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On the generalized Hermite-Hadamard inequalities

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Abstract. In this paper, we present a new definition which generalizes some significant well known fractional integral operators such as Riemann-Liouville fractional integral, k-Riemann-Liouville fractional integral, Katugampola fractional operators, conformable fractional integral, Hadamard fractional integrals, etc. Then, using a general class of this generalized fractional integral operator, we establish new generalized fractional integral inequalities of Hermite-Hadamard type which cover the previously published results.

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1. Introduction

The subject of the fractional calculus (integrals and derivatives) has gained considerable popularity and importance during the past there decades or so, due mainly to its demonstrated applications in numerous seemingly diverse and widespread fields of science and engineering. The fractional integral does indeed provide several potentially useful tools for various problems involving special functions of mathematical science as well as their extensions and generalizations in one and more variables. This subject is still being studied extensively by many authors, see for instance ([1], [2], [4], [6], [8]-[25]). One of the important applications of fractional integrals is Hermite-Hadamard integral inequality, see [10], [19]-[21]. First, let's recall the basic expressions of the classical Hermite-Hadamard inequality as follows:

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex mapping defined on the interval I of real numbers and $a, b \in I$, with $a < b$. The following double inequality is well known in the literature as the Hermite-Hadamard inequality [7]:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

The most well-known inequalities related to the integral mean of a convex function are the Hermite Hadamard inequalities.

In [3], Dragomir and Agarwal proved the following results connected with the right part of (1).

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Lemma 1.1. *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$. If $f' \in L[a,b]$, then the following equality holds:*

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx = \frac{b-a}{2}\int_0^1 (1-2t)f'(ta+(1-t)b)dt. \tag{2}$$

Theorem 1.2. *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$. If $|f'|$ is convex on $[a,b]$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{(b-a)}{8}\left(|f'(a)|+|f'(b)|\right). \tag{3}$$

In [5], Kirmaci proved the following results connected with the left part of (1). In [5] some inequalities of Hermite-Hadamard type for differentiable convex mappings were proved using the following lemma.

Lemma 1.3. *Let $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$, be a differentiable mapping on I°, $a, b \in I^\circ$ (I° is the interior of I) with $a < b$. If $f' \in L([a,b])$, then we have*

$$\begin{aligned}&\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\\ &= (b-a)\left[\int_0^{\frac{1}{2}} tf'(ta+(1-t)b)dt + \int_{\frac{1}{2}}^1 (t-1)f'(ta+(1-t)b)dt\right].\end{aligned} \tag{4}$$

Theorem 1.4. *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$. If $|f'|$ is convex on $[a,b]$, then we have*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \le \frac{(b-a)}{8}\left(|f'(a)|+|f'(b)|\right). \tag{5}$$

Now, let's recall the basic expressions of Hermite-Hadamard inequality for fractional integrals is proved by Sarikaya et al. in [10] as follows:

Theorem 1.5. *Let $f : [a,b] \to \mathbb{R}$ be a positive function with $a < b$ and $f \in L_1([a,b])$. If f is a convex function on $[a,b]$, then the following inequalities for fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right) \le \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)\right] \le \frac{f(a)+f(b)}{2} \tag{6}$$

with $\alpha > 0$.

Meanwhile, in [10], Sarikaya et al. gave the following interesting Trapeozid identity for Riemann-Liouville integral:

Lemma 1.6. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) with $a < b$. If $f' \in L[a,b]$, then the following equality for fractional integrals holds:*

$$\begin{aligned}&\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)\right]\\ &= \frac{b-a}{2}\int_0^1 \left[(1-t)^\alpha - t^\alpha\right] f'(ta+(1-t)b)\,dt.\end{aligned} \tag{7}$$

Theorem 1.7. *Let $f : [a, b] \to \mathbb{R}$ be a differentiable mapping on (a, b) with $a < b$. If $|f'|$ is convex on $[a, b]$, then the following inequality for fractional integrals holds:*

$$\left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a) \right] \right| \leq \frac{b-a}{2(\alpha+1)} \left(1 - \frac{1}{2^{\alpha}} \right) \left[f'(a) + f'(b) \right]. \tag{8}$$

On the other hand, in [19] Iqbal et al. gave the following results connected with the left part of Riemann-Liouville integral inequalities of Hermite-Hadamard type (6) by using the following Midpoint identity as follows.

Lemma 1.8. *Let $f : [a, b] \to \mathbb{R}$ be a differentiable function on (a, b). If $f' \in L^1[a, b]$, then the following identity for Riemann-Liouville fractional integrals holds:*

$$f\left(\frac{a+b}{2} \right) - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a) \right] = \frac{b-a}{2} \sum_{k=1}^{4} I_k, \tag{9}$$

where

$$I_1 = \int_0^{\frac{1}{2}} t^{\alpha} f'(tb + (1-t)a)\, dt, \qquad I_2 = \int_0^{\frac{1}{2}} (-t^{\alpha}) f'(ta + (1-t)b)\, dt,$$

$$I_3 = \int_{\frac{1}{2}}^{1} (t^{\alpha} - 1) f'(tb + (1-t)a)\, dt, \quad I_4 = \int_{\frac{1}{2}}^{1} (1 - t^{\alpha}) f'(ta + (1-t)b)\, dt.$$

Many papers study the Riemann-Liouville fractionals integrals and give new and interesting generalizations of Hermite-Hadamard type inequalities using these kind of integrals, see for instance see ([10]-[13]).

The purpose of this paper is to introduce a more general integral definition which generalizes some significant well known fractional integral operators such as Riemann-Liouville fractional integral, k-Riemann-Liouville fractional integral, Katugampola fractional integrals, conformable fractional integral, Hadamard fractional integrals, etc. In second section, using a general class of this generalized fractional integral operator, we establish new Hermite-Hadamard type inequalities. In third section, using functions whose first derivatives absolute values are convex, we obtained new trapezoid inequalities that are connected with the celebrated Hermite-Hadamard type which cover the previously puplished results. In the last section, we extend some estimates of the left hand side of a Hermite-Hadamard type inequality for functions whose first derivatives absolute values are convex.

2. Generalized fractional integral operators

In this section, we state the following new integral definition which are useful in the proofs of main theorems:

Let's define a function $\varphi : [0, \infty) \to [0, \infty)$ satisfiying the following conditions:

$$\int_0^1 \frac{\varphi(t)}{t} dt < \infty, \tag{10}$$

$$\frac{1}{A_1} \leq \frac{\varphi(s)}{\varphi(r)} \leq A_1 \text{ for } \frac{1}{2} \leq \frac{s}{r} \leq 2 \tag{11}$$

$$\frac{\varphi(r)}{r^2} \leq A_2 \frac{\varphi(s)}{s^2} \text{ for } s \leq r \tag{12}$$

$$\left| \frac{\varphi(r)}{r^2} - \frac{\varphi(s)}{s^2} \right| \leq A_3 \left| r - s \right| \frac{\varphi(r)}{r^2} \text{ for } \frac{1}{2} \leq \frac{s}{r} \leq 2 \tag{13}$$

where $A_1, A_2, A_3 > 0$ are independent of $r, s > 0$. If $\varphi(r)r^\alpha$ is increasing for some $\alpha \geq 0$ and $\frac{\varphi(r)}{r^\beta}$ is decreasing for some $\beta \geq 0$, then φ satisfies (10)-(13), see [26]. Therefore, we define the following left-sided and right-sided generalized fractional integral operators, respectively, as follows:

$$_{a^+}I_\varphi f(x) = \int_a^x \frac{\varphi(x-t)}{x-t} f(t)dt, \quad x > a, \tag{14}$$

$$_{b^-}I_\varphi f(x) = \int_x^b \frac{\varphi(t-x)}{t-x} f(t)dt, \quad x < b. \tag{15}$$

The most important feature of generalized fractional integrals is that they generalize some types of fractional integrals such as Riemann-Liouville fractional integral, k-Riemann-Liouville fractional integral, Katugampola fractional integrals, conformable fractional integral, Hadamard fractional integrals, etc. These important special cases of the integral operators (14) and (15) are mentioned below.

i) If we take $\varphi(t) = t$, the operators (14) and (15) reduce to the Riemann integral as follows:

$$I_{a^+} f(x) = \int_a^x f(t)dt, \quad x > a,$$

$$I_{b^-} f(x) = \int_x^b f(t)dt, \quad x < b.$$

ii) If we take $\varphi(t) = \frac{t^\alpha}{\Gamma(\alpha)}$, the operators (14) and (15) reduce to the Riemann-Liouville fractional integral as follows:

$$I_{a^+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a,$$

$$I_{b^-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \quad x < b.$$

iii) If we take $\varphi(t) = \frac{1}{k\Gamma_k(\alpha)} t^{\frac{\alpha}{k}}$, the operators (14) and (15) reduce to the k-Riemann-Liouville fractional integral as follows:

$$I_{a^+,k}^\alpha f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_a^x (x-t)^{\frac{\alpha}{k}-1} f(t)dt, \quad x > a,$$

$$I_{b^-,k}^\alpha f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_x^b (t-x)^{\frac{\alpha}{k}-1} f(t)dt, \quad x < b$$

where

$$\Gamma_k(\alpha) = \int_0^\infty t^{\alpha-1} e^{-\frac{t^k}{k}} dt, \quad \mathcal{R}(\alpha) > 0$$

and

$$\Gamma_k(\alpha) = k^{\frac{\alpha}{k}-1} \Gamma\left(\frac{\alpha}{k}\right), \quad \mathcal{R}(\alpha) > 0; k > 0$$

are given by Mubeen and Habibullah in [20].

iv) If we take

$$\varphi(t)=\frac{1}{\Gamma(\alpha)}t(x-t)^{s}\left(x^{s+1}-t^{s+1}\right)^{\alpha-1}$$

and

$$\varphi(t)=\frac{1}{\Gamma(\alpha)}t(t-x)^{s}\left(t^{s+1}-x^{s+1}\right)^{\alpha-1},$$

in the operators (14) and (15), respectively, then the (14) and (15) reduce to the Katugampola fractional operators as follows: for $\alpha>0$ and $s\neq-1$ is a real numbers

$$I^{\alpha}_{a+,s}f(x)=\frac{(s+1)^{1-\alpha}}{\Gamma(\alpha)}\int_a^x\left(x^{s+1}-t^{s+1}\right)^{\alpha-1}t^sf(t)dt,\quad x>a,$$

$$I^{\alpha}_{b-,s}f(x)=\frac{(s+1)^{1-\alpha}}{\Gamma(\alpha)}\int_x^b\left(x^{s+1}-t^{s+1}\right)^{\alpha-1}t^sf(t)dt,\quad x<b,$$

are given by Katugampola in [24].

v) If we take $\varphi(t)=t(x-t)^{\alpha-1}$, the operator (14) reduces to the conformable fractional operators as follows:

$$I^{\alpha}_{a}f(x)=\int_a^x t^{\alpha-1}f(t)dt=\int_a^x f(t)d_{\alpha}t,\quad x>a,\ \alpha\in(0,1),$$

is given by Khalil et.al in [25].

vi) If we take

$$\varphi(t)=\frac{1}{\Gamma(\alpha)}\frac{[(\log x-\log(x-t)]^{\alpha-1}}{x-t}$$

and

$$\varphi(t)=\frac{1}{\Gamma(\alpha)}t\frac{[(\log(t-x)-\log x]^{\alpha-1}}{t-x},$$

in the operators (14) and (15), respectively, then operators (14) and (15) reduce to the right-sided and left-sided Hadamard fractional integrals:

$$I^{\alpha}_{a+}f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x(\log x-\log t)^{\alpha-1}\frac{f(t)}{t}dt,\quad 0<a<x<b,$$

$$I^{\alpha}_{b-}f(x)=\frac{1}{\Gamma(\alpha)}\int_x^b(\log t-\log x)^{\alpha-1}\frac{f(t)}{t}dt,\quad 0<a<x<b.$$

vii) If we take $\varphi(t)=\frac{t}{\alpha}\exp\left(-\frac{1-\alpha}{\alpha}t\right)$ in the operators (14) and (15), respectively, then operators (14) and (15) reduce to the right-sided and left-sided fractional integral operators with exponential kernel for $\alpha\in(0,1)$, as follows:

$$\mathcal{I}^{\alpha}_{a+}f(x)=\frac{1}{\alpha}\int_a^x\exp\left(-\frac{1-\alpha}{\alpha}(x-t)\right)f(t)dt,\quad a<x,$$

$$\mathcal{I}^{\alpha}_{b-}f(x)=\frac{1}{\alpha}\int_x^b\exp\left(-\frac{1-\alpha}{\alpha}(t-x)\right)f(t)dt,\quad x<b$$

are defined by Kirane and Torebek in [18].

3. Hermite-Hadamard inequalities for generalized fractional integrals

Throughout this study, for brevity, we define

$$\Lambda(y) = \int_0^y \frac{\varphi((b-a)u)}{u}du < \infty, \ \ \Delta(y) = \int_y^1 \frac{\varphi((b-a)u)}{u}du < \infty.$$

In this section, using generalized fractional integral operators, we begin by the following theorem:

Theorem 3.1. *Let $f : [a,b] \to \mathbb{R}$ be a convex function on $[a,b]$ with $a < b$, then the following inequalities for generalized fractional integral hold:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{2\Lambda(1)}\left[{}_{a^+}I_\varphi f(b) + {}_{b^-}I_\varphi f(a)\right] \leq \frac{f(a)+f(b)}{2}. \tag{16}$$

Proof. For $t \in [0,1]$, let $x = ta + (1-t)b$, $y = (1-t)a + tb$. The convexity of φ yields

$$f\left(\frac{a+b}{2}\right) = f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2} \tag{17}$$

i.e.,

$$2f\left(\frac{a+b}{2}\right) \leq f(ta+(1-t)b) + f((1-t)a+tb). \tag{18}$$

Multiplying both sides of (18) by $\frac{\varphi((b-a)t)}{t}$, then integrating the resulting inequality with respect to t over $(0,1]$, we obtain

$$2f\left(\frac{a+b}{2}\right)\int_0^1 \frac{\varphi((b-a)t)}{t}dt \leq \int_0^1 \frac{\varphi((b-a)t)}{t} f(ta+(1-t)b)\,dt + \int_0^1 \frac{\varphi((b-a)t)}{t} f((1-t)a+tb)\,dt.$$

As consequence, we obtain

$$2f\left(\frac{a+b}{2}\right)\int_0^1 \frac{\varphi((b-a)t)}{t}dt \leq \left[{}_{a^+}I_\varphi f(b) + {}_{b^-}I_\varphi f(a)\right] \tag{19}$$

and the first inequality is proved.

To prove the other half of the inequality in (16), since f is convex, for every $t \in [0,1]$, we have,

$$f(ta+(1-t)b) + f((1-t)a+tb) \leq f(a) + f(b). \tag{20}$$

Then multiplying both sides of (20) by $\frac{\varphi((b-a)t)}{t}$ and integrating the resulting inequality with respect to t over $(0,1]$, we obtain

$$\left[{}_{a^+}I_\varphi f(b) + {}_{b^-}I_\varphi f(a)\right] \leq [f(a)+f(b)]\int_0^1 \frac{\varphi((b-a)t)}{t}dt$$

and the second inequality is proved. □

Remark 3.1. If in Theorem 3.1, we get $\varphi(t) = t$, then the inequalities (16) become the inequalities (1).

Remark 3.2. If in Theorem 3.1, we get $\varphi(t) = \frac{t^\alpha}{\Gamma(\alpha)}$, then the inequalities (16) become the inequalities (6) of Theorem 1.5.

Remark 3.3. If in Theorem 3.1, we get $\varphi(t) = \frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, then the inequalities (16) become the inequalities

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma_k(\alpha+k)}{(b-a)^{\frac{\alpha}{k}}}\left[I^{\alpha}_{a+,k} f(b) + I^{\alpha}_{b-,k} f(a)\right] \leq \frac{f(a)+f(b)}{2}.$$

which are proved by Hussain et. al. in [22].

Corollary 3.2. *Under the assumption of Theorem 4.5 with* $\varphi(t) = t(b-t)^{\alpha-1}$, *and* f *is a symmetric to* $\frac{(a+b)}{2}$, *then we have*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\alpha}{b^{\alpha}-a^{\alpha}}\int_a^b f(t)d_{\alpha}t \leq \frac{f(a)+f(b)}{2}$$

which are Hermite-Hadamard type inequalities for conformable fractional integrals.

Corollary 3.3. *Under the assumption of Theorem 4.5 with* $\varphi(t) = \frac{t}{\alpha}\exp\left(-\frac{1-\alpha}{\alpha}t\right)$, $\alpha \in (0,1)$, *we have*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1-\alpha}{2(1-\exp(-A))}\left[\mathcal{I}^{\alpha}_{a+} f(b) + \mathcal{I}^{\alpha}_{b-} f(a)\right] \leq \frac{f(a)+f(b)}{2}$$

where $A = \frac{1-\alpha}{\alpha}(b-a)$.

This result is given by Kirane and Torebek in [18].

4. Trapezoid inequalities for generalized fractional integrals

Before starting and proving our next result in this section, we need the following lemma.

Lemma 4.1. *Let* $f:[a,b]\to\mathbb{R}$ *be a differentiable mapping on* (a,b) *with* $a<b$. *If* $f' \in L[a,b]$, *then the following equalities for generalized fractional integrals hold:*

$$\begin{aligned}
&\frac{f(a)+f(b)}{2} - \frac{1}{2\Lambda(1)}\left[{}_{a^+}I_{\varphi}f(b) + {}_{b^-}I_{\varphi}f(a)\right] \\
&\quad = \frac{(b-a)}{2\Lambda(1)}\int_0^1 [\Lambda(1-t)-\Lambda(t)]\, f'(ta+(1-t)b)\,dt \\
&\quad = \frac{(b-a)}{2\Lambda(1)}\int_0^1 \Lambda(t)\left[f'(tb+(1-t)a) - f'(ta+(1-t)b)\right]dt. \qquad (21)
\end{aligned}$$

Proof. Here, we apply integration by parts in integrals of right part of (21), then we have

$$\begin{aligned}
S_1 &= \int_0^1 \left[\int_0^t \frac{\varphi((b-a)u)}{u}du\right] f'(ta+(1-t)b)\,dt \\
&= -\frac{f(a)}{b-a}\int_0^1 \frac{\varphi((b-a)u)}{u}du + \frac{1}{b-a}\int_a^b \frac{\varphi((b-a)x)}{x}f(x)dx.
\end{aligned}$$

And similarly, we obtain

$$\begin{aligned}
S_2 &= \int_0^1 \left[\int_0^t \frac{\varphi((b-a)u)}{u} du \right] f'(tb+(1-t)a)\,dt \\
&= \frac{f(b)}{b-a} \int_0^1 \frac{\varphi((b-a)u)}{u} du - \frac{1}{b-a} \int_a^b \frac{\varphi((b-a)x)}{x} f(x)dx.
\end{aligned}$$

If we subtract S_1 from S_2 and multiply by $(b-a)$, we obtain proof of the (21). $\square$

Remark 4.1. If in Lemma 4.1, we get $\varphi(t) = t$, then the inequalities (21) become the identity (2) of Lemma 1.1.

Remark 4.2. If in Lemma 4.1, we get $\varphi(t) = \frac{t^\alpha}{\Gamma(\alpha)}$, then the inequalities (21) become the identity (7) of Lemma 1.6.

Remark 4.3. If in Lemma 4.1, we get $\varphi(t) = \frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, then the identity (21) reduces to

$$\begin{aligned}
&\frac{f(a)+f(b)}{2} - \frac{\Gamma_k(\alpha+k)}{(b-a)^{\frac{\alpha}{k}}} \left[I^\alpha_{a^+,k} f(b) + I^\alpha_{b^-,k} f(a) \right] \\
&= \frac{b-a}{2} \int_0^1 \left[(1-t)^{\frac{\alpha}{k}} - t^{\frac{\alpha}{k}} \right] f'(ta+(1-t)b)\,dt
\end{aligned}$$

which are proved by Hussain et. al. in [22].

Corollary 4.2. *Under the assumption of Lemma 4.1 with $\varphi(t) = t(b-t)^{\alpha-1}$ and f is a symmetric to $\frac{(a+b)}{2}$, then we have*

$$\begin{aligned}
&\frac{f(a)+f(b)}{2} - \frac{\alpha}{b^\alpha - a^\alpha} \int_a^b f(t)d_\alpha t \\
&= \frac{(b-a)}{2(b^\alpha - a^\alpha)} \int_0^1 \left([b-(b-a)(1-t)]^\alpha - [b-(b-a)t]^\alpha \right) f'(ta+(1-t)b)\,dt \\
&= \frac{(b-a)}{2(b^\alpha - a^\alpha)} \int_0^1 \left(b^\alpha - [b-(b-a)t]^\alpha \right) \left[f'(tb+(1-t)a) - f'(ta+(1-t)b) \right] dt.
\end{aligned}$$

Remark 4.4. If in Lemma 4.1, we get $\varphi(t) = \frac{t}{\alpha} \exp\left(-\frac{1-\alpha}{\alpha} t\right)$, $\alpha \in (0,1)$, then we have the following identity

$$\begin{aligned}
&\frac{f(a)+f(b)}{2} - \frac{1-\alpha}{2(1-\exp(-A))} \left[\mathcal{I}^\alpha_{a^+} f(b) + \mathcal{I}^\alpha_{b^-} f(a) \right] \\
&= \frac{b-a}{2(1-\exp(-A))} \int_0^1 \left[\exp(-At) - \exp(-A(1-t)) \right] f'(ta+(1-t)b)\,dt
\end{aligned}$$

is proved by Kirane and Torebek in [18].

Now, we extend some estimates of the right hand side of a Hermite-Hadamard type inequality for functions whose first derivatives absolute values are convex as follows:

Theorem 4.3. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) with $a < b$. If $|f'|$ is convex on $[a,b]$, then the following inequality for generalized fractional integrals* *hold:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{2\Lambda(1)}\left[{}_{a^+}I_{\varphi}f(b)+{}_{b^-}I_{\varphi}f(a)\right]\right| \leq \frac{(b-a)}{\Lambda(1)}\int_0^1 t\left|\Lambda(1-t)-\Lambda(t)\right|dt\left(\frac{|f'(a)|+|f'(b)|}{2}\right). \tag{22}$$

Proof. Using Lemma 4.1 and the convexity of $|f'|$, we find that

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{2\Lambda(1)}\left[{}_{a^+}I_{\varphi}f(b)+{}_{b^-}I_{\varphi}f(a)\right]\right| \\
&\leq \frac{(b-a)}{2\Lambda(1)}\int_0^1\left[\int_0^{1-t}\frac{\varphi((b-a)u)}{u}du-\int_0^t\frac{\varphi((b-a)u)}{u}du\right]\left|f'(ta+(1-t)b)\right|dt \\
&\leq \frac{(b-a)}{2\Lambda(1)}\left|f'(a)\right|\int_0^1 t\left|\int_0^{1-t}\frac{\varphi((b-a)u)}{u}du-\int_0^t\frac{\varphi((b-a)u)}{u}du\right|dt \\
&+\frac{(b-a)}{2\Lambda(1)}\left|f'(b)\right|\int_0^1(1-t)\left|\int_0^{1-t}\frac{\varphi((b-a)u)}{u}du-\int_0^t\frac{\varphi((b-a)u)}{u}du\right|dt
\end{aligned}$$

which this completes the proof of the (22). This completes the proof. □

Remark 4.5. If in Theorem 4.3, we get $\varphi(t)=t$, then, the inequalities (22) become the inequalities (3) of Theorem 1.2.

Remark 4.6. If in Theorem 4.3, we get $\varphi(t)=\frac{t^{\alpha}}{\Gamma(\alpha)}$, then the inequalities (22) become the inequalities (8) of Theorem 1.7.

Remark 4.7. If in Theorem 4.3, we get $\varphi(t)=\frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, then the inequalities (22) become

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma_k(\alpha+k)}{(b-a)^{\frac{\alpha}{k}}}\left[I^{\alpha}_{a^+,k}\,f(b)+\,I^{\alpha}_{b^-,k}\,f(a)\right]\right| \leq \frac{(b-a)}{\left(\frac{\alpha}{k}+1\right)}\left(1-\frac{1}{2^{\frac{\alpha}{k}}}\right)\left(\frac{|f'(a)|+|f'(b)|}{2}\right)$$

which are proved by Hussain et. al. in [22].

Corollary 4.4. *Under the assumption of Theorem 4.3 with* $\varphi(t)=t(b-t)^{\alpha-1}$ *and* f *is a symmetric to* $\frac{(a+b)}{2}$, *then we have*

$$\left|\frac{f(a)+f(b)}{2}-\frac{\alpha}{b^{\alpha}-a^{\alpha}}\int_a^b f(t)d_{\alpha}t\right| \leq \frac{1}{(b^{\alpha}-a^{\alpha})(\alpha+1)}\left[a^{\alpha+1}+b^{\alpha+1}-\frac{(a+b)^{\alpha+1}}{2^{\alpha}}\right]\left[\frac{|f'(a)|+|f'(b)|}{2}\right].$$

Proof. In (22), if we take $\varphi(t)=t(b-t)^{\alpha-1}$, then it follows that

$$\left|\frac{f(a)+f(b)}{2}-\frac{\alpha}{b^{\alpha}-a^{\alpha}}\int_a^b f(t)d_{\alpha}t\right| \leq \frac{(b-a)\alpha}{(b^{\alpha}-a^{\alpha})}\int_0^1 t\left|\Lambda(1-t)-\Lambda(t)\right|dt\left(\frac{|f'(a)|+|f'(b)|}{2}\right). \tag{23}$$

By computing $M=\int_0^1 t\left|\Lambda(1-t)-\Lambda(t)\right|dt$ in (23), we get

$$\begin{aligned} M &= \frac{1}{\alpha}\int_0^1 t\left|\left[b-(b-a)(1-t)\right]^\alpha-\left[b-(b-a)t\right]^\alpha\right|dt \\ &= \frac{1}{\alpha(b-a)^2}\int_a^b (u-a)\left|u^\alpha-\left[a+b-u\right]^\alpha\right|du \\ &= \frac{1}{\alpha(b-a)^2}\int_a^{\frac{a+b}{2}} (u-a)\left(\left[a+b-u\right]^\alpha-u^\alpha\right)du \\ &\quad +\frac{1}{\alpha(b-a)^2}\int_{\frac{a+b}{2}}^b (u-a)\left(u^\alpha-\left[a+b-u\right]^\alpha\right)du \\ &= \frac{1}{\alpha(b-a)(\alpha+1)}\left[a^{\alpha+1}+b^{\alpha+1}-\frac{(a+b)^{\alpha+1}}{2^\alpha}\right] \end{aligned}$$

which completes the proof. □

Remark 4.8. If in Corollary 4.4, we take $\alpha=1$, then we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)dt\right|\le\frac{(b-a)}{4}\left[\frac{|f'(a)|+|f'(b)|}{2}\right]$$

which are proved by Dragomir and Agarwal in [3].

Remark 4.9. If in Theorem 4.3, we get $\varphi(t)=\frac{t}{\alpha}\exp\left(-\frac{1-\alpha}{\alpha}t\right),\ \alpha\in(0,1)$, then we have

$$\begin{aligned} &\left|\frac{f(a)+f(b)}{2}-\frac{1-\alpha}{2\left(1-\exp(-A)\right)}\left[\mathcal{I}_{a^+}^\alpha f(b)+\mathcal{I}_{b^-}^\alpha f(a)\right]\right| \\ &\qquad\le\frac{b-a}{2A}\tanh\left(\frac{A}{4}\right)\left(\frac{|f'(a)|+|f'(b)|}{2}\right) \end{aligned}$$

is proved by Kirane and Torebek in [18].

Theorem 4.5. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$. If $|f'|^q$ is convex on $[a,b]$ for some $q>1$, then the following inequality for fractional integrals holds:*

$$\begin{aligned} &\left|\frac{f(a)+f(b)}{2}-\frac{1}{2\Lambda(1)}\left[{}_{a^+}I_\varphi f(b)+{}_{b^-}I_\varphi f(a)\right]\right| \\ &\qquad\le\frac{(b-a)}{2\Lambda(1)}\left(\int_0^1\left|\Lambda(1-t)-\Lambda(t)\right|^p dt\right)^{\frac{1}{p}}\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}}. \quad (24) \end{aligned}$$

Proof. Using Lemma 4.1, Hölder's inequality and the convexity of $|f'|^q$, we find that

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{2\Lambda(1)}\left[{}_{a^+}I_\varphi f(b)+{}_{b^-}I_\varphi f(a)\right]\right|$$

$$\leq \frac{(b-a)}{2\Lambda(1)}\left(\int_0^1\left|\int_0^{1-t}\frac{\varphi((b-a)u)}{u}du-\int_0^t\frac{\varphi((b-a)u)}{u}du\right|^p dt\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^1\left|f'(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}$$

$$\leq \frac{(b-a)}{2\Lambda(1)}\left(\int_0^1\left|\int_0^{1-t}\frac{\varphi((b-a)u)}{u}du-\int_0^t\frac{\varphi((b-a)u)}{u}du\right|^p dt\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^1 t\left|f'(a)\right|^q+(1-t)\left|f'(b)\right|^q dt\right)^{\frac{1}{q}}$$

which this completes the proof of the (24). This completes the proof. □

Remark 4.10. If in Theorem 4.5, we get $\varphi(t)=t$, then, the inequalities (24) become the inequalities (2.4) of Theorem 2.3 in [3].

Remark 4.11. If in Theorem 4.5, we get $\varphi(t)=\frac{t^\alpha}{\Gamma(\alpha)}$, then the inequalities (24) become the inequalities (2.7) of Theorem 8 in [14].

Remark 4.12. If in Theorem 4.5 we get $\varphi(t)=\frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, then the inequalities (24) become

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma_k(\alpha+k)}{(b-a)^{\frac{\alpha}{k}}}\left[I^\alpha_{a^+,k}f(b)+I^\alpha_{b^-,k}f(a)\right]\right|$$

$$\leq\frac{(b-a)}{2\left(\frac{\alpha}{k}p+1\right)^{\frac{1}{p}}}\left(\frac{\left|f'(a)\right|^q+\left|f'(b)\right|^q}{2}\right)^{\frac{1}{q}}$$

where $\frac{1}{p}+\frac{1}{q}=1$, $\frac{\alpha}{k}\in[0,1]$, which are proved Hussain et. al. in [22].

Corollary 4.6. *Under the assumption of Theorem 4.5 with* $\varphi(t)=t(b-t)^{\alpha-1}$ *and* f *is a symmetric to* $\frac{(a+b)}{2}$, *then we have*

$$\left|\frac{f(a)+f(b)}{2}-\frac{\alpha}{b^\alpha-a^\alpha}\int_a^b f(t)d_\alpha t\right|$$

$$\leq\frac{(b-a)^{\frac{1}{q}}}{2^{\frac{1}{q}}(b^\alpha-a^\alpha)(\alpha p+1)^{\frac{1}{p}}}\left[a^{\alpha p+1}+b^{\alpha p+1}-\frac{(a+b)^{\alpha p+1}}{2^{\alpha p}}\right]^{\frac{1}{p}}\left(\frac{\left|f'(a)\right|^q+\left|f'(b)\right|^q}{2}\right)^{\frac{1}{q}}.$$

Proof. In (24), we take $\varphi(t)=t(b-t)^{\alpha-1}$, then we get

$$\left|\frac{f(a)+f(b)}{2}-\frac{\alpha}{b^\alpha-a^\alpha}\int_a^b f(t)d_\alpha t\right|$$

$$\leq\frac{(b-a)\alpha}{2(b^\alpha-a^\alpha)}\left(\int_0^1\left|\Lambda(1-t)-\Lambda(t)\right|^p dt\right)^{\frac{1}{p}}\left(\frac{\left|f'(a)\right|^q+\left|f'(b)\right|^q}{2}\right)^{\frac{1}{q}}. \tag{25}$$

By computing right-side integral in (25), using the inequality $A \geq B > 0$ and $q > 1$, $(A-B)^q \leq A^q - B^q$, we get

$$\int_0^1 |\Lambda(1-t) - \Lambda(t)|^p \, dt = \frac{1}{\alpha^p} \int_0^1 |[b-(b-a)(1-t)]^\alpha - [b-(b-a)t]^\alpha|^p \, dt$$

$$= \frac{1}{\alpha^p (b-a)} \int_a^b |u^\alpha - [a+b-u]^\alpha|^p \, du$$

$$= \frac{1}{\alpha^p (b-a)} \int_a^{\frac{a+b}{2}} ([a+b-u]^\alpha - u^\alpha)^p \, du + \frac{1}{\alpha^p (b-a)} \int_{\frac{a+b}{2}}^b (u^\alpha - [a+b-u]^\alpha)^p \, du$$

$$\leq \frac{1}{\alpha^p (b-a)} \int_a^{\frac{a+b}{2}} ([a+b-u]^{\alpha p} - u^{\alpha p}) \, du + \frac{1}{\alpha^p (b-a)} \int_{\frac{a+b}{2}}^b (u^{\alpha p} - [a+b-u]^{\alpha p}) \, du$$

$$= \frac{2}{\alpha^p (b-a)(\alpha p + 1)} \left[a^{\alpha p+1} + b^{\alpha p+1} - \frac{(a+b)^{\alpha p+1}}{2^{\alpha p}} \right]$$

which completes the proof. □

5. Midpoint inequalities for generalized fractional integrals

Before starting and proving our next result, we need the following lemma.

Lemma 5.1. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) with $a < b$. If $f' \in L[a,b]$, then the following equalities for generalized fractional integrals holds:*

$$f\left(\frac{a+b}{2}\right) - \frac{1}{2\Lambda(1)} \left[{}_{a^+}I_\varphi f(b) + {}_{b^-}I_\varphi f(a) \right] = \frac{b-a}{2\Lambda(1)} \sum_{k=1}^{4} J_k \tag{26}$$

where

$$J_1 = \int_0^{\frac{1}{2}} \Lambda(t) f'(tb + (1-t)a) \, dt, \qquad J_2 = \int_0^{\frac{1}{2}} (-\Lambda(t)) f'(ta + (1-t)b) \, dt,$$
$$J_3 = \int_{\frac{1}{2}}^1 (-\Delta(t)) f'(tb + (1-t)a) \, dt, \quad J_4 = \int_{\frac{1}{2}}^1 \Delta(t) f'(ta + (1-t)b) \, dt.$$

Proof. In the proof of (26), we apply integration by parts, then we have

$$J_1 = \int_0^{\frac{1}{2}} \left(\int_0^t \frac{\varphi((b-a)u)}{u} du \right) f'(tb + (1-t)a) \, dt$$

$$= \frac{1}{b-a} \left(\int_0^{\frac{1}{2}} \frac{\varphi((b-a)u)}{u} du \right) f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_0^{\frac{1}{2}} \frac{\varphi((b-a)t)}{t} f(tb + (1-t)a) \, dt,$$

$$J_2 = \int_0^{\frac{1}{2}} \left(-\int_0^t \frac{\varphi((b-a)u)}{u} du \right) f'(ta + (1-t)b) \, dt$$

$$= \frac{1}{b-a} \left(\int_0^{\frac{1}{2}} \frac{\varphi((b-a)u)}{u} du \right) f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_0^{\frac{1}{2}} \frac{\varphi((b-a)t)}{t} f(ta + (1-t)b) \, dt,$$

$$J_3=\int_{\frac{1}{2}}^{1}\left(-\int_{t}^{1}\frac{\varphi\left((b-a)\,u\right)}{u}du\right)f'\left(tb+(1-t)a\right)dt$$
$$=\frac{1}{b-a}\left(\int_{\frac{1}{2}}^{1}\frac{\varphi\left((b-a)\,u\right)}{u}du\right)f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_{\frac{1}{2}}^{1}\frac{\varphi\left((b-a)\,t\right)}{t}f\left(tb+(1-t)a\right)dt,$$

$$J_4=\int_{\frac{1}{2}}^{1}\left(\int_{t}^{1}\frac{\varphi\left((b-a)\,u\right)}{u}du\right)f'\left(ta+(1-t)b\right)dt$$
$$=\frac{1}{b-a}\left(\int_{\frac{1}{2}}^{1}\frac{\varphi\left((b-a)\,u\right)}{u}du\right)f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_{\frac{1}{2}}^{1}\frac{\varphi\left((b-a)\,t\right)}{t}f\left(ta+(1-t)b\right)dt.$$

Thus, by the above expressions, the desired identity (26) is obtained. □

Remark 5.1. If in Lemma 5.1, we get $\varphi\left(t\right)=t$, then, the identity (26) become the identity (4) of Lemma 1.3 by Kirmaci in [5].

Remark 5.2. If in Lemma 5.1, we get $\varphi\left(t\right)=\frac{t^{\alpha}}{\Gamma(\alpha)}$, then the identity (26) become the inequalities (9) of Lemma 1.8 by Iqbal et al. in [19] .

Corollary 5.2. *Under the assumption of Lemma 5.1 with* $\varphi\left(t\right)=\frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, *then the identity (26) become*

$$f\left(\frac{a+b}{2}\right)-\frac{\Gamma_k\left(\alpha+k\right)}{(b-a)^{\frac{\alpha}{k}}}\left[I^{\alpha}_{a^+,k}\ f(b)+\ I^{\alpha}_{b^-,k}\ f(a)\right]=\frac{b-a}{2}\sum_{k=1}^{4}I^1_k,$$

where

$$I^1_1=\int_0^{\frac{1}{2}}t^{\frac{\alpha}{k}}f'\left(tb+(1-t)a\right)dt,\qquad I^1_2=\int_0^{\frac{1}{2}}\left(-t^{\frac{\alpha}{k}}\right)f'\left(ta+(1-t)b\right)dt,$$
$$I^1_3=\int_{\frac{1}{2}}^{1}\left(t^{\frac{\alpha}{k}}-1\right)f'\left(tb+(1-t)a\right)dt,\quad I^1_4=\int_{\frac{1}{2}}^{1}\left(1-t^{\frac{\alpha}{k}}\right)f'\left(ta+(1-t)b\right)dt.$$

Corollary 5.3. *Under the assumption of Lemma 5.1 with* $\varphi\left(t\right)=t\left(b-t\right)^{\alpha-1}$ *and* f *is a symmetric to* $\frac{(a+b)}{2}$, *then we have*

$$f\left(\frac{a+b}{2}\right)-\frac{\alpha}{b^{\alpha}-a^{\alpha}}\int_a^b f(t)d_{\alpha}t=\frac{b-a}{2\left(b^{\alpha}-a^{\alpha}\right)}\sum_{k=1}^{4}J^1_k$$

where

$$J^1_1=\frac{1}{\alpha}\int_0^{\frac{1}{2}}\left(b^{\alpha}-\left[b-\left(b-a\right)t\right]^{\alpha}\right)f'\left(tb+(1-t)a\right)dt,$$

$$J^1_2=-\frac{1}{\alpha}\int_0^{\frac{1}{2}}\left(b^{\alpha}-\left[b-\left(b-a\right)t\right]^{\alpha}\right)f'\left(ta+(1-t)b\right)dt,$$

$$J^1_3=-\frac{1}{\alpha}\int_{\frac{1}{2}}^{1}\left(\left[b-\left(b-a\right)t\right]^{\alpha}-a^{\alpha}\right)f'\left(tb+(1-t)a\right)dt,$$

$$J^1_4=\frac{1}{\alpha}\int_{\frac{1}{2}}^{1}\left(\left[b-\left(b-a\right)t\right]^{\alpha}-a^{\alpha}\right)f'\left(ta+(1-t)b\right)dt.$$

Proof. In (26), if we take $\varphi(t) = t(b-t)^{\alpha-1}$, then we get

$$\Lambda(t) = \frac{1}{\alpha}\left(b^{\alpha} - [b-(b-a)t]^{\alpha}\right)$$

and

$$\Delta(t) = \frac{1}{\alpha}\left([b-(b-a)t]^{\alpha} - a^{\alpha}\right).$$

Thus, we obtain

$$f\left(\frac{a+b}{2}\right) - \frac{\alpha}{b^{\alpha}-a^{\alpha}}\int_a^b f(t)d_{\alpha}t = \frac{b-a}{2(b^{\alpha}-a^{\alpha})}\sum_{k=1}^{4} J_k^1$$

which completes the proof. □

Corollary 5.4. *Under the assumption of Lemma 5.1 with* $\varphi(t) = \frac{t}{\alpha}\exp\left(-\frac{1-\alpha}{\alpha}t\right)$, $\alpha \in (0,1)$, *then we have*

$$f\left(\frac{a+b}{2}\right) - \frac{1-\alpha}{2(1-\exp(-A))}\left[\mathcal{I}_{a^+}^{\alpha} f(b) + \mathcal{I}_{b^-}^{\alpha} f(a)\right] = \frac{b-a}{2(1-\exp(-A))}\sum_{k=1}^{4} J_k^1$$

where $A = \frac{1-\alpha}{\alpha}(b-a)$ *and*

$$J_1^1 = \int_0^{\frac{1}{2}} (\exp(-At) - 1)\, f'(tb+(1-t)a)\, dt,$$

$$J_2^1 = \int_0^{\frac{1}{2}} (1-\exp(-At))\, f'(ta+(1-t)b)\, dt,$$

$$J_3^1 = \int_{\frac{1}{2}}^{1} (\exp(-At) - \exp(-A))\, f'(tb+(1-t)a)\, dt,$$

$$J_4^1 = \int_{\frac{1}{2}}^{1} (\exp(-A) - \exp(-At))\, f'(ta+(1-t)b)\, dt.$$

Finally, we extend some estimates of the left hand side of a Hermite-Hadamard type inequality for functions whose first derivatives absolute values are convex as follows:

Theorem 5.5. *Let* $f : [a,b] \to \mathbb{R}$ *be a differentiable mapping on* (a,b) *with* $a < b$. *If* $|f'|$ *is convex on* $[a,b]$, *then the following inequality for generalized fractional integrals hold:*

$$\begin{aligned} &\left| f\left(\frac{a+b}{2}\right) - \frac{1}{2\Lambda(1)}\left[{}_{a^+}I_{\varphi}f(b) + {}_{b^-}I_{\varphi}f(a)\right]\right| \\ &\leq \frac{(b-a)}{\Lambda(1)}\left[\frac{|f'(a)|+|f'(b)|}{2}\right]\left(\int_0^{\frac{1}{2}}|\Lambda(t)|\,dt + \int_{\frac{1}{2}}^{1}|\Delta(t)|\,dt\right). \qquad (27) \end{aligned}$$

Proof. By using the convexity of $|f'|$, then the inequality (27) is obtained as follows

$$\begin{aligned} &\left| f\left(\frac{a+b}{2}\right) - \frac{1}{2\Lambda(1)}\left[{}_{a^+}I_{\varphi}f(b) + {}_{b^-}I_{\varphi}f(a)\right]\right| \\ &\leq \frac{(b-a)}{2\Lambda(1)}\left[\int_0^{\frac{1}{2}}\left|\int_0^t \frac{\varphi((b-a)u)}{u}du\right|dt + \int_{\frac{1}{2}}^{1}\left|\int_t^1 \frac{\varphi((b-a)u)}{u}du\right|dt\right]\left[|f'(a)|+|f'(b)|\right]. \end{aligned}$$

This completes the proof. □

Remark 5.3. If in Theorem 5.5, we get $\varphi(t) = t$, then, the inequality (27) reduces to the inequality (2.3) of Theorem 2.3 by Kirmaci in [5].

Remark 5.4. If in Theorem 5.5, we get $\varphi(t) = \frac{t^{\alpha}}{\Gamma(\alpha)}$, then the inequality (27) reduces to the inequality (3) of Theorem 2 by Iqbal et al. in [19] .

Corollary 5.6. *Under the assumption of Theorem 5.5 with* $\varphi(t) = \frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, *then the inequality (27) become*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}} \left[I^{\alpha}_{a+,k} f(b) + I^{\alpha}_{b-,k} f(a) \right] \right| \leq (b-a)\left[\frac{1}{2} - \frac{1}{\left(\frac{\alpha}{k}+1\right)} + \frac{1}{\left(\frac{\alpha}{k}+1\right)2^{\frac{\alpha}{k}}}\right]\left(\frac{|f'(a)|+|f'(b)|}{2}\right)$$

where $\frac{\alpha}{k} \in [0,1]$.

Proof. In (27), if we take $\varphi(t) = \frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, then we have

$$\left| f\left(\frac{a+b}{2}\right) - \frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}} \left[I^{\alpha}_{a+,k} f(b) + I^{\alpha}_{b-,k} f(a) \right] \right| \leq (b-a)\frac{\Gamma_k(\alpha+k)}{(b-a)^{\frac{\alpha}{k}}}\left[\frac{|f'(a)|+|f'(b)|}{2}\right]\left(\int_0^{\frac{1}{2}} |\Lambda(t)|\,dt + \int_{\frac{1}{2}}^1 |\Delta(t)|\,dt\right). \quad (28)$$

By computing two integrals in (28), since $\Lambda(t) = \frac{(b-a)^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)} t^{\frac{\alpha}{k}}$, we get

$$\int_0^{\frac{1}{2}} |\Lambda(t)|\,dt = \frac{(b-a)^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\int_0^{\frac{1}{2}} t^{\frac{\alpha}{k}}\,dt = \frac{k(b-a)^{\frac{\alpha}{k}}}{(\alpha+k)\Gamma_k(\alpha+k)2^{\frac{\alpha}{k}+1}}$$

and since $\Delta(t) = \frac{(b-a)^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\left(1 - t^{\frac{\alpha}{k}}\right)$, we have

$$\int_{\frac{1}{2}}^1 |\Delta(t)|\,dt = \frac{(b-a)^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\int_{\frac{1}{2}}^1 \left(1-t^{\frac{\alpha}{k}}\right)dt = \frac{(b-a)^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\left(\frac{1}{2} - \frac{k}{(\alpha+k)} + \frac{k}{(\alpha+k)2^{\frac{\alpha}{k}+1}}\right).$$

Thus, we get

$$\int_0^{\frac{1}{2}} |\Lambda(t)|\,dt + \int_{\frac{1}{2}}^1 |\Delta(t)|\,dt = \frac{(b-a)^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\left[\frac{1}{2} - \frac{k}{(\alpha+k)} + \frac{k}{(\alpha+k)2^{\frac{\alpha}{k}}}\right]$$

which completes the proof. □

Corollary 5.7. *Under the assumption of Theorem 5.5 with* $\varphi(t) = t(b-t)^{\alpha-1}$ *and* f *is a symmetric to* $\frac{(a+b)}{2}$, *then we have*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{\alpha}{b^\alpha - a^\alpha}\int_a^b f(t)d_\alpha t \right|$$
$$\leq \frac{1}{(b^\alpha - a^\alpha)}\left[\left(a^{\alpha+1}+b^{\alpha+1}\right)\frac{\alpha}{\alpha+1} - (a^\alpha + b^\alpha)\frac{a+b}{2} + \frac{(a+b)^{\alpha+1}}{2^\alpha(\alpha+1)}\right]\left[\frac{|f'(a)|+|f'(b)|}{2}\right].$$

Proof. In (27), if we take $\varphi(t) = t(b-t)^{\alpha-1}$, it follows that

$$\left| f\left(\frac{a+b}{2}\right) - \frac{\alpha}{b^\alpha - a^\alpha}\int_a^b f(t)d_\alpha t \right|$$
$$\leq (b-a)\frac{\alpha}{b^\alpha - a^\alpha}\left[\frac{|f'(a)|+|f'(b)|}{2}\right]\left(\int_0^{\frac{1}{2}}|\Lambda(t)|\,dt + \int_{\frac{1}{2}}^1 |\Delta(t)|\,dt\right). \tag{29}$$

By computing two integrals in (29), we get

$$\begin{aligned}
\int_0^{\frac{1}{2}}|\Lambda(t)|\,dt &= \frac{1}{\alpha}\int_0^{\frac{1}{2}}\left|b^\alpha - [b-(b-a)t]^\alpha\right|dt \\
&= \frac{1}{\alpha(b-a)}\int_{\frac{a+b}{2}}^b |b^\alpha - s^\alpha|\,ds \\
&= \frac{1}{\alpha(b-a)}\left[\frac{\alpha}{\alpha+1}b^{\alpha+1} + \frac{(a+b)^{\alpha+1}}{2^{\alpha+1}(\alpha+1)} - b^\alpha\frac{a+b}{2}\right].
\end{aligned}$$

and

$$\begin{aligned}
\int_{\frac{1}{2}}^1|\Delta(t)|\,dt &= \frac{1}{\alpha}\int_{\frac{1}{2}}^1\left|[b-(b-a)t]^\alpha - a^\alpha\right|dt \\
&= \frac{1}{\alpha(b-a)}\int_a^{\frac{a+b}{2}} |s^\alpha - a^\alpha|\,ds \\
&= \frac{1}{\alpha(b-a)}\left[\frac{\alpha}{\alpha+1}a^{\alpha+1} + \frac{(a+b)^{\alpha+1}}{2^{\alpha+1}(\alpha+1)} - a^\alpha\frac{a+b}{2}\right].
\end{aligned}$$

Thus, we get

$$\int_0^{\frac{1}{2}}|\Lambda(t)|\,dt + \int_{\frac{1}{2}}^1|\Delta(t)|\,dt$$
$$= \frac{1}{\alpha(b-a)}\left[\left(a^{\alpha+1}+b^{\alpha+1}\right)\frac{\alpha}{\alpha+1} - (a^\alpha+b^\alpha)\frac{a+b}{2} + \frac{(a+b)^{\alpha+1}}{2^\alpha(\alpha+1)}\right]$$

which completes the proof. □

Remark 5.5. If in Corollary 5.7, we take $\alpha = 1$, then we have

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leq \frac{(b-a)}{4}\left[\frac{|f'(a)|+|f'(b)|}{2}\right]$$

which is given by Kirmaci in [5].

Corollary 5.8. *Under the assumption of Theorem 5.5 with* $\varphi(t) = \frac{t}{\alpha}\exp\left(-\frac{1-\alpha}{\alpha}t\right)$, $\alpha \in (0,1)$, *then we have*

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1-\alpha}{2(1-\exp(-A))}\left[\mathcal{I}^{\alpha}_{a^+} f(b) + \mathcal{I}^{\alpha}_{b^-} f(a)\right]\right| \leq \frac{b-a}{1-\exp(-A)}$$
$$\times \left(\frac{1}{2} - \frac{1}{2}\exp(-A) + \frac{1}{A}\left(2\exp\left(-\frac{A}{2}\right) - \exp(-A) - 1\right)\right)\frac{|f'(a)| + |f'(b)|}{2}.$$

Proof. In (27), we take $\varphi(t) = \frac{t}{\alpha}\exp\left(-\frac{1-\alpha}{\alpha}t\right)$, $\alpha \in (0,1)$, we have

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1-\alpha}{2(1-\exp(-A))}\left[\mathcal{I}^{\alpha}_{a^+} f(b) + \mathcal{I}^{\alpha}_{b^-} f(a)\right]\right| \leq \frac{b-a}{1-\exp(-A)}$$

$$\times \frac{|f'(a)| + |f'(b)|}{2}\left(\int_0^{\frac{1}{2}} |1 - \exp(-At)|\, dt + \int_{\frac{1}{2}}^1 |\exp(-A) - \exp(-At)|\, dt\right)$$
$$= \frac{b-a}{1-\exp(-A)}\frac{|f'(a)| + |f'(b)|}{2}\left(\frac{1}{2} - \frac{1}{A} + \frac{2}{A}\exp\left(-\frac{A}{2}\right) - \left(\frac{1}{2} + \frac{1}{A}\right)\exp(-A)\right)$$

which is completed the proof. ☐

Remark 5.6. If in Corollary 5.8, for $\alpha = 1$, $A \to 0$, it follows that

$$(b-a)\lim_{A\to 0}\left[\frac{\frac{1}{2} - \frac{1}{2}\exp(-A)}{(1-\exp(-A))} + \frac{2\exp\left(-\frac{A}{2}\right) - \exp(-A) - 1}{A(1-\exp(-A))}\right] = \frac{b-a}{4}$$

then we have

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{b-a}{4}\left[\frac{|f'(a)| + |f'(b)|}{2}\right]$$

which is given by Kirmaci in [5].

Theorem 5.9. *Let* $f : [a,b] \to \mathbb{R}$ *be a differentiable mapping on* (a,b) *with* $a < b$. *If* $|f'|^q$ *is convex on* $[a,b]$ *for some* $q > 1$, *then the following inequality for fractional integrals holds:*

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{2\Lambda(1)}\left[{}_{a^+}I_{\varphi} f(b) + {}_{b^-}I_{\varphi} f(a)\right]\right|$$
$$\leq \frac{(b-a)S_p}{2^{\frac{1}{q}}\Lambda(1)}\left[\left(\frac{3|f'(a)|^q + |f'(b)|^q}{4}\right)^{\frac{1}{q}} + \left(\frac{|f'(a)|^q + 3|f'(b)|^q}{4}\right)^{\frac{1}{q}}\right], \tag{30}$$

where

$$S_p = \left(\int_0^{\frac{1}{2}} |\Lambda(t)|^p\, dt\right)^{\frac{1}{p}} + \left(\int_{\frac{1}{2}}^1 |\Delta(t)|^p\, dt\right)^{\frac{1}{p}}.$$

Proof. By using the convexity of $|f'|^q$, then the inequality (30) is obtained as follows

$$\begin{aligned}
&\left| f\left(\frac{a+b}{2}\right) - \frac{1}{2\Lambda(1)}\left[{}_{a^+}I_\varphi f(b) + {}_{b^-}I_\varphi f(a)\right] \right| \\
&\leq \frac{b-a}{2^{\frac{1}{q}}\Lambda(1)}\left[\left(\int_0^{\frac{1}{2}}\left|\int_0^t \frac{\varphi((b-a)u)}{u}du\right|^p dt\right)^{\frac{1}{p}} + \left(\int_{\frac{1}{2}}^1\left|\int_t^1 \frac{\varphi((b-a)u)}{u}du\right|^p dt\right)^{\frac{1}{p}}\right] \\
&\quad \times \left[\left(\frac{3|f'(a)|^q + |f'(b)|^q}{4}\right)^{\frac{1}{q}} + \left(\frac{|f'(a)|^q + 3|f'(b)|^q}{4}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

This completes the proof. □

Remark 5.7. If in Theorem 5.9, we get $\varphi(t) = t$, then, the inequality (30) reduces to the inequality (5) of Theorem 1.4 by Kirmaci in [5].

Remark 5.8. If in Theorem 5.9, we get $\varphi(t) = \frac{t^\alpha}{\Gamma(\alpha)}$, then the inequality (30) reduces to the inequality (4) of Theorem 3 by Iqbal et al. in [19] .

Corollary 5.10. *Under the assumption of Theorem 5.9 with* $\varphi(t) = \frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, *then the inequalities (30) become*

$$\begin{aligned}
&\left| f\left(\frac{a+b}{2}\right) - \frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}}\left[I^\alpha_{a^+,k} f(b) + I^\alpha_{b^-,k} f(a)\right] \right| \\
&\qquad \leq \frac{b-a}{\left(\frac{\alpha}{k}p+1\right)^{\frac{1}{p}} 2^{\frac{\alpha}{k}}}\left[\left(\frac{3|f'(a)|^q + |f'(b)|^q}{4}\right)^{\frac{1}{q}} + \left(\frac{|f'(a)|^q + 3|f'(b)|^q}{4}\right)^{\frac{1}{q}}\right]
\end{aligned}$$

where $\frac{\alpha}{k} \in [0,1]$.

Proof. In (30), if we take $\varphi(t) = \frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, it follows that

$$\begin{aligned}
&\left| f\left(\frac{a+b}{2}\right) - \frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}}\left[I^\alpha_{a^+,k} f(b) + I^\alpha_{b^-,k} f(a)\right] \right| \\
&\leq (b-a)\frac{\Gamma_k(\alpha+k)}{2^{\frac{1}{q}}(b-a)^{\frac{\alpha}{k}}}\left[\left(\frac{3|f'(a)|^q + |f'(b)|^q}{4}\right)^{\frac{1}{q}} + \left(\frac{|f'(a)|^q + 3|f'(b)|^q}{4}\right)^{\frac{1}{q}}\right] \\
&\quad \times \left[\left(\int_0^{\frac{1}{2}} |\Lambda(t)|^p dt\right)^{\frac{1}{p}} + \left(\int_{\frac{1}{2}}^1 |\Delta(t)|^p dt\right)^{\frac{1}{p}}\right]. \qquad (31)
\end{aligned}$$

By computing two integrals in (31), since $\Lambda(t) = \frac{(b-a)^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}t^{\frac{\alpha}{k}}$, we get

$$\begin{aligned}
\int_0^{\frac{1}{2}} |\Lambda(t)|^p dt &= \left[\frac{(b-a)^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\right]^p \int_0^{\frac{1}{2}} t^{\frac{\alpha}{k}p} dt \\
&= \left[\frac{(b-a)^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\right]^p \frac{1}{\left(\frac{\alpha}{k}p+1\right)2^{\frac{\alpha}{k}p+1}}.
\end{aligned}$$

Let $\alpha \in (0,1]$ and $\forall t_1, t_2 \in [0,1]$, $|t_1^\alpha - t_2^\alpha| \le |t_1 - t_2|^\alpha$. Then, since $\Delta(t) = \frac{(b-a)^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\left(1 - t^{\frac{\alpha}{k}}\right)$ it follows that

$$\begin{aligned}\int_{\frac{1}{2}}^{1} |\Delta(t)|^p \, dt &\le \left[\frac{(b-a)^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\right]^p \int_{\frac{1}{2}}^{1} (1-t)^{\frac{\alpha}{k}p} \, dt \\ &\le \left[\frac{(b-a)^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\right]^p \frac{1}{\left(\frac{\alpha}{k}p+1\right) 2^{\frac{\alpha}{k}p+1}},\end{aligned}$$

which is completed the proof. □

Corollary 5.11. *Under the assumption of Theorem 5.9 with $\varphi(t) = t(b-t)^{\alpha-1}$ and f is a symmetric to $\frac{(a+b)}{2}$, then we have*

$$\begin{aligned}&\left| f\left(\frac{a+b}{2}\right) - \frac{\alpha}{b^\alpha - a^\alpha} \int_a^b f(t) d_\alpha t \right| \\ &\quad \le \frac{(b-a)^{\frac{1}{q}} T(a,b;\alpha;p)}{2^{\frac{1}{q}}(b^\alpha - a^\alpha)} \left[\left(\frac{3|f'(a)|^q + |f'(b)|^q}{4}\right)^{\frac{1}{q}} + \left(\frac{|f'(a)|^q + 3|f'(b)|^q}{4}\right)^{\frac{1}{q}}\right],\end{aligned}$$

where

$$\begin{aligned}T(a,b;\alpha;p) &= \left[\left(\frac{\alpha p}{\alpha p+1} - \frac{a+b}{2}\right) b^{\alpha p+1} + \frac{(a+b)^{\alpha p+1}}{2^{\alpha p+1}(\alpha p+1)}\right]^{\frac{1}{p}} \\ &\quad + \left[\left(\frac{\alpha p}{\alpha p+1} - \frac{a+b}{2}\right) a^{\alpha p+1} + \frac{(a+b)^{\alpha p+1}}{2^{\alpha p+1}(\alpha p+1)}\right]^{\frac{1}{p}}.\end{aligned}$$

Proof. In (27), if we take $\varphi(t) = t(b-t)^{\alpha-1}$, it follows that

$$\begin{aligned}&\left| f\left(\frac{a+b}{2}\right) - \frac{\alpha}{b^\alpha - a^\alpha} \int_a^b f(t) d_\alpha t \right| \\ &\quad \le \frac{\alpha(b-a)}{2^{\frac{1}{q}}(b^\alpha - a^\alpha)} \left[\left(\frac{3|f'(a)|^q + |f'(b)|^q}{4}\right)^{\frac{1}{q}} + \left(\frac{|f'(a)|^q + 3|f'(b)|^q}{4}\right)^{\frac{1}{q}}\right] \\ &\quad\quad \times \left[\left(\int_0^{\frac{1}{2}} |\Lambda(t)|^p \, dt\right)^{\frac{1}{p}} + \left(\int_{\frac{1}{2}}^{1} |\Delta(t)|^p \, dt\right)^{\frac{1}{p}}\right]. \qquad (32)\end{aligned}$$

By computing two integrals in (32), using the inequality $A \ge B > 0$ and $q > 1$, $(A-B)^q \le A^q - B^q$, we get

$$\begin{aligned}\int_0^{\frac{1}{2}} |\Lambda(t)|^p \, dt &= \frac{1}{\alpha^p} \int_0^{\frac{1}{2}} |b^\alpha - [b-(b-a)t]^\alpha|^p \, dt \\ &\le \frac{1}{\alpha^p(b-a)} \int_{\frac{a+b}{2}}^{b} (b^{\alpha p} - s^{\alpha p}) \, ds \\ &\le \frac{1}{\alpha^p(b-a)} \left[\left(\frac{\alpha p}{\alpha p+1} - \frac{a+b}{2}\right) b^{\alpha p+1} + \frac{(a+b)^{\alpha p+1}}{2^{\alpha p+1}(\alpha p+1)}\right]\end{aligned}$$

and

$$\begin{aligned}\int_{\frac{1}{2}}^{1}|\Delta(t)|^{p}dt &= \frac{1}{\alpha^{p}}\int_{\frac{1}{2}}^{1}\left|[b-(b-a)t]^{\alpha}-a^{\alpha}\right|^{p}dt \\ &\leq \frac{1}{\alpha^{p}(b-a)}\int_{a}^{\frac{a+b}{2}}\left(s^{\alpha p}-a^{\alpha p}\right)ds \\ &= \frac{1}{\alpha^{p}(b-a)}\left[\left(\frac{\alpha p}{\alpha p+1}-\frac{a+b}{2}\right)a^{\alpha p+1}+\frac{(a+b)^{\alpha p+1}}{2^{\alpha p+1}(\alpha p+1)}\right].\end{aligned}$$

Thus, we get

$$\begin{aligned}&\left(\int_{0}^{\frac{1}{2}}|\Lambda(t)|^{p}dt\right)^{\frac{1}{p}}+\left(\int_{\frac{1}{2}}^{1}|\Delta(t)|^{p}dt\right)^{\frac{1}{p}} \\ &\leq\frac{1}{\alpha(b-a)^{\frac{1}{p}}}\left\{\left[\left(\frac{\alpha p}{\alpha p+1}-\frac{a+b}{2}\right)b^{\alpha p+1}+\frac{(a+b)^{\alpha p+1}}{2^{\alpha p+1}(\alpha p+1)}\right]^{\frac{1}{p}}\right. \\ &\quad\left.+\left[\left(\frac{\alpha p}{\alpha p+1}-\frac{a+b}{2}\right)a^{\alpha p+1}+\frac{(a+b)^{\alpha p+1}}{2^{\alpha p+1}(\alpha p+1)}\right]^{\frac{1}{p}}\right\},\end{aligned}$$

which is completed the proof. □

6. Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

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