results of our numerical simulations of the LES equations using the selective version of the Smagorinsky model. For both cases treated here we used a 3D spectral code in a periodic box. As it is customary the spatial derivatives are computed in the Fourier space and the nonlinear terms in the physical space.A third order Runge-Kutta method is used as time- advancing scheme for the nonlinear and SGS terms. To eliminate the aliasing errors the two-third rule is used (see [3] for more details on the spectral methods). We compare the selective Smagorinsky model with the classical Smagorinsky model and reference DNS results.

3.1. The 3D vortex tubes reconnection problem. We first show the good behavior of the selective version of the Smagorinsky model in a "non turbulent flow" simulation: the 3D vortex tubes reconnection case. For a detailed presentation and a complete physical analysis of this phenomena we refer to [19]. For this flow, direct numerical simulations using 240 and 120 discretization grid points in each direction of the 3D space give similar results. We denote them by DNS1 and DNS2 respectively,

[2] i.e., which verifies the Constantin and Fefferman hypothesis

and we use them as reference for the results of LES computations. The Reynolds number for this flow is Re= 3500.

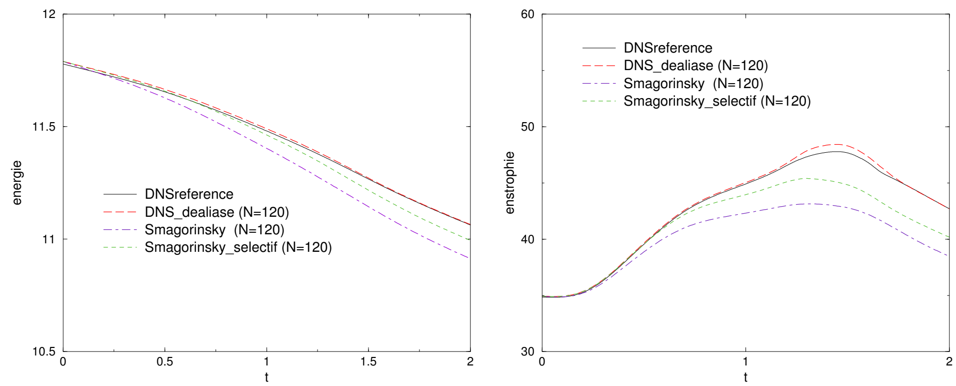

FIGURE 1. *Energy decay (left) and enstrophy decay (right) for DNS1 (solid line), DNS2 (long dashed), and LES with classical Smagorinsky model (dot-dashed line) and its selective variante (dashed line)*

In figure 1 and 2 it clearly appears from the energy, enstrophy curves and energy spectra that the selective Smagorinsky model is less dissipative than the Smagorinsky model. It is interesting to remark the shrinking of the eddy viscosity for the selective Smagorinsky model: figure 3 reflects well that the selective model adds dissipation only in the reconnection ring, which corresponds to a "turbulent area" for this kind of flow.

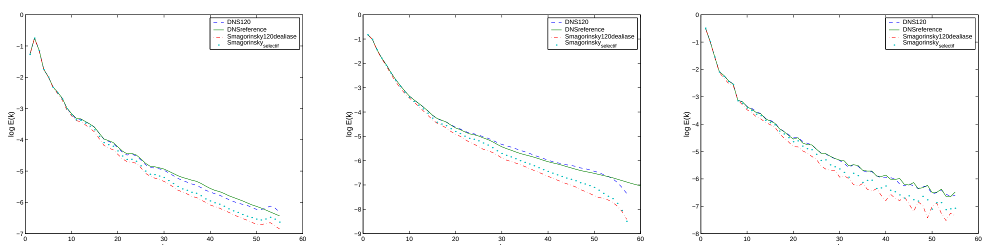

FIGURE 2. *From left to right: decay of directional spectra (in x, y and z directions; the same legend as in Figure 1 is used)*

Figure 4 shows the level curves of the vorticity field at nondimensionalized time $t = 1.5$ obtained with DNS1, classical and selective Smagorinsky models. These plots indicate that the results with the selective model are very closed to reference results, while considering the classical Smagorinsky model too many "details" of the flow are lost.

3.2. The 3D isotropic turbulence case. A second case chosen as validation test for the selective Smagorinsky model is the three dimensional decaying isotropic turbulence. The reference simulation for our comparison tests is the 512^3 DNS performed

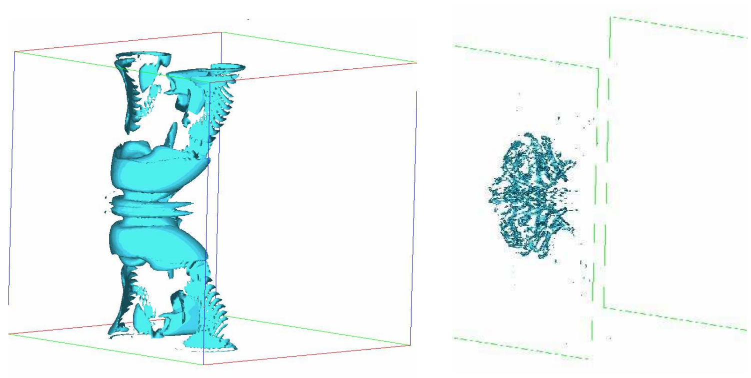

FIGURE 3. *Isosurfaces of the eddy viscosity* ν_t *given by the classical Smagorinsky model (left) and its selective version (right)*

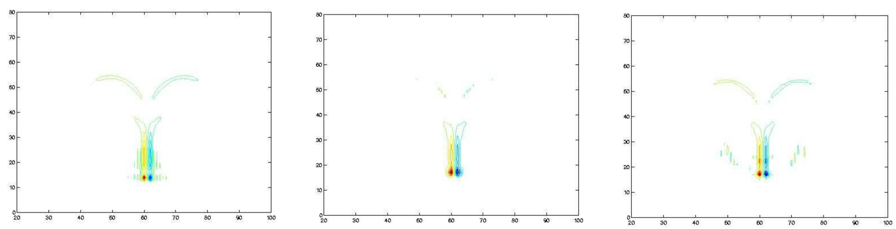

FIGURE 4. *From left to right : isovalues of the vorticity field in the reconnection plane at time* $t = 1.5$*, given by DNS1, LES with classical Smagorinsky and selective Smagorinsky models*

by A. Wray and reported in [10]. For the LES runs we used as initial condition a 32^3 data obtained by a sharptruncation in the Fourier space of the 512^3 DNS. All computations were performed with a Taylor-Reynolds number $Re_\lambda = 104.5$. We mention that this is a sever case for the LES validations.

Figure 6 illustrate the evolution of filtered kinetic energy and the energy spectra of the reference data in contrast with LES simulations using the Smagorinsky model, the classical and the selective variant models, for a 32^3 resolution.
The two models are considered with the same constant, chosen such that it yields an energy decay which matches the reference DNS result (fig. 5). The difficulty of this test is in fact to accurately recover the decay of the energy spectra. We notice that both models *satisfactorily* predict this decay rate even if, after $t = 2.45$ both of them are too dissipative. For further numerical results and comments on this test problem we refer to [6]. From this test it is clear that the selective model works efficiently for turbulent flows too, the excessive dissipation of Smagorinsky model beeing cured in the selective variant.

4. Concluding remarks

We analysed a selective procedure for the classical Smagorinsky model through numerical simulations of the three dimensional vortex tubes reconnection problem and

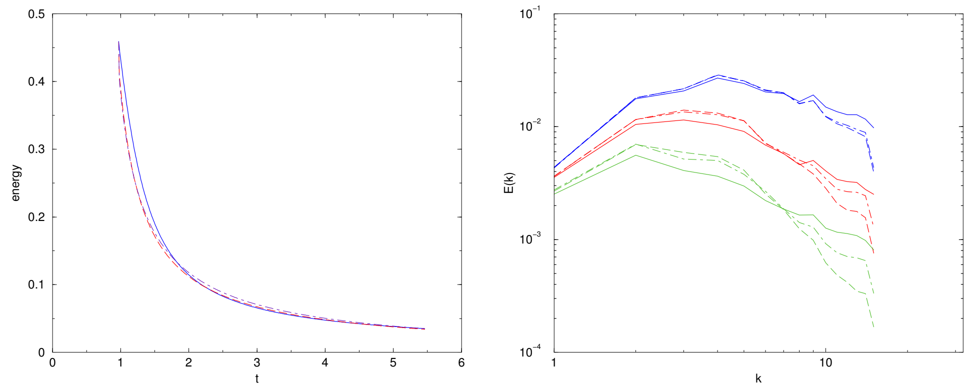

FIGURE 5. *Left: the filtered kinetic energy; Right: the spectrum of energy at different times: 1.28, 2.45 and 5.46; DNS512 (-), DNSfilt32 (-.-), Smagorinsky32 (- -) and selective Smagorinsky32 (..)*

the isotropic decaying turbulence. Numerical tests with the selective model confirm that this model "dissipates" only in flow regions where vorticity direction undergoes rapid variations, avoiding an excessive dissipation. These results show a clear improvement over those obtained with the original Smagorinsky model.

References

[1] J.T.Beale, T. Kato, A. Majda, Remarks on the Breakdown of Smooth Solutions for the 3-D Euler Equations, *Comm. Math. Phys.*, **94**, (1984).

[2] H. Beirao da Veiga, L. Berselli, On the Regularizing Effect of the Vorticity Diretion in Incompressible Viscous Flows, *Differential and Integral Equations*, **15**(3), (2002).

[3] C. Canuto, M. Y. Hussaini, A. Quarteroni, T. A. Zang, *Spectral Methods in Fluid Dynamics*, Springer Verlag, 1988.

[4] P. Constantin, Geometric Statistics in Turbulence, *SIAM Rev.*, **36**, (1994).

[5] P. Constantin, Ch. Fefferman, Direction of Vorticity and the Problem of Global Regularity for the Navier-Stokes Equations, *Indiana University Mathematics Journal*, **42**(3), (1993).

[6] G.-H. Cottet, D. Jiroveanu, B. Michaux, Vorticity Dynamics and Turbulence Models for Large-Eddy Simulations, Submitted in *Math. Model. and Num. Analysis*.

[7] E. David, *Modélisation des écoulements compressibles et hypersoniques: une approche instationnaire*, PhD, LEGI - Grenoble, 1993.

[8] Q. Du, M. D. Gunzburger, Analysis of a Ladyszenskaya model for incompressible viscous flow, *J. Math. Anal. Appl.*, **155**, 21-45 (1991).

[9] M. Germano, U. Piomelli, P. Moin, W. H. Cabot, A Dynamic Subrid- Scale Eddy Viscosity Model, *Phys. Fluids A*, **3**, (1991).

[10] J. Jimenez, An overview of LES validation, In *A Selection of Test Cases for the Validation of Large Eddy Simulation of Turbulent Flows*, NATO AGARD, 1999.

[11] O. A. Ladyszenskaya, New equations for the description of motion of viscous incompressible fluids and solvability in the large of boundary value problems for them, *Proc. Steklov Insts. Math.*, **102**, 95-11 (1967).

[12] O. A. Ladyszenskaya, *The Mathematical Theory of Viscous Incompressible Flow. Gordon and Breach*, 2nd edition, 1969.

[13] W. J. Layton, *A Mathematical Introduction to Large-Eddy Simulation*, VKI Lecture Notes, 2002.

[14] J. Leray, Sur le mouvement d'un fluid visqueux emplissant l'espace, *Acta Math.*, **63**, 193-248 (1934).

[15] M. Lesieur, Turbulence in Fluids, In *Fluid Mechanics and its Applications*, **40**, Kluwer Academic Publishers, 1997.
[16] J. L. Lions, *Quelques méthodes de résolution des problèmes aux limites non-linéaires*, Paris, Dunod, 1969.
[17] R.S. Rogallo, P. Moin, Numerical Simulations of Turbulent Flows, *Annu. Rev. Fluid Mechanics*, **16**, 99-137 (1984).
[18] P. G. Saffman, Problems and progress in the theory of turbulence, In *Proc. of the Symposium on Turbulence*, Technische Universitat, Berlin, 273-306 (1977), Berlin, Springer, 1978.
[19] M. J. Shelley, D. I. Meiron, S. A. Orszag, Dynamical Aspects of Vortex Reconnection of Perturbed Anti-parallel Vortex Tubes, *J. of Fluids Mechanics*, **246**, (1993).
[20] J. Smagorinsky, General circulation experiments with the primitive equations. I. The basic experiment, *Monthly Weather Review*, **91**, (1963).
[21] R. Temam, *Navier-Stokes Equations and Nonlinear Functional Analysis*, CBMS-NSF, 1995.

(Delia Jiroveanu) Departamento de Matematica Aplicada
Facultad de Ciencias
Universidad de Granada
Campus de Fuente Nueva, 18071, Granada, Spain
Tel/Fax: (9)58 249059/ (9)58248596
E-mail address: jirovean@ugr.es , Delia.Jiroveanu@imag.fr

(Georges-Henri Cottet) Laboratoire de Modélisation et Calcul - IMAG
Université Joseph Fourier
B.P.53- 38041 Grenoble cedex 9
Fax: (33) 4 76 63 12 63
E-mail address: Georges-Henri.Cottet@imag.fr

(Bertrand Michaux) Laboratoire de Modélisation et Calcul - IMAG
Université Joseph Fourier
B.P.53- 38041 Grenoble cedex 9
Fax: (33) 4 76 63 12 63
E-mail address: Bertrand.Michaux@imag.fr