Annals of the University of Craiova, Mathematics and Computer Science Series
Volume 45(2), 2018, Pages 290–303
ISSN: 1223-6934

Standing wave solutions of a perturbed Schrödinger equation in an Orlicz-Sobolev space setting

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Abstract. In this paper we study a class of non-linear Schrödinger-type equation on the whole space. The differential operator was introduced by A. Azzollini *et al.* in the papers [2] and [3] and it is described by a potential with a different growth near zero and at infinity.

We have the following perturbed problem:

$$\text{-div}\left[\phi'(|\nabla u|^2)\nabla u\right] + \gamma(x)|u|^{\alpha-2}u = K(x)|u|^{s-2}u + g(x) \qquad \text{in} \quad \mathbb{R}^N$$

Our aim is to show that if the perturbation $g(x)$ is not too large in a suitable topology, we can prove the existence of at least one nontrivial solution using the mountain pass theorem in the framework of an Orlicz-Sobolev space.

2010 Mathematics Subject Classification. Primary 35J60; Secondary 58E05.
Key words and phrases. Schrödinger equation, mountain-pass theorem, perturbation, unbounded potential.

1. Introduction

The Schrödinger equation plays a crucial role in the quantum mechanics theory. It describes how the quantum state of a quantum system changes in time. Problems of this type appear in the study of several physical phenomena: self-channeling of high-power laser in matter see [6], [7], [10], [27], in the theory of Heisenberg ferromagnets and magnons, for more details we refer to [5] and [22], in plasma physics, (e.g. the Kurihara superfluid film equation) studied in [14], [16] etc.

The linear Schrödinger equations are describing in a non-relativistic framework the time evolution of the system's wave function, which is also called state function.

P. Rabinowitz in [23] showed how variational arguments based on the mountain-pass theorem can be applied to obtain existence results for nonlinear Schrödinger-type equations with lack of compactness.

In his research P. Rabinowitz ([23]) studied the following problem

$$-\Delta u + a(x)u = f(x,u) \qquad \text{in} \quad \mathbb{R}^N \quad (N \geq 3) \tag{1}$$

where a is a positive potential and f has a subcritical growth. The existence of nontrivial standing waves of problem (1) is obtained as a result of the mountain-pass theorem.

For more details about the impact of the mountain-pass theorem we refer to [17], [21], [28], [31].

Received November 9, 2018. Accepted December 1, 2018.

Equations like (1) are deduced by taking:

$$\psi(x,t) = exp(-iEt/\hbar)u(x)$$

into Schrödinger's equation:

$$i\hbar\psi_t = -\frac{\hbar^2}{2m}\Delta\psi + V(x)\psi - \gamma|\psi|^{p-1}\psi \quad \text{in } \mathbb{R}^N \times (0,\infty),$$

where $1 < p < \frac{N+2}{N-2}$, and $\hbar$ is the Planck constant divided by 2π, ψ is the wave function, m the magnetic quantum number, V is the potential energy and γ is a constant that depends of the number of particles.

D. Repovš studied in [26] a new type of Schrödinger equation involving also variable exponents.

A. Azzollini in [2] and A. Azzollini, P. d'Avenia and A. Pomponio in [3] studied a new kind of nonhomogeneous operator with an associated functional framework.

They examined the following differential operator:

$$\operatorname{div}\left[\phi'(|\nabla u|^2)\nabla u\right],$$

where $\phi \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ has a different growth near zero and at infinity.

An example is constituted by the prescribed mean curvature operator (capillary surface operator), where the function ϕ is defined as

$$\phi(t) = 2\left(\sqrt{1+t} - 1\right)$$

and the differential operator becomes:

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right).$$

Furthermore, $\phi(t)$ behaves as $t^{q/2}$ for a sufficiently small t, and like $t^{p/2}$ when t is large enough.

These conditions are satisfied by the function

$$\phi(t) = \frac{2}{p}\left[\left(1 + t^{q/2}\right)^{p/q} - 1\right]$$

and we obtain a problem like

$$-\operatorname{div}\left[(1+|\nabla u|^q)^{\frac{p-q}{q}}|\nabla u|^{q-2}\nabla u\right] + \gamma(x)|u|^{\alpha-2}u = K(x)|u|^{s-2}u + g(x) \qquad \text{in} \quad \mathbb{R}^N.$$

Many ideas in the study of these types of general operators in divergence form are developed by G. Molica Bisci, D. Repovš in [20], V. Rădulescu in [24] and V.F. Uţă in [32].

The main goal of this paper is to study problem (1) in the new functional framework introduced by A. Azzollini in [2] and A. Azzollini, P. d'Avenia, A. Pomponio in [3].

More contributions in the study of quasilinear elliptic equations in an Orlicz-Sobolev space framework are also detailed in [25] by V. Rădulescu and D. Repovš.

In the next section we will set the hypotheses, the abstract setting and we will approach an existence result of a nontrivial solution due to the mountain-pass theorem.

2. The main result

We study the effect of a certain perturbation for the next non-linear Schrödinger equation

$$-\mathrm{div}\left[\phi'(|\nabla u|^2)\nabla u\right] + \gamma(x)|u|^{\alpha-2}u = K(x)|u|^{s-2}u + g(x) \qquad \text{in} \quad \mathbb{R}^N \tag{2}$$

In our study, the exponents α, p, q and s are real numbers, and they fulfill the following assumptions:

$$\begin{cases} 1 < p < q < N \\ 1 < \alpha < p^* \frac{q'}{p'} \\ \max\{\alpha, q\} < s < p^* := \frac{Np}{N-p}, \end{cases} \tag{3}$$

where p' and p are conjugate exponents, that is, $p' = \frac{p}{p-1}$. Analogous we have q and q', that is, $\frac{1}{q} + \frac{1}{q'} = 1$.

Throughout this paper we will denote by $\|\cdot\|_r$ the Lebesgue norm for all $1 \le r \le \infty$ and by $C_c^\infty(\mathbb{R}^N)$ the space of all C^∞ functions with a compact support.

We suppose that γ is a singular potential and it have the following properties:
(γ_1) $\gamma \in L^\infty_{loc}(\mathbb{R}^N \setminus \{0\})$ and $\mathrm{essinf}_{\mathbb{R}^N}\gamma > 0$;
(γ_2) $\lim\limits_{x\to 0}\gamma(x) = \lim\limits_{|x|\to\infty}\gamma(x) = +\infty$.

Remark 2.1. A function that satisfies these conditions is

$$\gamma(x) := \frac{e^{|x|}}{|x|}, \ \forall\, x \in \mathbb{R}^N \setminus \{0\}.$$

(K) The function $K : \mathbb{R}^N \to \mathbb{R}$ is strictly positive on $\mathbb{R}^N$ and it is from $L^\infty(\mathbb{R}^N)$.

The perturbation $g : \mathbb{R}^N \to \mathbb{R}$ has the following properties:
(g_1) $g \in L^{p_0}(\mathbb{R}^N)$, where p_0 is the conjugate of the critical exponent p^*, that is,

$$p_0 = (p^*)' = \frac{p^*}{p^*-1} = \frac{Np}{N(p-1)+p}.$$

(g_2) $\|g\|_{p_0} < \overline{C}$, where $\overline{C} > 0$ is an arbitrary small constant.
(g_3) $g(x) > 0$ for every $x \in \mathbb{R}^N$.

The function ϕ, that generates our nonhomogeneous differential operator is from $C^1(\mathbb{R}_+, \mathbb{R}_+)$ and it satisfies the following conditions:
(Φ_1) $\phi(0) = 0$;
(Φ_2) There exists $c_1 > 0$ such that

$$\begin{cases} \phi(t) \ge c_1 t^{p/2}, & \text{if } t \ge 1; \\ \phi(t) \ge c_1 t^{q/2}, & \text{if } 0 \le t \le 1. \end{cases}$$

(Φ_3) There exists $c_2 > 0$ such that

$$\begin{cases} \phi(t) \le c_2 t^{p/2}, & \text{if } t \ge 1; \\ \phi(t) \le c_2 t^{q/2}, & \text{if } 0 \le t \le 1. \end{cases}$$

(Φ_4) There exists $0 < \mu < \frac{1}{s}$ such that

$$2t\phi'(t) \le s\mu\phi(t), \ \forall\, t \ge 0;$$

(Φ_5) The application $t \mapsto \phi(t^2)$ is strictly convex.

As our assumptions enable ϕ' to approach 0, then our problem (2) is *degenerate* and no ellipticity condition is assumed.

In order to study our problem (2) we will define our functional framework in few steps.

Definition 2.1. We define the function space $L^p(\mathbb{R}^N) + L^q(\mathbb{R}^N)$ as the completion of $C_c^\infty(\mathbb{R}^N)$ in the norm:

$$\|u\|_{L^p+L^q} := \inf\{\|v\|_p + \|w\|_q;\ v \in L^p(\mathbb{R}^N),\ w \in L^q(\mathbb{R}^N),\ u = v + w\}.$$

The Orlicz space $L^p(\mathbb{R}^N) + L^q(\mathbb{R}^N)$ was intensive studied in the following papers: M. Badiale, L. Pisani and S. Rolando [4], A. Azzollini [2], A. Azzollini, P. d'Avenia and A. Pomponio [3]. More details about Orlicz-Sobolev type functional frameworks used in the study of quasilinear equations are developed by M. Mihăilescu, V. Rădulescu, D. Repovš in [19], I. Stăncuţ and C. Udrea in [29], I. Stăncuţ and I. Stîrcu in [30].

In this paper for simplicity we will use the following notation:

$$\|u\|_{p,q} = \|u\|_{L^p+L^q}.$$

Note that, an important role is taken by the function space

$$\mathcal{X} := \overline{C_c^\infty(\mathbb{R}^N)}^{\|\cdot\|}$$

with the norm

$$\|u\| := \|\nabla u\|_{p,q} + \left(\int_{\mathbb{R}^N} \gamma(x)|u|^\alpha dx\right)^{1/\alpha},$$

which is defined by N. Chorfi and V. Rădulescu in [11].

We also know from the above cited paper that the space $\mathcal{X}$ is continuously embedded in the reflexive Banach space $\mathcal{W}$, defined in A. Azzollini, P. d'Avenia and A. Pomponio [3] and $\mathcal{W}$ is the completion of $C_c^\infty(\mathbb{R}^N)$ in the norm

$$\|u\| = \|\nabla u\|_{p,q} + \|u\|_\alpha.$$

Definition 2.2. A weak solution of problem (2), is a function $u \in \mathcal{X} \setminus \{0\}$ such that

$$\int_{\mathbb{R}^N} \phi'(|\nabla u|^2)\nabla u \nabla \varphi dx + \int_{\mathbb{R}^N} \gamma(x)|u|^{\alpha-2}u\varphi dx = \int_{\mathbb{R}^N} K(x)|u|^{s-2}u\varphi dx + \int_{\mathbb{R}^N} g(x)\varphi dx$$

for all $\varphi \in \mathcal{X}$.

The energy functional associated to problem (2) is $J : \mathcal{X} \to \mathbb{R}$ define by:

$$J(u) := \frac{1}{2}\int_{\mathbb{R}^N} \phi(|\nabla u|^2)dx + \frac{1}{\alpha}\int_{\mathbb{R}^N} \gamma(x)|u|^\alpha dx - \frac{1}{s}\int_{\mathbb{R}^N} K(x)|u|^s dx - \int_{\mathbb{R}^N} u \cdot g(x)dx. \tag{4}$$

Remark 2.1. Our hypotheses reveals that:

$$\phi(|\nabla u|^2) \simeq \begin{cases} |\nabla u|^p, & \text{if } |\nabla u| \gg 1; \\ |\nabla u|^q, & \text{if } |\nabla u| \ll 1. \end{cases}$$

From A. Azzollini [2] and A. Azzollini, P. d'Avenia and A. Pomponio [3] we observe that J is well defined on $\mathcal{X}$, and of class C^1.

Furthermore, for all u, φ in $\mathcal{X}$ its Gâteaux directional derivative is given by

$$\begin{aligned} J'(u)(\varphi) &= \int_{\mathbb{R}^N} \phi'(|\nabla u|^2)\nabla u \nabla \varphi dx + \int_{\mathbb{R}^N} \gamma(x)|u|^{\alpha-2}u\varphi dx - \\ &- \int_{\mathbb{R}^N} K(x)|u|^{s-2}u\varphi dx - \int_{\mathbb{R}^N} \varphi \cdot g(x)dx. \end{aligned}$$

We can enunciate now the main result of this paper, the following existence property.

Theorem 2.1. *If the following assumptions:* (3), (γ_1), (γ_2), (K), (g_1), (g_2), (g_3) *and* $(\Phi_1) - (\Phi_5)$ *hold, then the problem* (2) *has at least one nontrivial weak solution.*

We notice that an existence property was obtained by A. Azzollini, P. d'Avenia and A. Pomponio in [3], but under the assumption that the potential γ reduces to a positive constant ($\gamma \equiv 1$), and by N. Chorfi and V. Rădulescu in [11], under the hypotheses that $K \equiv 1$ and the problem is not perturbed, with the property that the nonlinearity is not a power nonlinearity but it satisfies the Ambrosetti-Rabinowitz growth condition.

The effects of some perturbations in the nonlinear problems involving a singular potential was also studied by M. Cencelj, D. Repovš, Ž. Virk in [9].

The lack of compactness due to the unboundedness of the domain is handled by A. Azzollini, P. d'Avenia and A. Pomponio in [3], by restricting the study to the case of radially symmetric weak solutions.

The functional framework developed in this paper is more general and can not be reduced to radially symmetric solutions, due to the presence of the potentials γ and K. For more details about singular points we refer to Y. Fu and Y. Shan [12]. A key role is played in this paper by the fact that the function space $\mathcal{X}$ is continuously embedded in the function space $\mathcal{W}$ defined by A. Azzollini in [2] and A. Azzollini, P. d'Avenia and A. Pomponio in [3]. But the space $\mathcal{W}$ is continuously embedded in $L^{p^*}(\mathbb{R}^N)$, by the fact that we have $1 < p < \min\{q, N\}$, $1 < p^* \frac{q'}{p'}$, and $\alpha \in \left(1, p^* \frac{q'}{p'}\right)$. By interpolation we have the fact that the next continuous embedding

$$\mathcal{W} \hookrightarrow L^r(\mathbb{R}^N)$$

holds for every $r \in [\alpha, p^*]$.

They key argument in the proof of our main result is the mountain-pass lemma of A. Ambrosetti and P. Rabinowitz [1](we also studied Brézis and Nirenberg [8]).

Theorem 2.2. *Let $\mathcal{X}$ be a real Banach space and assume that $J : \mathcal{X} \to \mathbb{R}$ is a C^1-functional that satisfies the following geometric hypotheses:*
(i) $J(0) = 0$ and there exist positive numbers a and r such that $J(u) \geq a$ for all $u \in \mathcal{X}$ with $\|u\| = r$;
(ii) there exists $e \in \mathcal{X}$ with $\|e\| > r$ such that $J(e) < 0$.
Set

$$\mathcal{P} := \{p \in C([0,1]; \mathcal{X});\ p(0) = 0,\ p(1) = e\}$$

and

$$c := \inf_{p \in \mathcal{P}} \sup_{t \in [0,1]} J(p(t)).$$

Then there exists a sequence $(u_n) \subset \mathcal{X}$ such that

$$\lim_{n\to\infty} J(u_n) = c \quad \text{and} \quad \lim_{n\to\infty} \|J'(u_n)\|_{\mathcal{X}^*} = 0.$$

Moreover, if J satisfies the Palais-Smale condition at the level c, then c is a critical value of J.

3. Proof of the main result

Step 1. *Verification of the first geometrical condition.*

Let $r \in (0,1)$ and let $u \in \mathcal{X}$ such that $\|u\| = r$.

We have

$$\begin{aligned} J(u) &= \frac{1}{2}\int_{\mathbb{R}^N} \phi(|\nabla u|^2)dx + \frac{1}{\alpha}\int_{\mathbb{R}^N} \gamma(x)|u|^\alpha dx - \frac{1}{s}\int_{\mathbb{R}^N} K(x)|u|^s dx - \int_{\mathbb{R}^N} u \cdot g(x)dx \\ &\simeq \frac{1}{2}\int_{[|\nabla u|\le 1]} |\nabla u|^q dx + \frac{1}{2}\int_{[|\nabla u|>1]} |\nabla u|^p dx + \frac{1}{\alpha}\int_{\mathbb{R}^N} \gamma(x)|u|^\alpha dx \\ &- \frac{1}{s}\int_{\mathbb{R}^N} K(x)|u|^s dx - \int_{\mathbb{R}^N} u \cdot g(x)dx. \end{aligned}$$

Now using the assumptions (Φ_1) and (Φ_2) we obtain

$$\begin{aligned} J(u) &\ge \frac{c_1}{2}\int_{[|\nabla u|\le 1]} |\nabla u|^q dx + \frac{c_1}{2}\int_{[|\nabla u|>1]} |\nabla u|^p dx + \frac{1}{\alpha}\int_{\mathbb{R}^N} \gamma(x)|u|^\alpha dx \\ &- \frac{1}{s}\int_{\mathbb{R}^N} K(x)|u|^s dx - \int_{\mathbb{R}^N} u \cdot g(x)dx \\ &\ge \frac{c_1}{2}\max\left\{\int_{[|\nabla u|\le 1]} |\nabla u|^q dx, \int_{[|\nabla u|>1]} |\nabla u|^p dx\right\} + \frac{1}{\alpha}\int_{\mathbb{R}^N} \gamma(x)|u|^\alpha dx \\ &- \frac{\|K\|_\infty}{s}\int_{\mathbb{R}^N} |u|^s dx - \int_{\mathbb{R}^N} u \cdot g(x)dx \\ &\ge \frac{c_1}{2}\|\nabla u\|_{p,q}^q + \frac{1}{\alpha}\int_{\mathbb{R}^N} \gamma(x)|u|^\alpha dx - \frac{\|K\|_\infty}{s}\int_{\mathbb{R}^N} |u|^s dx - \int_{\mathbb{R}^N} u \cdot g(x)dx. \end{aligned}$$

Now, using the fact that $g \in L^{p_0}(\mathbb{R}^N)$ and $u \in \mathcal{X}$, which is continuously embedded in $L^{p^*}(\mathbb{R}^N)$, by Hölder's inequality we obtain

$$\int_{\mathbb{R}^N} u \cdot g(x)dx \le \|u\|_{p^*} \cdot \|g\|_{p_0}.$$

Moreover, by Young's inequality we have the following

$$\int_{\mathbb{R}^N} u \cdot g(x)dx \le \varepsilon\|u\|_{p^*}^{p^*} + C_\varepsilon\|g\|_{p_0}^{p_0},$$

where $\varepsilon > 0$, $C_\varepsilon = \varepsilon^{-1/(p^*-1)}$ are constants.

Therefore

$$J(u) \ge \frac{c_1}{2}\|\nabla u\|_{p,q}^q + \frac{1}{\alpha}\int_{\mathbb{R}^N} \gamma(x)|u|^\alpha dx - \frac{\|K\|_\infty}{s}\int_{\mathbb{R}^N} |u|^s dx - \varepsilon\|u\|_{p^*}^{p^*} - C_\varepsilon\|g\|_{p_0}^{p_0}.$$

Hence, using the following continuous embeddings we have:
$\mathcal{X} \hookrightarrow L^s(\mathbb{R}^N)$ yields there exists $\tilde{C}_1 > 0$ such that $\tilde{C}_1\|u\| \ge \|u\|_s$;
$\mathcal{X} \hookrightarrow L^\alpha(\mathbb{R}^N)$ yields there exists $\tilde{C}_2 > 0$ such that $\tilde{C}_2\|u\| \ge \|u\|_\alpha$;
$\mathcal{X} \hookrightarrow L^{p^*}(\mathbb{R}^N)$ yields there exists $C_{p^*} > 0$ such that $C_{p^*}\|u\| \ge \|u\|_{p^*}$.

Using the notation: $C_m = \min\{\frac{c_1}{2}, \frac{1}{\alpha}\}$ we have

$$\begin{aligned} J(u) &\geq C_m \left(\|\nabla u\|_{p,q}^q + \int_{\mathbb{R}^N} \gamma(x)|u|^\alpha dx \right) - \frac{\tilde{C}_1 \|K\|_\infty}{s} \|u\|^s - \varepsilon C_{p^*}^{p^*} \|u\|^{p^*} - C_\varepsilon \|g\|_{p_0}^{p_0} \\ &\geq C_m \|u\|^{\max\{\alpha,q\}} - \frac{\tilde{C}_1 \|K\|_\infty}{s} \|u\|^s - \varepsilon C_{p^*}^{p^*} \|u\|^{p^*} - C_\varepsilon \|g\|_{p_0}^{p_0}. \end{aligned}$$

Now making use of the hypothesis (g_2) it follows that

$$J(u) \geq C_m \|u\|^{\max\{\alpha,q\}} - \frac{\tilde{C}_1 \|K\|_\infty}{s} \|u\|^s - \varepsilon C_{p^*}^{p^*} \|u\|^{p^*} - C_\varepsilon \cdot \overline{C}^{p_0}. \tag{5}$$

Now, taking $\|u\| = r$, $r \in (0,1)$ small enough, and using the assumptions (3), (K) and (g_2) we observe that there exists $a > 0$ such that

$$J(u) \geq a > 0, \text{ for all } u \in \mathcal{X} \text{ with } \|u\| = r. \tag{6}$$

So, the first geometrical condition, the existence of a "mountain" around the origin is proved.

Step 2. *Verification of the second geometrical condition.*
We choose $w \in C_c^\infty(\mathbb{R}^N) \setminus \{0\}$ and $t > 0$.
Combining (4) with hypothesis (Φ_3) we have:

$$\begin{aligned} J(tw) &= \frac{1}{2} \int_{\mathbb{R}^N} \phi(|\nabla(tw)|^2) dx + \frac{1}{\alpha} \int_{\mathbb{R}^N} \gamma(x)|tw|^\alpha dx \\ &- \frac{1}{s} \int_{\mathbb{R}^N} K(x)|tw|^s dx - \int_{\mathbb{R}^N} tw \cdot g(x) dx \\ &\leq \frac{c_2}{2} \int_{[|\nabla(tw)| \leq 1]} |\nabla(tw)|^q dx + \frac{c_2}{2} \int_{[|\nabla(tw)| > 1]} |\nabla(tw)|^p dx \\ &+ \frac{1}{\alpha} \int_{\mathbb{R}^N} \gamma(x)|tw|^\alpha dx - \frac{1}{s} \int_{\mathbb{R}^N} K(x)|tw|^s dx - t \int_{\mathbb{R}^N} w \cdot g(x) dx \\ &\leq \frac{c_2}{2} \left(t^q \int_{\mathbb{R}^N} |\nabla w|^q dx + t^p \int_{\mathbb{R}^N} |\nabla w|^p dx \right) + \frac{t^\alpha}{\alpha} \int_{\mathbb{R}^N} \gamma(x)|w|^\alpha dx \\ &- \frac{t^s}{s} \int_{\mathbb{R}^N} K(x)|w|^s dx - t \int_{\mathbb{R}^N} w \cdot g(x) dx. \end{aligned} \tag{7}$$

Since w is fixed, taking account of relation (7) and hypothesis (3) we obtain $\lim_{t\to\infty} J(tw) = -\infty$. So, we can find a $t_0 > 0$ such that

$$J(t_0 w) < 0. \tag{8}$$

In order to set relation (8) in terms of Theorem 2.2 we can consider $e = tw$ with $\|e\| = \|tw\| = t\|w\| > r$, for t taken sufficiently large.

Therefore, the second geometrical condition, that is, the existence of a "valley" over the chain of mountains, is proved.

Step 3. We will show in what follows that the associated min-max value given by the Theorem 2.2 is positive.

We define

$$c := \inf_{p \in \mathcal{P}} \max_{t \in [0,1]} J(p(t)),$$

where

$$\mathcal{P} := \{p \in C([0,1];\mathcal{X});\ p(0) = 0,\ p(1) = t_0 w\}.$$

We see that for every $p \in \mathcal{P}$

$$c \geq J(p(0)) = J(0) = 0.$$

In what follows we claim that

$$c > 0. \tag{9}$$

Arguing by contradiction, we suppose that $c = 0$. So, this means that for all $\varepsilon > 0$, there exists $q \in \mathcal{P}$ such that

$$0 \leq \max_{t\in[0,1]} J(q(t)) < \varepsilon.$$

Set $\varepsilon < a$, where a is given by (6). Therefore we have $q(0) = 0$ and $q(1) = t_0 w$, hence

$$\|q(0)\| = 0 \qquad \text{and} \qquad \|q(1)\| > r.$$

By the continuity of q, there exists $t_1 \in (0,1)$ such that $\|q(t_1)\| = r$, so

$$\|J(q(t_1))\| = a > \varepsilon,$$

which is a contradiction. This shows that our claim (9) is true.

Using Theorem 2.2, we find a Palais-Smale sequence for the positive level $c > 0$, that is, a sequence $(u_n) \subset \mathcal{X}$ such that

$$\lim_{n\to\infty} J(u_n) = c \qquad \text{and} \qquad \lim_{n\to\infty} \|J'(u_n)\|_{\mathcal{X}^*} = 0. \tag{10}$$

Step 4. Next, we prove that the sequence (u_n) described in (10) is bounded in $\mathcal{X}$.

Using the conditions from (10), we find that

$$\begin{aligned}
c + O(1) + o(\|u_n\|) &= J(u_n) - \frac{1}{s}J'(u_n)u_n = \\
&= \frac{1}{2}\int_{\mathbb{R}^N} \phi(|\nabla u_n|^2)dx + \frac{1}{\alpha}\int_{\mathbb{R}^N} \gamma(x)|u_n|^\alpha dx - \frac{1}{s}\int_{\mathbb{R}^N} K(x)|u_n|^s dx \\
&- \int_{\mathbb{R}^N} u_n \cdot g(x)dx - \frac{1}{s}\int_{\mathbb{R}^N} \phi'(|\nabla u_n|^2)|\nabla u_n|^2 dx \\
&- \frac{1}{s}\int_{\mathbb{R}^N} \gamma(x)|u_n|^\alpha dx + \frac{1}{s}\int_{\mathbb{R}^N} K(x)|u_n|^s dx + \frac{1}{s}\int_{\mathbb{R}^N} u_n \cdot g(x)dx = \\
&= \int_{\mathbb{R}^N} \left[\frac{1}{2}\phi(|\nabla u_n|^2) - \frac{1}{s}\phi'(|\nabla u_n|^2)|\nabla u_n|^2\right] dx + \left(\frac{1}{\alpha} - \frac{1}{s}\right)\int_{\mathbb{R}^N} \gamma(x)|u_n|^\alpha dx \\
&+ \left(\frac{1}{s} - \frac{1}{s}\right)\int_{\mathbb{R}^N} K(x)|u_n|^s dx + \left(\frac{1}{s} - 1\right)\int_{\mathbb{R}^N} u_n \cdot g(x)dx.
\end{aligned}$$

Taking use of the assumption (Φ_4) we have:

$$\begin{aligned} c+O(1)+o(\|u_n\|) &\geq \frac{1-\mu}{2}\int_{\mathbb{R}^N}\phi(|\nabla u_n|^2)dx+\left(\frac{1}{\alpha}-\frac{1}{s}\right)\int_{\mathbb{R}^N}\gamma(x)|u_n|^\alpha dx+ \\ &+ \left(\frac{1}{s}-1\right)\int_{\mathbb{R}^N}u_n\cdot g(x)dx \\ &\geq c_0\left[\min\left\{\|\nabla u_n\|^q_{p,q},\|\nabla u_n\|^p_{p,q}\right\}+\int_{\mathbb{R}^N}\gamma(x)|u_n|^\alpha dx\right]+ \\ &+ \left(\frac{1}{s}-1\right)\int_{\mathbb{R}^N}u_n\cdot g(x)dx, \end{aligned}$$

where $c_0>0$ is a constant.

In what follows we will use some ideas developed in [18].

We claim that (u_n) is bounded in $\mathcal{X}$.

To this end we argue by contradiction and we suppose that $\|u_n\|\to+\infty$.

Using the previous estimates we may write that:

$$\begin{aligned} c+O(1)+\|u_n\| &\geq c_0\left[\min\left\{\|\nabla u_n\|^q_{p,q},\|\nabla u_n\|^p_{p,q}\right\}+\int_{\mathbb{R}^N}\gamma(x)|u_n|^\alpha dx\right]+ \\ &+ \left(\frac{1}{s}-1\right)\int_{\mathbb{R}^N}u_n\cdot g(x)dx. \end{aligned}$$

Since

$$\int_{\mathbb{R}^N}u_n\cdot g(x)dx\leq\|u_n\|_{p^*}\cdot\|g\|_{p_0}\leq C_{p^*}\overline{C}\|u_n\|,$$

we can say that

$$c+O(1)+\|u_n\|\geq c_0\|u_n\|^{\min\{\alpha,p,q\}}-\left(1-\frac{1}{s}\right)\overline{C}C_{p^*}\|u_n\|.$$

Dividing by $\|u_n\|$ and passing to the limit, we obtain a contradiction.

We can say now that $(u_n)\subset\mathcal{X}$ is bounded.

Like we said before, because $\mathcal{X}$ is a closed subset of $\mathcal{W}$, using Proposition 2.5 in [3] we get that the sequence (u_n) (up to a subsequence) converges weakly in $\mathcal{X}$ and strongly in $L^s_{loc}(\mathbb{R}^N)$ to some u_0:

$$\begin{aligned} u_n &\rightharpoonup u_0 && \text{in } \mathcal{X} \\ u_n &\to u_0 && \text{in } L^s_{loc}(\mathbb{R}^N). \end{aligned}$$

We prove in what follows that u_0 is a solution for the problem (2).

Fix $\theta\in C_c^\infty(\mathbb{R}^N)$ and set $\Omega:=\mathrm{supp}(\theta)$. Define

$$A(u)=\frac{1}{2}\int_\Omega\phi(|\nabla u|^2)dx+\frac{1}{\alpha}\int_\Omega\gamma(x)|u|^\alpha dx$$

and

$$B(u)=\frac{1}{s}\int_\Omega K(x)|u|^s dx+\int_\Omega g(x)\cdot u\,dx.$$

Considering that (10) holds we deduce that

$$A'(u_n)(\theta)-B'(u_n)(\theta)\to 0 \qquad \text{as } n\to\infty. \tag{11}$$

Since

$$u_n \to u_0 \text{ in } L^s(\Omega).$$

it ensures that

$$B(u_n) \to B(u_0) \qquad \text{and} \qquad B'(u_n)(\theta) \to B'(u_0)(\theta) \text{ as } n \to \infty. \tag{12}$$

Using (11) and (12) together we have that

$$A'(u_n)(\theta) \to B'(u_0)(\theta), \qquad \text{as } n \to \infty. \tag{13}$$

Knowing that (Φ_5) holds, we deduce that the nonlinear mapping A is convex. Hence

$$A(u_n) \leq A(u_0) + A'(u_n)(u_n - u_0), \ \forall\, n \in \mathbb{N}. \tag{14}$$

Using (13) and the fact that $u_n \rightharpoonup u_0$ in $\mathcal{X}$, relation (14) yields

$$\limsup_{n\to\infty} A(u_n) \leq A(u_0).$$

Now, by the fact that A is convex and continuous, it follows that it is lower semicontinuous, and we obtain that

$$A(u_0) \leq \liminf_{n\to\infty} A(u_n).$$

Finally we get that

$$A(u_n) \to A(u_0), \qquad \text{as } n \to \infty.$$

In what follows we use the same arguments like A. Azzollini, P. d'Avenia and A. Pomponio [3], and Lieb, Loss [15], by the fact that $u_n \rightharpoonup u_0$ in $\mathcal{X}$ we obtain

$$\begin{aligned} \nabla u_n &\rightharpoonup \nabla u_0 && \text{in } L^p(\mathbb{R}^N) + L^q(\mathbb{R}^N) \\ u_n &\rightharpoonup u_0 && \text{in } L^\alpha(\mathbb{R}^N). \end{aligned}$$

Therefore using same details as in [11] we obtain $\nabla u_n \to \nabla u_0$ as $n \to \infty$ in $L^p(\mathbb{R}^N) + L^q(\mathbb{R}^N)$ and

$$\int_{\mathbb{R}^N} \gamma(x)|u_n|^\alpha dx \to \int_{\mathbb{R}^N} \gamma(x)|u_0|^\alpha dx, \qquad \text{as } n \to \infty$$

$$\int_{\mathbb{R}^N} K(x)|u_n|^s dx \to \int_{\mathbb{R}^N} K(x)|u_0|^s dx, \qquad \text{as } n \to \infty.$$

Hence

$$\int_\Omega \phi'(|\nabla u_0|^2)\nabla u_0 \nabla\theta dx + \int_\Omega \gamma(x)|u_0|^{\alpha-2}u_0\theta dx = \int_\Omega K(x)|u_0|^{s-2}u_0\theta dx + \int_\Omega g(x)\cdot\theta dx.$$

By standard density arguments, we deduce that this identity holds for all $\theta \in \mathcal{X}$, therefore u_0 is a weak solution of problem (2).

Final step. As we are on an unbounded domain, the problem lacks compactness; by the Step 4, the weak limit u_0 of the Palais-Smale sequence is a weak solution of the problem (2), and the main problem is that it could be $u_0 = 0$. As a final step we will clarify that $u_0 \neq 0$. To this end we studied some ideas developed in [11], [13] and [18].

Using the properties of the Palais-Smale sequence (u_n), in relation (10) we obtain for a sufficiently large positive integer n, that:

$$\begin{aligned} \frac{c}{2} &\leq J(u_n) - \frac{1}{2}J'(u_n)u_n = \qquad (15)\\ &= \int_{\mathbb{R}^N} \left[\phi(|\nabla u_n|^2) - \phi'(|\nabla u_n|^2)|\nabla u_n|^2\right] dx + \left(\frac{1}{\alpha} - \frac{1}{2}\right)\int_{\mathbb{R}^N} \gamma(x)|u_n|^\alpha dx + \\ &+ \left(\frac{1}{2} - \frac{1}{s}\right)\int_{\mathbb{R}^N} K(x)|u_n|^s dx - \frac{1}{2}\int_{\mathbb{R}^N} u_n \cdot g(x)dx \end{aligned}$$

Taking account of the condition (Φ_5) that the application $t \mapsto \phi(t^2)$ is strictly convex we deduce that

$$\phi(t^2) - \phi(0) \leq \phi'(t^2)t^2.$$

Therefore:

$$\phi(|\nabla u_n|^2) \leq \phi'(|\nabla u_n|^2)|\nabla u_n|^2. \qquad (16)$$

Hence, we have

$$\frac{c}{2} \leq \left(\frac{1}{\alpha} - \frac{1}{2}\right)\int_{\mathbb{R}^N} \gamma(x)|u_n|^\alpha dx + \left(\frac{1}{2} - \frac{1}{s}\right)\int_{\mathbb{R}^N} K(x)|u_n|^s dx - \frac{1}{2}\int_{\mathbb{R}^N} u_n \cdot g(x)dx.$$

For now on, due to the perturbation term $g(x)$ we have to split this step into two cases.

(I) We suppose that $\operatorname{sgn}(u_n) \neq \operatorname{sgn}(g(x))$. Considering that, we set

$$u_n^- = \begin{cases} 0, & \text{if } u_n \geq 0; \\ u_n, & \text{if } u_n < 0. \end{cases}$$

We rearrange the terms in the following way:

$$\frac{c}{2} + \frac{1}{2}\int_{\mathbb{R}^N} u_n^- \cdot g(x)dx \leq \left(\frac{1}{\alpha} - \frac{1}{2}\right)\int_{\mathbb{R}^N} \gamma(x)|u_n^-|^\alpha dx + \left(\frac{1}{2} - \frac{1}{s}\right)\int_{\mathbb{R}^N} K(x)|u_n^-|^s dx$$

and we can say using hypothesis (g_2) that

$$\frac{c}{2} + \frac{1}{2}\int_{\mathbb{R}^N} u_n^- \cdot g(x)dx \geq \frac{c}{2} - \frac{\overline{C}C_{p^*}}{2}\|u_n^-\|.$$

Since $\|u_n\|$ is bounded, due to Step 4, $C_{p^*} > 0$ is a constant and $\overline{C}$ is supposed to be arbitrary small, we can find $\underline{c} > 0$ such that

$$\begin{aligned} \underline{c} &\leq \frac{c}{2} + \frac{1}{2}\int_{\mathbb{R}^N} u_n^- \cdot g(x)dx \leq \\ &\leq \left(\frac{1}{\alpha} - \frac{1}{2}\right)\int_{\mathbb{R}^N} \gamma(x)|u_n^-|^\alpha dx + \left(\frac{1}{2} - \frac{1}{s}\right)\int_{\mathbb{R}^N} K(x)|u_n^-|^s dx. \end{aligned}$$

(II) We suppose now that $\operatorname{sgn}(u_n) = \operatorname{sgn}(g(x))$, so we set

$$u_n^+ = \begin{cases} u_n, & \text{if } u_n > 0; \\ 0, & \text{if } u_n \leq 0. \end{cases}$$

Then, we obtain

$$\int_{\mathbb{R}^N} u_n^+ \cdot g(x)dx \geq 0,$$

so one have

$$\frac{c}{2} \leq \left(\frac{1}{\alpha} - \frac{1}{2}\right) \int_{\mathbb{R}^N} \gamma(x)|u_n^+|^\alpha dx + \left(\frac{1}{2} - \frac{1}{s}\right) \int_{\mathbb{R}^N} K(x)|u_n^+|^s dx.$$

We proceed now to show that $u_0 \neq 0$ for this case, the previous could be handled analogous.

Firstly, we suppose that $\alpha \geq 2$ which yields that $s \geq 2$, so we have

$$\frac{c}{2} \leq \left(\frac{1}{2} - \frac{1}{s}\right) \int_{\mathbb{R}^N} K(x)|u_n^+|^s dx$$

and using the hypothesis (K) we obtain

$$\frac{c}{2} \leq \left(\frac{1}{2} - \frac{1}{s}\right) \|K\|_\infty \|u_n^+\|_s^s.$$

To point out that $u_0 \neq 0$ we will use some ideas developed by F. Gazzola, V. Rădulescu in [13]. To this end we will prove the following technical result:

Lemma 3.1. There exists a constant $C_0 > 0$ such that $\|u_n^+\| \geq C_0$.

Proof. Since $\alpha \geq 2$ and $s \geq 2$, it yields that

$$\frac{c}{2} \leq \left(\frac{1}{2} - \frac{1}{s}\right) \|K\|_\infty \|u_n^+\|_s^s.$$

Taking use of the continuous embedding $\mathcal{X} \hookrightarrow L^s(\mathbb{R}^N)$, we have $\tilde{C}_1 > 0$ such that $\tilde{C}_1\|u_n^+\| \geq \|u_n^+\|_s$, for any $n \in \mathbb{N}$. We have that

$$\|u_n^+\|_s^s \leq \tilde{C}_1^s \|u_n^+\|^s.$$

Therefore if using the fact that (u_n^+) is a Palais-Smale sequence, for $n \in \mathbb{N}$ large enough we obtain that

$$\frac{c}{2} \leq \left(\frac{1}{2} - \frac{1}{s}\right) \|K\|_\infty \cdot \tilde{C}_1^s \|u_n^+\|^s.$$

So we can choose $C_0 = \left(\dfrac{c}{2\left(\frac{1}{2} - \frac{1}{s}\right)\|K\|_\infty \cdot \tilde{C}_1^s}\right)^{1/s}$. □

We treat now the case: $\alpha \leq 2$, $s \leq 2$ and $\alpha < s$. So we have

$$\begin{aligned} \frac{c}{2} &\leq J(u_n^+) - \frac{1}{2} J'(u_n^+)u_n^+ \\ &\leq \left(\frac{1}{\alpha} - \frac{1}{2}\right) \int_{\mathbb{R}^N} \gamma(x)|u_n^+|^\alpha dx + \left(\frac{1}{2} - \frac{1}{s}\right) \int_{\mathbb{R}^N} K(x)|u_n^+|^s dx \\ &\leq \left(\frac{1}{\alpha} - \frac{1}{2}\right) \int_{\mathbb{R}^N} \gamma(x)|u_n^+|^\alpha dx. \end{aligned}$$

We argue by contradiction and we suppose that $u_0 = 0$. We obtain that

$$u_n^+ \to 0, \qquad \text{in } L^\alpha_{loc}(\mathbb{R}^N). \tag{17}$$

Let k belong to $\mathbb{N}^*$, we define the set

$$\Gamma := \left\{ x \in \mathbb{R}^N ; \frac{1}{k} < |x| < k \right\}.$$

By (17) we have that

$$C_\Gamma \int_\Gamma |u_n^+|^\alpha dx < \frac{c}{2}, \text{ for } k \text{ large enough },$$

where $C_\Gamma > 0$ is a constant. We may arbitrary pick $C_\Gamma = \left(\frac{1}{\alpha} - \frac{1}{2}\right) > 0$.

Hence

$$\begin{aligned} \frac{c}{2} &\leq C_\Gamma \int_{\mathbb{R}^N \setminus \Gamma} |u_n^+|^\alpha dx \\ &\leq \frac{C_\Gamma}{\inf_{|x|\leq 1/k} \gamma(x)} \int_{[|x|\leq 1/k]} \gamma(x)|u_n^+|^\alpha dx + \frac{C_\Gamma}{\inf_{|x|\geq k} \gamma(x)} \int_{[|x|\geq k]} \gamma(x)|u_n^+|^\alpha dx \\ &\leq C_\Gamma N \left[\frac{1}{\inf_{|x|\leq 1/k} \gamma(x)} + \frac{1}{\inf_{|x|\geq k} \gamma(x)}\right], \end{aligned} \tag{18}$$

where $N = \sup_n \int_{\mathbb{R}^N} \gamma(x)|u_n^+|^\alpha dx$.

Taking k sufficiently large and using the hypothesis (γ_2) in (18), yields that $c = 0$ which is a contradiction.

For the case where $\alpha \leq 2$ and $s \geq 2$ we obtain that

$$\frac{c}{2} \leq \left(\frac{1}{\alpha} - \frac{1}{2}\right) \int_{\mathbb{R}^N} \gamma(x)|u_n^+|^\alpha dx + \left(\frac{1}{2} - \frac{1}{s}\right) \int_{\mathbb{R}^N} K(x)|u_n^+|^s dx.$$

With the same arguments like the ones used in Lemma 3.1 and choosing Γ like in the case studied before, we obtain that $u_0 \neq 0$ which leads to the end of our last step.

In conclusion, if we go through all our steps we obtain that u_0 is a nontrivial solution for the problem (2).

□

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