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Periodic solutions for some second-order impulsive Hamiltonian systems

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Abstract. In this article, we study the existence of periodic solutions for a class of second-order impulsive Hamiltonian systems. Some new existence theorems are obtained by the least action principle.

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1. Introduction

Consider the following second-order impulsive Hamiltonian systems

$$\begin{cases} -\ddot{u}(t) + A(t)u(t) = \nabla F(x, u(t)) + \nabla H(u(t)), & \text{a.e. } t \in [0, T], \\ \Delta(\dot{u}_i(t_j)) = I_{ij}(u_i(t_j)), \quad i = 1, 2, ..., N, \ j = 1, 2, ..., p, \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0, \end{cases} \tag{1.1}$$

where $N \geq 1, p \geq 2$, $u = (u_1, ..., u_N)$, $T > 0$, $A : [0, T] \to \mathbb{R}^{N\times N}$ is a continuous map from the interval $[0, T]$ to the set of $N \times N$ symmetric matrices, t_j, $j = 1, 2, ..., p$, are the instants at which the impulses occur, $0 = t_0 < t_1 < ... < t_p < t_{p+1} = T$, and

$$\Delta(\dot{u}_i(t_j)) = \dot{u}_i(t_j^+) - \dot{u}_i(t_j^+) = \lim_{t\to t_j^+} \dot{u}_i(t) - \lim_{t\to t_j^-} \dot{u}_i(t).$$

The following conditions are assumed to hold throughout the remainder of this article. The functions $I_{ij} : \mathbb{R} \to \mathbb{R}$ are Lipschitz continuous with the Lipschitz constants $L_{ij} > 0$, i.e.,

$$|I_{ij}(s_1) - I_{ij}(s_2)| \leq L_{ij}|s_1 - s_2| \tag{1.2}$$

for every $s_1, s_2 \in \mathbb{R}$, and $I_{ij}(0) = 0$ for $i = 1, 2, ..., N$, $j = 1, 2, ..., p$. In addition, $F : [0, T] \times \mathbb{R}^N \to \mathbb{R}$ satisfies the following assumption:

(A_0) $F(t, x)$ is measurable in t for every $x \in \mathbb{R}^N$ and continuously differentiable in x for a.e. $t \in [0, T]$ and there exist $a \in C(\mathbb{R}^+, \mathbb{R}^+)$, $b \in L^1([0, T]; \mathbb{R}^+)$ such that

$$|F(t, x)| \leq a(|x|)b(t), \quad |\nabla F(t, x)| \leq a(|x|)b(t)$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0, T]$.

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The function $H : \mathbb{R}^N \to \mathbb{R}$ is continuously differentiable, ∇H is Lipschitz continuous with the Lipschitz constant $L > 0$, i.e.,

$$|\nabla H(\xi_1) - \nabla H(\xi_2)| \leq L|\xi_1 - \xi_2| \tag{1.3}$$

for every $\xi_1, \xi_2 \in \mathbb{R}^N$,

$$H(0, ..., 0) = 0, \quad \text{and} \quad \nabla H(0, ..., 0) = 0. \tag{1.4}$$

Assuming that $\nabla F : [0, T] \times \mathbb{R}^N \to \mathbb{R}$ is continuous, it implies that the condition (A_0) is satisfied.

The corresponding functions I on H_T^1 given by

$$I(u) = \Phi(u) + \Psi(u) = \frac{1}{2}\|u\|^2 + \sum_{j=1}^{p}\sum_{i=1}^{N}\int_0^{u_i(t_j)} I_{ij}(s)\, ds - \int_0^T H(u(t))\, dt + \int_0^T F(t, u(t))\, dt$$

is continuously differentiable and weakly lower semicontinuous on H_T^1 (see [11, Theorem 3.1]), where

$H_T^1 := \{u : [0, T] \to \mathbb{R}^N \mid u \text{ is absolutely continuous, } u(0) = u(T) \text{ and } \dot{u} \in L^2([0, T])\}$

is a Hilbert space with the inner product defined by

$$< u, v >_0 = \int_0^T \ [(\dot{u}(t), \dot{v}(t)) + (u(t), v(t))] \ dt.$$

The corresponding norm is defined by

$$\|u\|_0 = \left(\int_0^T \ \left(|\dot{u}(t)|^2 + |u(t)|^2\right) \ dt\right)^{\frac{1}{2}} \quad \text{for all } u \in H_T^1.$$

Moreover, one has

$$I'(u)(v) = \Phi'(u)(v) + \Psi'(u)(v) = \int_0^T [(\dot{u}(t), \dot{v}(t)) + (A(t)u(t), v(t)) - (\nabla H(u(t)), v(t))]\, dt$$
$$+ \sum_{j=1}^{p}\sum_{i=1}^{N} I_{ij}(u_i(t_j))v_i(t_j) + \int_0^T \ (\nabla F(t, u(t)), v(t)) \ dt$$

for every $u, v \in H_T^1$. It is well known that the solutions of the problem (1.1) correspond to the critical points of the functional $I = \Phi + \Psi$ (see [11, Definition 2.4]).

We assume throughout that the matrix A satisfies the following conditions:

(M_1) $A(t) = (a_{kl}(t))$, $k = 1, ..., N$, $l = 1, ..., N$, is a symmetric matrix with $a_{kl} \in L^\infty[0, T]$ for any $t \in [0, T]$;

(M_2) There exists $\delta > 0$ such that $(A(t)\xi, \xi) \geq \delta|\xi|^2$ for any $\xi \in \mathbb{R}^N$ and a.e. $t \in [0, T]$, where $(., .)$ denotes the inner product in $\mathbb{R}^N$.

For every $u, v \in H_T^1$, we define

$$< u, v > = \int_0^T \ [(\dot{u}(t), \dot{v}(t)) + (A(t)u(t), v(t))] \ dt,$$

and we observe that conditions (M_1) and (M_2) ensure that this defines an inner product in H_T^1. Then H_T^1 is a separable and reflexive Banach space with the norm

$$\|u\| = < u, u >^{\frac{1}{2}} \quad \text{for all } u \in H_T^1.$$

Clearly, H_T^1 is an uniformly convex Banach space.

A simple computation shows that

$$(A(t)\xi,\xi)=\sum_{k,l=1}^{N} a_{kl}(t)\xi_k\xi_l \leq \sum_{k,l=1}^{N} \|a_{kl}\|_{L^\infty}\,|\xi|^2$$

for every $t\in[0,T]$ and $\xi\in\mathbb{R}^N$. Along with condition (M_2), this implies

$$\sqrt{m}\|u\|_0 \leq \|u\| \leq \sqrt{M}\|u\|_0, \tag{1.5}$$

where $m=\min\{1,\delta\}$ and $M=\max\{1,\sum_{k,l}^{N}\|a_{kl}\|_\infty\}$, which means the norm $\|\cdot\|$ is equivalent to the norm $\|\cdot\|_0$. Since $(H_T^1,\|\cdot\|)$ is compactly embedded in $C([0,T],\mathbb{R}^N)$ (see [18]), there exists a positive constant c such that

$$\|u\|_\infty \leq c\|u\|,$$

where $\|u\|_\infty=\max_{t\in[0,T]}|u(t)|$ and $c=\sqrt{\frac{2}{m}}\max\{\frac{1}{\sqrt{T}},\sqrt{T}\}$ (see [4]).

For $u\in H_T^1$, let $\bar{u}=\frac{1}{T}\int_0^T u(t)\,dt$ and $\widetilde{u}(t)=u(t)-\bar{u}$. Then one has

$$\|\widetilde{u}\|_\infty^2 \leq \frac{T}{12}\int_0^T |\dot{u}(t)|^2\,dt \quad \text{(Sobolev's inequality)},$$

$$\|\widetilde{u}\|_{L^2}^2 \leq \frac{T^2}{4\pi^2}\int_0^T |\dot{u}(t)|^2\,dt \quad \text{(Wirtinger's inequality)}$$

(see [18, Proposition 1.3]).

When $I_{ij}=A=H\equiv 0$, the problem (1.1) reduces to the second order Hamiltonian system, it has been proved that the problem (1.1) has at least one solution by the least action principle and the minimax methods (see [27, 33, 34]). Many solvability conditions are given, such as the coercive condition (see [2]), the periodicity condition (see [26]); the convexity condition (see [17]); the subadditive condition (see [23]); the bounded condition (see [18]).

When the nonlinearity $\nabla F(t,x)$ is bounded sub-linearly, that is, there exist $f,g\in L^1([0,T],\mathbb{R}^+)$ and $\alpha\in[0,1)$ such that

$$|\nabla F(t,x)|\leq f(t)|x|^\alpha+g(t)$$

for all $x\in\mathbb{R}^N$ and a.e. $t\in[0,T]$, Tang [24] also proved the existence of solutions for the problem (1.1) when $I_{ij}=A=H\equiv 0$ under the condition

$$\lim_{|x|\to+\infty}|x|^{-2\alpha}\int_0^T F(t,x)\,dt\to+\infty, \tag{1.6}$$

or

$$\lim_{|x|\to+\infty}|x|^{-2\alpha}\int_0^T F(t,x)\,dt\to-\infty, \tag{1.7}$$

which generalizes Mawhin-Willem's results under bounded condition (see [18]). Wu and Tang [27, 28] also proved the existence of solutions for the problem (1.1) with cited conditions under a convenient condition and (1.6) or (1.7) with $\alpha=1$.

For $I_{ij}\not\equiv 0$, $i\in\{1,2,...,N\}$, $j\in\{1,2,...,p\}$, the problem (1.1) is an impulsive differential problem. Impulsive differential equations arising from the real world describe the dynamics of processes in which sudden, discontinuous jumps occur. For the background, theory and applications of impulsive differential equations, we refer the readers to the monographs and some recent contributions as [1, 6, 14, 19, 26].

Some classical tools such as fixed point theorems in cones [1, 15, 25], the method of lower and upper solutions [6, 30] have been widely used to study impulsive differential equations.

For notations and definitions, and for a thorough account on the subject and related problems concerning the variational analysis of solutions of some classes of boundary value problems we refer the reader to [4, 5, 9, 10, 11, 12, 20, 32], and reference therein.

It has been shown by the least action principle that the problem (1.1) has at least one solution which minimizes the functional I on H_T^1 in many papers. When $F(t,\cdot)$ is convex for a.e. $t \in [0,T]$, Mawhin and Willem [18] have studied the existence of solution which minimizes I on H_T^1 for the problem (1.1), by choosing $A = H \equiv 0$ and without impulsive condition. For non-convex potential cases, using the least action principle, the existence of solution which minimizes I on H_T^1 has been researched by many people; for example, see [16, 21, 22, 23, 28, 29, 34] and their references.

Inspired and motivated by the results in [9, 16, 21, 28, 29, 32, 34], we obtain some new results for the problem (1.1) by using the least action principle.

2. Main results

In this section, we establish the main abstract results of this paper. Before introducing our results and without further mention, we first will assume throughout that

$$K := c^2 \left(2LT + \sum_{j=1}^{p} \sum_{i=1}^{N} L_{ij} \right) < 1,$$

and recall a definition due to Wu and Tang [28]:

A function $G : \mathbb{R}^N \to \mathbb{R}$ is called (λ,μ)-subconvex if

$$G(\lambda(x+y)) \le \mu(G(x) + G(y))$$

for some $\lambda, \mu > 0$ and all $x, y \in \mathbb{R}^N$. A function is called γ-subadditive if it is $(1,\gamma)$-subconvex. A function is called subadditive if it is 1-subadditive. The convex and subadditive functions are special cases of subconvex functions.

Now, we present our first main result as follows:

Theorem 2.1. *Suppose that* $F(t,x) = F_1(t,x) + F_2(x)$, *where* F_1 *and* F_2 *satisfy assumption* (A_0) *and the following conditions:*

(i) $F_1(t,x)$ *is* (λ,μ)*-subconvex for a.e. respect to* $t \in [0,T]$, *where* $\lambda > \frac{1}{2}$ *and* $\mu < 2\lambda^2$;

(ii) there exist constants $0 \le r_1 < \frac{4\pi^2 m(1-K)}{T^2}$, $r_2 \in [0,+\infty)$ *such that*

$$|\nabla F_2(x) - \nabla F_2(y)| \le r_1|x-y| + r_2$$

for all $x, y \in \mathbb{R}^N$;

(iii)

$$\left(\frac{1}{\mu} \int_0^T F_1(t,\lambda x)\, dt + T\, F_2(x) \right) \to +\infty \quad \text{as } |x| \to +\infty.$$

Then the problem (1.1) *has at least one solution which minimizes the functional* I *on* H_T^1.

Proof. Let $\beta = \log_{2\lambda}^{2\mu}$, then since $2\lambda > 1$, one has $2\mu < 4\lambda^2$ and $\log_{2\lambda}^{2\mu} < 2$, so $\beta < 2$. In a similar way to Wu and Tang [28], by the (λ,μ)-subconvexity of $G(\cdot)$ and assumption (A_0), one can prove that

$$F_1(t,x) \le \left(2\mu|x|^\beta + 1\right) a_0\, b(t)$$

for a.e. $t \in [0,T]$ and all $x \in \mathbb{R}^N$, where $\beta < 2$, $a_0 = \max_{0\le s\le 1} a(s)$. Thus it follows from (i) and Sobolev's inequality that

$$\begin{aligned}
\int_0^T F_1(t,u(t))\,dt &\ge \frac{1}{\mu}\int_0^T F_1(t,\lambda\bar{u})\,dt - \int_0^T F_1(t,-\widetilde{u}(t))\,dt \\
&\ge \frac{1}{\mu}\int_0^T F_1(t,\lambda\bar{u})\,dt - \left(2\mu\|\widetilde{u}\|_\infty^\beta + 1\right) a_0 \int_0^T b(t)\,dt \\
&\ge \frac{1}{\mu}\int_0^T F_1(t,\lambda\bar{u})\,dt - C_1\|\dot{u}\|_2^\beta - C_2 \qquad (2.8)
\end{aligned}$$

for some constants C_1 and C_2. It follows from assumption (ii), Wirtinger's inequality and Sobolev's inequality that

$$\begin{aligned}
\left|\int_0^T [F_2(u(t)) - F_2(\bar{u})]\,dt\right| &= \left|\int_0^T\int_0^1 (\nabla F_2(\bar{u}+s\widetilde{u}(t)), \widetilde{u}(t))\,ds\,dt\right| \\
&= \left|\int_0^T\int_0^1 (\nabla F_2(\bar{u}+s\widetilde{u}(t)) - \nabla F_2(\bar{u}), \widetilde{u}(t))\,ds\,dt\right| \\
&\le \int_0^T\int_0^1 r_1 s|\widetilde{u}(t)|^2\,ds\,dt + r_2\int_0^T |\widetilde{u}(t)|\,dt \\
&\le \frac{r_1}{2}\int_0^T |\widetilde{u}(t)|^2\,dt + r_2 T\|\widetilde{u}(t)\|_\infty \\
&\le \frac{r_1 T^2}{8\pi^2}\|\dot{u}\|_2^2 + C_3\|\dot{u}\|_2 \qquad (2.9)
\end{aligned}$$

for all $u \in H_T^1$ and some positive constant C_3. It follows from (1.2), (1.5), (2.8), (2.9) and the fact that $I_{ij}(0) = 0$,

$$\begin{aligned}
I(u) = \Phi(u) + \Psi(u) &= \frac{1}{2}\|u\|^2 + \sum_{j=1}^{p}\sum_{i=1}^{N}\int_0^{u_i(t_j)} I_{ij}(s)\,ds \\
&\quad - \int_0^T H(u(t))\,dt + \int_0^T F(t,u(t))\,dt \\
&\ge \frac{1}{2}(1-K)\|u\|^2 + \int_0^T F_(t,u(t))\,dt \\
&\quad + \int_0^T [F_2(u(t)) - F_2(\bar{u})]\,dt + \int_0^T F_2(\bar{u})\,dt \\
&\ge \frac{1}{2}(1-K)m\|u\|_0^2 + \left[\frac{1}{\mu}\int_0^T F_1(t,\lambda\bar{u})\,dt + \int_0^T F_2(\bar{u})\,dt\right]
\end{aligned}$$

$$
\begin{aligned}
&- C_1\|\dot{u}\|_2^{\beta} - C_2 - \frac{r_1T^2}{8\pi^2}\|\dot{u}\|_2^2 - C_3\|\dot{u}\|_{L^2} \\
&\geq \left[\frac{1}{2}(1-K)m - \frac{r_1T^2}{8\pi^2}\right]\|\dot{u}\|_2^2 + \left[\frac{1}{\mu}\int_0^T F_1(t,\lambda\bar{u})\,dt + T\,F_2(\bar{u})\right] \\
&\qquad - C_1\|\dot{u}\|_2^{\beta} - C_2 - C_3\|\dot{u}\|_{L^2}
\end{aligned}
$$

for all $u \in H_T^1$, which implies that

$$I(u) \to +\infty$$

as $\|u\| \to \infty$ by (iii) because $r_1 < \frac{4\pi^2 m(1-K)}{T^2}$, $\beta < 2$ and

$$\|u\| \to \infty \ \Leftrightarrow\ \left(|\bar{u}|^2 + \|\dot{u}\|_2^2\right)^{\frac{1}{2}} \to \infty.$$

By Theorem 1.1 and Theorem 1.4 in Mawhin and Willem [18], and the fact that the functional I is weakly lower semi continuous, the proof is completed. $\square$

Theorem 2.2. *Suppose that $F(t,x) = F_1(t,x) + F_2(x)$, where F_1 and F_2 satisfy assumption (A_0) and the following conditions:*

(i) *there exist $k_0, m_0 \in L^1([0,T];\mathbb{R}^+)$ and a positive constant γ with $\gamma < 1$ such that $|\nabla F_1(t,x)| \leq k_0(t)|x|^{\gamma} + m_0(t)$ for all $x \in \mathbb{R}^N$ and a.e. $t \in [0,T]$;*

(ii) *there exist constants $0 \leq r_1 < \frac{2\pi^2[2m(1-K)-1]}{T^2}$, $r_2 \in [0,+\infty)$ such that*

$$|\nabla F_2(x) - \nabla F_2(y)| \leq r_1|x-y| + r_2$$

for all $x, y \in \mathbb{R}^N$;

(iii)

$$\frac{1}{|x|^{2\gamma}}\int_0^T F(t,x)\,dt \to +\infty \quad \text{as } |x| \to +\infty.$$

Then the problem (1.1) *has at least one solution which minimizes the functional I on H_T^1.*

Proof. By condition (i), Sobolev's inequality and $\gamma < 1$, one has

$$
\begin{aligned}
&\left|\int_0^T [F_1(t,u(t)) - F_1(t,\bar{u})]\,dt\right| = \left|\int_0^T\int_0^1 (\nabla F_1(t,\bar{u}+s\widetilde{u}(t)),\widetilde{u}(t))\,ds\,dt\right| \\
&\qquad \leq \int_0^T\int_0^1 k_0(t)|\bar{u}+s\widetilde{u}(t)|^{\gamma}|\widetilde{u}(t)|\,ds\,dt + \int_0^T m_0(t)|\widetilde{u}(t)|\,dt \\
&\qquad \leq 2\left(|\bar{u}|^{\gamma} + \|\widetilde{u}\|_{\infty}^{\gamma}\right)\|\widetilde{u}\|_{\infty}\int_0^T k_0(t)\,dt + \|\widetilde{u}\|_{\infty}\int_0^T m_0(t)\,dt \\
&\qquad \leq \frac{3}{T}\|\widetilde{u}\|_{\infty}^2 + \frac{T}{3}|\bar{u}|^{2\gamma}\left(\int_0^T k_0(t)\,dt\right)^2 + 2\|\widetilde{u}\|_{\infty}^{\gamma+1}\int_0^T k_0(t)\,dt + \|\widetilde{u}\|_{\infty}\int_0^T m_0(t)\,dt \\
&\qquad \leq \frac{1}{4}\|\dot{u}\|_2^2 + C_4|\bar{u}|^{2\gamma} + C_5\|\dot{u}\|_2^{\gamma+1} + C_6\|\dot{u}\|_2 \qquad (2.10)
\end{aligned}
$$

for all $u \in H_T^1$ and some positive constants C_4, C_5 and C_6. It follows from (2.9) and (2.10) that

$$\begin{aligned}
I(u) &= \Phi(u) + \Psi(u) \geq \frac{1}{2}(1-K)\|u\|^2 + \int_0^T F(t,u(t))\,dt \\
&\geq \frac{m}{2}(1-K)\|\dot{u}\|_2^2 + \int_0^T [F_1(t,u(t)) - F_1(t,\bar{u})]\,dt + \int_0^T [F_2(u(t)) - F_2(\bar{u})]\,dt \\
&\qquad + \int_0^T F(t,\bar{u})\,dt \geq \frac{m}{2}(1-K)\|\dot{u}\|_2^2 - \frac{r_1 T^2}{8\pi^2}\|\dot{u}\|_2^2 - C_3\|\dot{u}\|_2 \\
&\quad - \frac{1}{4}\|\dot{u}\|_2^2 - C_4|\bar{u}|^{2\gamma} - C_5\|\dot{u}\|_2^{\gamma+1} - C_6\|\dot{u}\|_2 + \int_0^T F(t,\bar{u})\,dt \\
&= \frac{1}{2}\left[m(1-K) - \frac{r_1 T^2}{4\pi^2} - \frac{1}{2}\right]\|\dot{u}\|_2^2 - (C_3 + C_6)\|\dot{u}\|_2 \\
&\quad - C_5\|\dot{u}\|_2^{\gamma+1} + |\bar{u}|^{2\gamma}\left[\frac{1}{|\bar{u}|^{2\gamma}}\int_0^T F(t,\bar{u})\,dt - C_4\right]
\end{aligned}$$

for all $u \in H_T^1$, which implies that $I(u) \to +\infty$ as $\|u\| \to +\infty$ by (iii) because $\gamma < 1$, $r_1 < \frac{2\pi^2[2m(1-K)-1]}{T^2}$ and

$$\|u\| \to +\infty \;\Leftrightarrow\; \left(|\bar{u}|^2 + \|\dot{u}\|_2^2\right)^{\frac{1}{2}} \to +\infty.$$

By Theorem 1.1 and Theorem 1.4 in Mawhin and Willem [18], the proof is completed. □

Remark 2.1. (see [32, Remark 2.1]) By choosing $A = H \equiv 0$ and $I_{ij} \equiv 0$, $\dot{u}_i(t_j) = u_i(t_j) = 0$, for every $i = 1,2,...,N$, $j = 1,2,...,p$, Theorem 1 in [29] is the direct corollary of our Theorem 2.1, where the condition (ii) of Theorem 1 in [29] implies the one of our Theorem 2.1. In the same reason, with choosing suitable $\epsilon > 0$, in condition (ii) of Theorem 2, it follows that Theorem 2 in [29] is also the direct corollary of our Theorem 2.2.

Theorem 2.3. *Suppose that $F(t,x) = F_1(t,x) + F_2(x)$, where F_1 and F_2 satisfy assumption (A_0) and the following conditions:*

(i) *there exist some $g \in L^1([0,T];\mathbb{R})$ and some $h \in L^1([0,T];\mathbb{R}^N)$ such that*

$$F_1(t,x) \geq (h(t),x) + g(t)$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0,T]$;

(ii) *there exist constants $0 \leq r_1 < \frac{4m\pi^2(1-K)}{T^2}$, $r_2 \in [0,+\infty)$ such that*

$$|\nabla F_2(x) - \nabla F_2(y)| \leq r_1|x-y| + r_2$$

for all $x, y \in \mathbb{R}^N$;

(iii) $\frac{F_2(x)}{|x|} \to +\infty$ *as* $|x| \to +\infty$.

Then the problem (1.1) has at least one solution which minimizes the functional I on H_T^1.

Proof. By condition (i) and Sobolev's inequality, one has

$$\begin{aligned}\int_0^T F_1(t,u(t))\,dt &\geq \int_0^T \left[(h(t),\bar{u}+\widetilde{u}(t))+g(t)\right]\,dt\\ &\geq -|\bar{u}|\int_0^T |h(t)|\,dt-\|\widetilde{u}\|_\infty\int_0^T |h(t)|\,dt+\int_0^T g(t)\,dt\\ &\geq -D_1|\bar{u}|-D_2\|\dot{u}\|_2+D_3, \end{aligned}\tag{2.11}$$

for some constants D_1, D_2 and D_3. By (2.9) and (2.11), one has

$$\begin{aligned}&I(u)=\Phi(u)+\Psi(u)\geq\frac{1}{2}(1-K)\|u\|^2+\int_0^T F(t,u(t))\,dt\\ &\geq\frac{m}{2}(1-K)\|\dot{u}\|_2^2+\int_0^T F_1(t,u(t))\,dt+\int_0^T\left[F_2(u(t))-F_2(\bar{u})\right]\,dt+\int_0^T F_2(\bar{u})\,dt\\ &\geq\frac{m}{2}(1-K)\|\dot{u}\|_2^2-D_2\|\dot{u}\|_2+D_3-\frac{r_1T^2}{8\pi^2}\|\dot{u}\|_2^2-C_3\|\dot{u}\|_2+|\bar{u}|\left(\frac{\int_0^T F_2(\bar{u})\,dt}{|\bar{u}|}-D_1\right)\end{aligned}$$

for all $u\in H_T^1$, which implies that $I(u)\to+\infty$ as $\|u\|\to+\infty$ by (iii) because $0\leq r_1<\frac{4m\pi^2(1-K)}{T^2}$ and

$$\|u\|\to+\infty \;\Leftrightarrow\; \left(|\bar{u}|^2+\|\dot{u}\|_2^2\right)^{\frac{1}{2}}\to+\infty.$$

By Theorem 1.1 and Theorem 1.4 in Mawhin and Willem [18], the proof is completed. □

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