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A fixed point theorem of Sehgal-Guseman in $b_v(s)$-metric spaces

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Abstract. In this paper, we establish a proof for Sehgal-Guseman type fixed point theorem in $b_v(s)$-metric spaces. Some suitable examples are presented to substantiate our main result. We deduce several analogous results in usual metric spaces, rectangular b-metric spaces and b-metric spaces as corollaries of our finding.

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1. Introduction and preliminaries

Fixed point theorem due to Banach in metric spaces is regarded as one of the most important applicable results in nonlinear analysis, which has been extended by a great number of authors over the last several decades. One such extension is the following result due to Bryant [3].

Theorem 1.1. [3] *If T is a self-mapping on a complete metric space and if, for some positive integer k, T^k is a contraction, then T has a unique fixed point.*

In the above result, the mapping T satisfies the following contraction condition

$$d(T^k x, T^k y) \leq \alpha d(x, y), \ \textit{where}\ 0 < \alpha < 1. \tag{1.1}$$

In this direction, it is an interesting fact to observe that the positive integer k depends only on T and works for all $x, y \in X$. So it is a very natural question to ask whether it is possible to find some contraction condition on T in which the positive integer k depends on the points $x, y \in X$, i.e., for different $x, y \in X$, there exist different positive integers k satisfying the contraction condition (1.1) and T possesses fixed point. The following example justifies the necessity of investigating of this problem.

Example 1.1. Let $X = \{\ln n : n \geq 2\}$ with the usual metric and let $T : X \to X$ be defined by $T(\ln n) = \ln(n+1)$. Then

$$T^2(\ln n) = T(\ln(n+1)) = \ln(n+2), ..., T^k(\ln n) = \ln(n+k).$$

For fixed $x = \ln n$ and $y = \ln m$, we have

$$d\left(T^k x, T^k y\right) = \left|\ln \frac{n+k}{m+k}\right| \to 0 \text{ as } k \to \infty.$$

Hence, for any fixed $\alpha < 1$ and every $x, y \in X$, there exists $k(x, y)$ such that (1.1) holds. But, $Tx \neq x$ for each $x \in X$.

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The above example shows that in the contraction condition (1.1), if the positive integer k varies with x and y, then T may be fixed point free. However, in 1969, Sehgal [14] proved the following interesting partial solution of the above mentioned question.

Theorem 1.2. *Let (X, d) be a complete metric space, $q \in [0, 1)$ and $T : X \to X$ be a continuous mapping. If for each $x \in X$ there exists a positive integer $k = k(x)$ such that*

$$d(T^{k(x)}x, T^{k(x)}y) \leq qd(x, y) \tag{1.2}$$

for all $y \in X$, then T has a unique fixed point $u \in X$. Moreover, for any $x \in X$, $u = \lim_{n\to\infty} T^n x$.

In 1970, Guseman [8] extended the result of Sehgal to mappings which are both necessarily continuous and which have a contractive iterate at each point in a (possibly proper) subset of the space.

In the literature, there are a lot of generalizations of metric spaces concerning the study of existence and uniqueness of various types of non-linear contraction maps. Some of these results appear to be simple reformulations of the known results from the framework of metric spaces, with just slightly modified proofs, or even their direct consequences. However, the work in some of the generalized spaces is practically harder and can not be simply reduced to the metric case. This is a major challenge to prove some classical results in a generalized setting, which may be applicable in many other abstract spaces including metric spaces. We mention here two of such types of spaces.

Bakhtin [1] and Czerwik [4] introduced *b-metric spaces*, modifying the triangle inequality to the following form

$$d(x, z) \leq s[d(x, y) + d(y, z)], \tag{1.3}$$

where $s \geq 1$ is a fixed real number. On the other hand, Branciari [2] substituted the triangle inequality by a *polygonal* inequality of the form

$$d(x, z) \leq d(x, y_1) + d(y_1, y_2) + \cdots + d(y_v, z), \tag{1.4}$$

for arbitrary x, z and for all distinct points $y_1, y_2, \ldots, y_v$, each of them different from x and z (in particular, for $v = 2$, the inequality (1.4) is called *rectangular*). Further, a lot of fixed point results for single and multi-valued mappings were obtained in both kind of spaces by various authors , see [5, 6].

George et al. [7], as well as Roshan et al. [13], independently introduced *b-rectangular metric spaces*, by combining inequalities (1.3) and (1.4) (in the case $v = 2$). Finally, Mitrović and Radenović defined in [9] the concept of $b_v(s)$-metric space for arbitrary positive integer v (see the definition in the next section), thus generalizing all the mentioned types of spaces. They obtained some fixed point results in this new framework. It should be noted that these spaces might not be Hausdorff, that a $b_v(s)$-metric need not be continuous and that a convergent sequence might not be a Cauchy one.

Definition 1.1. [9] Let X be a non-empty set, $s \geq 1$ be a real number, $v \in \mathbb{N}$ and let d be a function from $X \times X$ into $[0, \infty)$. Then (X, d) is said to be a $b_v(s)$-metric space if for all $x, y, z \in X$ and for all distinct points $y_1, y_2, \ldots, y_v \in X$, each of them different from x and z the following hold:

(B1) $d(x, y) = 0$ if and only if $x = y$;

(B2) $d(x,y) = d(y,x)$;
(B3) $d(x,z) \le s[d(x,y_1) + d(y_1,y_2) + \cdots + d(y_v,z)]$.

Example 1.2. Consider the set $X = \{\frac{1}{n} : n \in \mathbb{N},\ n \ge 2\}$. Define $d : X \times X \to [0,\infty)$ by

$$d\left(\frac{1}{k}, \frac{1}{m}\right) = \begin{cases} |k-m|, & \text{if } |k-m| \ne 1, \\ \frac{1}{2}, & \text{if } |k-m| = 1. \end{cases}$$

It is an easy task to verify that (X,d) is a $b_3(3)$-metric space.

The notions of convergent sequence, Cauchy sequence, and completeness of a $b_v(s)$-metric space are introduced in the same way as in standard metric spaces.

This paper is a continuation of the recent works see [10] and [11]. The aim of this paper is to obtain a version of Sehgal-Guseman type theorem in $b_v(s)$-metric spaces. As applications of our obtained results, we deduce the analogous versions of Sehgal-Guseman theorem in the framework of usual metric space, b-metric space and rectangular metric space.

2. Main result

Before coming to the main result we first give the following lemma.

Lemma 2.1. *Let (X,d) be a complete $b_v(s)$-metric space and $T : X \to X$ a mapping satisfying the condition: for each $x \in X$ there exists $k(x) \in \mathbb{N}$ such that*

$$d\left(T^{k(x)}x, T^{k(x)}y\right) \le \lambda d(x,y), \tag{2.1}$$

for all $y \in X$, where $\lambda \in (0,1)$. Then for each $x \in X, r(x) = \sup\{d(T^n(x),x) : n \in \mathbb{N}\}$ is finite or T has a unique fixed point.

Proof. We will consider that $v > 1$. If $v = 1$, we refer to see the paper [10]. Let $x \in X$ and further assume

$$l(x) = \sup\{d(T^k(x),x) : k \in \{1,\ldots,k_1+k_2+\cdots+k_{n_0+v-1}\}\},$$

where $n_0 \in \mathbb{N}$ such that $\lambda^{n_0} < \frac{1}{2s(v-1)}$ and

$$k_1 = k(x), k_2 = k(T^{k_1}x), k_3 = k(T^{k_2+k_1}x), \ldots, k_{n_0+v} = k(T^{k_{n_0+v-1}+\ldots+k_1}x).$$

Let $S_1 = k_1 + k_2 + \cdots + k_{n_0}$ and $S_{i+1} = S_i + k_{n_0+i}, i \in \{1,2,\ldots,v-1\}$. We have,

$$\begin{aligned} d(T^{S_i}x, T^{S_{i+1}}x) &= d(T^{k_1+k_2+\ldots+k_{n_0+i-1}}x, T^{k_1+k_2+\ldots+k_{n_0+i}}x) \\ &\le \lambda d(T^{k_1+k_2+\ldots+k_{n_0+i-2}}x, T^{k_1+k_2+\ldots+k_{n_0+i-2}}(T^{k_{n_0+i}})x) \\ &\ \vdots \\ &\le \lambda^{n_0+i-1} d(x, T^{k_{n_0+i}}x), \end{aligned}$$

for all $i \in \{1,2,\ldots,v-1\}$. Now since

$$d(x, T^{k_{n_0+i}}x) \le l(x) \text{ for all } i \in \{1,2,\ldots,v-1\},$$

we obtain

$$d(T^{S_i}x, T^{S_{i+1}}x) \le \lambda^{n_0+i-1} l(x), \text{ for all } i \in \{1,2,\ldots,v-1\}. \tag{2.2}$$

We have the following

$$d(x, T^{S_1}x) \leq l(x) \tag{2.3}$$

and for $n > S_v$,

$$\begin{aligned} d(T^{S_v}x, T^n x) &= d(T^{S_v}x, T^{S_v}(T^{n-S_v}x)) && (2.4)\\ &\leq \lambda^{n+v-1} d(x, T^{n-S_v}x). && (2.5) \end{aligned}$$

Let $n \in \mathbb{N}$. We give the proof in the following three cases:
1. If $T^n x = T^{S_i}x$ for any $i \in \{1, 2, \ldots, v\}$ then $d(x, T^n x) \leq l(x)$ and proof holds.
2. If $T^{S_i}x = T^{S_j}x$ for any $i < j$ then $T^{S_i}x = T^{S_i+1}x$ and $T^{S_i}x$ is a fixed point of T and proof is done. Namely, if $T^{S_i}x \neq T^{S_i+1}x$, we obtain

$$\begin{aligned} d(T^{S_i}x, T^{S_i+1}x) &= d(T^{S_j}x, T^{S_j+1}x) \\ &\leq \lambda^{j-i} d(T^{S_i}x, T^{S_i+1}x) \\ &< d(T^{S_i}x, T^{S_i+1}x). \end{aligned}$$

It is a contradiction.
3. If $T^{S_i}x = x$ for any $i \in \{1, 2, \ldots, v\}$ then $Tx = x$ and proof is done. Namely, if $Tx \neq x$ then we have

$$\begin{aligned} d(x, Tx) &= d(T^{S_i}x, T^{S_i+1}x) \\ &\leq \lambda^{n_0+i} d(x, Tx) \\ &< d(x, Tx). \end{aligned}$$

It is a contradiction.

So, $T^{S_i}x, i \in \{1, 2, \ldots, v\}$ are different point and $T^{S_i}x \in X \backslash \{T^n x, x\}, i \in \{1, 2, \ldots, v\}$. If $n > S_v$ then there exists an integer $t \geq 0$ such that $tS_v < n \leq (t+1)S_v$. From (B3), (2.2), (2.3) and (2.5), we obtain

$$\begin{aligned} d(T^n x, x) &\leq s[d(T^{S_v+(n-S_v)}x, T^{S_v}x) + d(T^{S_v}x, T^{S_{v-1}}x) + \cdots \\ &+ d(T^{S_2}x, T^{S_1}x) + d(T^{S_1}x, x)] \\ &\leq s\left[\lambda^{n_0+v-1} d(T^{n-S_v}x, x) + \lambda^{n_0+v-2} l(x) + \cdots + \lambda^{n_0} l(x) + l(x)\right] \\ &\leq s\left[\lambda^{n_0} d(T^{n-S_v}x, x) + (v-1)\lambda^{n_0} l(x) + l(x)\right] \\ &\leq \frac{1}{2} d(T^{n-S_v}x, x) + \left(\frac{1}{2} + s\right) l(x). \end{aligned}$$

Similarly, we obtain

$$d(T^{n-S_v}x, x) \leq \frac{1}{2} d(T^{n-2S_v}x, x) + \left(\frac{1}{2} + s\right) l(x).$$

So,

$$d(T^n x, x) \leq \frac{1}{2^2} d(T^{n-2S_v}x, Tx) + \left(1 + \frac{1}{2}\right)\left(\frac{1}{2} + s\right) l(x). \tag{2.6}$$

Continuing in this process we obtain

$$\begin{aligned} d(T^n x, x) &\leq \frac{1}{2^t} d(T^{n-tS_v}x, Tx) + \left(1 + \frac{1}{2} + \cdots + \frac{1}{2^{t-1}}\right)\left(\frac{1}{2} + s\right) l(x) \\ &\leq \frac{1}{2^t} l(x) + 2\left(\frac{1}{2} + s\right) l(x) \\ &\leq 2(1+s) l(x) \end{aligned}$$

and $r(x)$ is finite. □

Theorem 2.2. *Let (X,d) be a complete $b_v(s)$-metric space and $T : X \to X$ a mapping satisfying the condition: for each $x \in X$ there exists $k(x) \in \mathbb{N}$ such that*

$$d\left(T^{k(x)}x, T^{k(x)}y\right) \le \lambda d(x,y), \tag{2.7}$$

for all $y \in X$, where $\lambda \in (0,1)$. Then T has a unique fixed point, say $u \in X$, and $T^n x \to u$ for each $x \in X$.

Proof. Let $x_0 \in X$ be arbitrary. Let $k_1 = k(x_0), x_1 = T^{k_1}x_0$ and inductively $k_{i+1} = k(x_i), x_{i+1} = T^{k_{i+1}}x_i, i \in \mathbb{N}$. Let $n, p \in \mathbb{N}$. We have

$$\begin{aligned} d(x_{n+p}, x_n) &= d(T^{k_{n+p}}x_{n+p-1}, T^{k_n}x_{n-1}) \\ &= d(T^{k_{n+p}+k_{n+p-1}}x_{n+p-2}, T^{k_n+k_{n-1}}x_{n-2}) \\ &\vdots \\ &= d(T^{k_{n+p}+k_{n+p-1}+\cdots+k_{p+1}}x_p, T^{k_n+k_{n-1}+\cdots+k_1}x_0) \\ &= d(T^{k_{n+p}+k_{n+p-1}+\cdots+k_{p+1}+k_p}x_{p-1}, T^{k_n+k_{n-1}+\cdots+k_1}x_0) \\ &\vdots \\ &= d(T^{k_{n+p}+k_{n+p-1}+\cdots+k_n+\cdots+k_1}x_0, T^{k_n+k_{n-1}+\cdots+k_1}x_0) \\ &= d(T^{k_n+k_{n-1}+\cdots+k_1}(T^{k_{n+1}+\cdots+k_{n+p}}x_0), T^{k_n+k_{n-1}+\cdots+k_1}x_0) \\ &\le \lambda^n d(T^{k_{n+1}+\cdots+k_{n+p}}x_0, x_0). \end{aligned}$$

Therefore,

$$d(x_{n+p}, x_n) \le \lambda^n r(x_0). \tag{2.8}$$

If $r(x_0)$ is not finite, from Lemma 2.1, we conclude that T has a fixed point and the proof is done. On the other hand, if $r(x_0)$ is finite, we infer that (x_n) is Cauchy. From the completeness of (X,d) we have $x_n \to u$, for some $u \in X$. Now, we show that $Tu = u$. For this u, there is $k(u) \in \mathbb{N}$ such that

$$d(T^{k(u)}u, T^{k(u)}x_n) \le \lambda d(x_n, u). \tag{2.9}$$

Hence,

$$\lim_{n\to\infty} d(T^{k(u)}x_n, T^{k(u)}u) = 0. \tag{2.10}$$

Now, from (2.7) we have

$$d(T^{k(u)}x_n, x_n) = d(T^{k(u)+k_{n-1}}x_{n-1}, T^{k_{n-1}}x_{n-1}) \le \lambda d(T^{k(u)}x_{n-1}, x_{n-1}) \tag{2.11}$$

it follows that

$$d(T^{k(u)}x_n, x_n) \le \lambda^n d(T^{k(u)}x_0, x_0) \le \lambda^n r(x_0). \tag{2.12}$$

From Lemma 2.1, we obtain

$$\lim_{n\to\infty} d(T^{k(u)}x_n, x_n) = 0. \tag{2.13}$$

From inequality (B3) we obtain

$$\begin{aligned} d(T^{k(u)}u, u) \le\ & s[d(T^{k(u)}u, T^{k(u)}x_n) + d(T^{k(u)}x_n, x_n) \\ +\ & d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{n+v-2}, u)] \end{aligned}$$

and together with (2.10) and (2.13) we obtain $d(T^{k(u)}u, u) = 0$. By (2.7), u is the unique fixed point for $T^{k(u)}$. Then $Tu = T(T^{k(u)})u = T^{k(u)}(Tu)$. This shows that Tu is also a fixed point of $T^{k(u)}$ and hence we have $Tu = u$. But then u is the unique fixed point of T. □

Example 2.1. Let $X = [0, \infty)$ and define a function $d : X \times X \to \mathbb{R}$ by

$$d(x, y) = \begin{cases} 0, \text{ if } x = y; \\ 1 + 2x + 2y, \text{ if } x, y > 0; \\ x + y, \text{ if one of } x, y \text{ is } 0. \end{cases}$$

Then, it is easy to verify that (X, d) is a complete $b_2(2)$-metric space. Next, we define a function $T : X \to X$ by

$$Tx = \begin{cases} 0, \text{ if } x \in [0, \frac{1}{16}]; \\ 2x, \text{ if } x \in (\frac{1}{16}, 1]; \\ 5x + 1, \text{ if } x \in (1, 3] \\ \frac{1}{16}, \text{ if } x > 3. \end{cases}$$

Therefore, for any $x \in X$, if we choose $k(x) = 8$, then it easy to check that

$$d(T^{k(x)}x, T^{k(x)}y) \leq \frac{1}{2} d(x, y)$$

for all $y \in X$. Hence, all conditions of Theorem 2.2 are satisfied. So by that theorem T has a unique fixed point. Indeed, 0 is the only fixed point of T.

Example 2.2. Let $X = \left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right\}$ and define $d : X \times X \to [0, +\infty)$ as follows

$$d\left(\frac{1}{m}, \frac{1}{n}\right) = \begin{cases} |m - n|, & \text{if } |m - n| \neq 1, \\ \frac{1}{2}, & \text{if } |m - n| = 1. \end{cases}$$

Then (X, d) is a $b_3(3)$-metric space. Let $T : X \to X$ be defined by

$$Tx = \begin{cases} \frac{1}{6}, & x = \frac{1}{2}, \\ \frac{1}{2}, & x = \frac{1}{3}, \\ \frac{1}{4}, & x \in \{\frac{1}{4}, \frac{1}{5}, \ldots\}. \end{cases}$$

Let $k(x) = 3$ for all $x \in X$. The mapping T is not a contraction because $d(T\frac{1}{2}, T\frac{1}{4}) = d(\frac{1}{2}, \frac{1}{4}) = 2$, but satisfies the hypothesis of Theorem 2.2 and hence T has a fixed point $u = \frac{1}{4}$.

Remark 2.1. Theorem 2.2 generalizes Banach contraction principle in $b_v(s)$-metric spaces obtained in the paper [9, Theorem 2.1].

Taking $v = 1, s = 1$ in Theorem 2.2, we have the following corollary.

Corollary 2.3. *Let $S : X \to X$ be a mapping on a complete metric space (X, d). If for each $u \in X$ there exists a natural number $k = k(u)$ such that*

$$d(S^{k(u)}u, S^{k(u)}t) \leq qd(u, t)$$

for all $t \in X$, then S has a unique fixed point $u_0 \in X$. Moreover, for any $u \in X$, $u_0 = \lim_{n\to\infty} S^n u$.

If we take $v = 1$ in Theorem 2.2, we have the following corollary.

Corollary 2.4. *Let $S : X \to X$ be a mapping on a complete b-metric space (X, d). If for each $u \in X$ there exists a natural number $k = k(u)$ such that*

$$d(S^{k(u)}u, S^{k(u)}t) \leq qd(u, t)$$

for all $t \in X$, then S has a unique fixed point $u_0 \in X$. Moreover, for any $u \in X$, $u_0 = \lim_{n\to\infty} S^n u$.

Again taking $v = 2, s = 1$ in Theorem 2.2, we have the following corollary.

Corollary 2.5. *Let $S : X \to X$ be a mapping on a complete rectangular metric space (X, d). If for each $u \in X$, there exists a natural number $k = k(u)$ such that*

$$d(S^{k(u)}u, S^{k(u)}t) \leq qd(u, t)$$

for all $t \in X$, then S has a unique fixed point $u_0 \in X$. Moreover, for any $u \in X$, $u_0 = \lim_{n\to\infty} S^n u$.

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