Annals of the University of Craiova, Mathematics and Computer Science Series
Volume 45(2), 2018, Pages 312–322
ISSN: 1223-6934

Integrability conditions for a cubic differential system with two invariant straight lines and one invariant cubic

Anatoli Dascalescu

Abstract. We find conditions for a singular point $O(0,0)$ of a center or a focus type to be a center, in a cubic differential system with two invariant straight lines and one invariant cubic. The presence of a center at $O(0,0)$ is proved by constructing Darboux first integrals.

2010 Mathematics Subject Classification. Primary 34C05; Secondary 37G10.
Key words and phrases. Cubic differential system, invariant algebraic curves, Darboux integrability, the problem of the center.

1. Introduction

We consider the cubic differential system

$$\begin{aligned} \dot{x} &= y + ax^2 + cxy + fy^2 + kx^3 + mx^2y + pxy^2 + ry^3 \equiv P(x,y), \\ \dot{y} &= -(x + gx^2 + dxy + by^2 + sx^3 + qx^2y + nxy^2 + ly^3) \equiv Q(x,y), \end{aligned} \tag{1}$$

where $P(x,y)$ and $Q(x,y)$ are real and coprime polynomials in the variables x and y. The origin $O(0,0)$ is a singular point of a center or a focus type for (1). It arises the problem of distinguishing between a center and a focus, i.e. of finding the coefficient conditions under which $O(0,0)$ is a center.

The problem of the center was solved for quadratic differential systems and for cubic symmetric differential systems. If the cubic system (1) contains both quadratic and cubic nonlinearities, then the problem of finding a finite number of necessary and sufficient conditions for the center is still open. It was possible to find a finite number of conditions for the center only in some particular cases (see, for example, [9]–[15]).

The problem of the center was solved for cubic differential systems (1) with at least three invariant straight lines ([3], [4], [16]) and for cubic differential systems (1) with two invariant straight lines and one invariant conic ([5], [6], [8]). It was proved that every center in the cubic differential system (1) with two invariant straight lines and one invariant conic comes from a Darboux integrability.

It is known [1] that a singular point $O(0,0)$ is a center for system (1) if and only if the system has a holomorphic first integral of the form $F(x,y) = C$ in some neighborhood of $O(0,0)$.

The integrability conditions for some families of cubic differential systems having invariant algebraic curves were found in [2], [7], [8], [11], [12], [17].

The goal of this paper is to obtain the center conditions for a cubic differential system (1) with two invariant straight lines and one irreducible invariant cubic by using the method of Darboux integrability. Our main result is the following one.

Theorem 1.1. *The origin is a center for cubic differential system* (1), *with two invariant straight lines and one irreducible invariant cubic, if one of the conditions* (i)–(xiv) *hold.*

The paper is organized as follows. In Section 2 we present the known results concerning relation between invariant algebraic curves and Darboux integrability. In Sections 3, 4 and 5 we determine the integrability conditions for cubic differential system (1) with two invariant straight lines and one invariant cubic by constructing Darboux first integrals. Finally in Section 6 we prove the Theorem 1.1.

2. Algebraic solutions and Darboux first integrals

One of the most important problem for differential system (1) is whether the trajectories to (1) can be described by an algebraic formula, for example, $\Phi(x, y) = 0$, where Φ is a polynomial.

Definition 2.1. An algebraic invariant curve of (1) is the solution set in $\mathbb{C}^2$ of an equation $\Phi(x, y) = 0$, where Φ is a polynomial in x, y with complex coefficients such that

$$\frac{\partial \Phi}{\partial x} P(x, y) + \frac{\partial \Phi}{\partial y} Q(x, y) = K(x, y)\Phi(x, y)$$

for some polynomial in x, y, $K = K(x, y)$ with complex coefficients, called the cofactor of the invariant algebraic curve $\Phi = 0$.

We say that the invariant algebraic curve $\Phi(x, y) = 0$ is an *algebraic solution* of (1) if and only if $\Phi(x, y)$ is an irreducible polynomial in $\mathbb{C}[x, y]$.

We shall study the problem of the center for cubic differential system (1) assuming that (1) has algebraic solutions: two invariant straight lines and one invariant cubic.

By Definition 2.1 a straight line

$$1 + Ax + By = 0, \ A, B \in \mathbb{C} \tag{2}$$

is said to be invariant for (1), if there exists a polynomial with complex coefficients $K(x, y)$ such that the following identity holds

$$AP(x, y) + BQ(x, y) \equiv (1 + Ax + By)K(x, y).$$

If the cubic system (1) has complex invariant straight lines then obviously they occur in complex conjugated pairs $1 + Ax + By = 0$ and $1 + \overline{A}x + \overline{B}y = 0$.

Let the cubic system (1) have two distinct invariant straight lines $l_1 = 0$ and $l_2 = 0$ real or complex ($l_2 = \overline{l_1}$) of the form (2). Assume that the invariant straight lines $l_1 = 0$, $l_2 = 0$ intersect at a real singular point (x_0, y_0). By rotating the system of coordinates ($x \to x\cos\varphi - y\sin\varphi, y \to x\sin\varphi + y\cos\varphi$) and rescaling the axes of coordinates ($x \to \alpha x$, $y \to \alpha y$), we obtain $l_1 \cap l_2 = (0, 1)$. In this case the invariant straight lines can be written as

$$l_1 \equiv 1 + a_1 x - y = 0, \ l_2 \equiv 1 + a_2 x - y = 0, \ a_1, a_2 \in \mathbb{C}, \ a_1 - a_2 \neq 0. \tag{3}$$

Assume now that the invariant straight lines (2) are parallel, then by a rotation of axes we can make them parallel to the axis of ordinates (Oy)

$$l_{1,2} \equiv 1 + \frac{c \pm \sqrt{c^2 - 4m}}{2} x = 0. \tag{4}$$

In [7] there were proved the following assertions

Lemma 2.1. *The cubic differential system (1) has two distinct invariant straight lines of the form* (3) *if and only if the following coefficient conditions are satisfied*

$$\begin{aligned} &k=(a-1)(a_1+a_2)+g,\ l=-b,\ r=-f-1,\ s=(1-a)a_1a_2,\\ &m=(a_1+a_2)(c-a_1-a_2)+a_1a_2-a+d+2,\ q=(a_1+a_2-c)a_1a_2-g,\\ &p=(f+2)(a_1+a_2)+b-c,\ n=-(f+2)a_1a_2-(d+1). \end{aligned} \quad (5)$$

Lemma 2.2. *The cubic differential system (1) has two parallel invariant straight lines of the form (4) if and only if the following conditions hold*

$$a=f=k=p=r=0,\ m(c^2-4m)\neq 0. \quad (6)$$

Let us consider the cubic curve

$$\Phi(x,y)\equiv x^2+y^2+a_{30}x^3+a_{21}x^2y+a_{12}xy^2+a_{03}y^3=0 \quad (7)$$

with $(a_{30},\, a_{21},\, a_{12},\, a_{03})\neq 0$ and $a_{30},\, a_{21},\, a_{12},\, a_{03}\in\mathbb{R}$.

By Definition 2.1, the cubic curve (7) is said to be an invariant cubic for (1), if there exists a polynomial with real coefficients $K(x,y)=c_{10}x+c_{01}y+c_{20}x^2+c_{11}xy+c_{02}y^2$ such that the following identity holds

$$\frac{\partial\Phi}{\partial x}P(x,y)+\frac{\partial\Phi}{\partial y}Q(x,y)\equiv\Phi(x,y)K(x,y).$$

Definition 2.2. System (1) is integrable on an open set D of R^2 if there exists a nonconstant analytic function $F: D\to R$ which is constant on all solution curves $(x(t),y(t))$ in D, i.e. $F(x(t),y(t))$=constant for all values of t where the solution is defined. Such an F is called a *first integral* of the system on D.

When F exists in D, all the solutions of the differential system in D are known since every solution is given by $F(x,y)=C$, for some $C\in\mathbb{R}$. Clearly F is a first integral of (1) on D if and only if

$$P\frac{\partial F}{\partial x}+Q\frac{\partial F}{\partial y}\equiv 0. \quad (8)$$

A first integral constructed from invariant algebraic curves $f_j(x,y)=0,\ j=\overline{1,q}$

$$F(x,y)\equiv f_1^{\alpha_1}f_2^{\alpha_2}\cdots f_q^{\alpha_q}=C \quad (9)$$

with $\alpha_j\in\mathbb{C}$ not all zero is called *a Darboux first integral.*

In this paper we find the conditions under which the cubic differential system (1) has Darboux first integrals of the form

$$F(x,y)\equiv l_1^{\alpha_1}l_2^{\alpha_2}\Phi^{\alpha_3}=C \quad (10)$$

composed of two invariant straight lines $l_1=0$, $l_2=0$ and one irreducible invariant cubic $\Phi=0$ of the form (7), where $\alpha_j\in\mathbb{C}$.

3. Two parallel invariant straight lines and one invariant cubic

In this section we find the center conditions for cubic differential system (1) having two parallel invariant straight lines and one invariant cubic.

Lemma 3.1. *The cubic differential system (1) with two parallel invariant straight lines (4) and one invariant cubic (7) has a Darboux first integral of the form (10) if and only if one of the sets of conditions* (i)–(iv) *is satisfied:*

(i) $a = d = f = k = l = p = q = r = 0$, $n = (-3m)/2$, $s = (4bg - 4b^2 - 10bc - 6c^2 + 6cg - 3m)/6$;

(ii) $a = d = f = k = l = p = q = r = 0$, $c = 2b$, $m = n$, $n = [8u(3bv - 2gu)(g - 3b)]/[9(2u - v)^2]$, $s = (nv)/u$;

(iii) $a = d = f = k = l = p = q = r = 0$, $m = [((u-2v)c+2bu)(c-2b)u]/[4(u-v)^2]$, $n = [((u - 2v)c + 2bu)(c - 2b)v]/[4(u - v)^2]$, $s = [((2b + 4g - c)u + (4b - 2c - 4g)v)(u + v)(c - 2b)]/[12(u - v)^2]$;

(iv) $a = f = k = l = p = r = 0$, $c = -3b$, $g = [b(b^2 - d^2)]/(2d^2)$, $m = 3(b^2 + d^2)$, $n = -2(b^2 + d^2)$, $q = [b(b^2 + d^2)]/(2d)$, $s = [-b^2(b^2 + d^2)]/(2d^2)$.

Proof. Let the cubic system (1) have two invariant straight lines $l_1 = 0$, $l_2 = 0$ of the form (4) and an invariant cubic $\Phi = 0$ of the form (7). In this case the system (1) will have a Darboux first integral of the form (10) if and only if the identity (8) holds. Identifying the coefficients of the monomials x^iy^j in (8), we obtain a system of fifteen equations

$$\{U_{ij} = 0,\ \ i + j = 3,\ 4,\ 5\} \tag{11}$$

for the unknowns a_{30}, a_{21}, a_{12}, a_{03}, α_1, α_2, α_3 and the coefficients of system (1).

When $i + j = 3$, we express a_{21}, a_{03}, α_1, a_{30} from the equations of (11). Next express α_2 from $U_{13} = 0$, q from $U_{22} = 0$, l from $U_{04} = 0$ and reduce the equations of (11) by s from $U_{31} = 0$. Then $U_{50} = 0$ and $U_{05} \equiv (a_{12} + b)d^2 = 0$.

I. Let $d = 0$, then $U_{05} = U_{14} = U_{32} = 0$ and $U_{23} = a_{12}(a_{12}^2 - ca_{12} + m) = 0$.

We assume that $g \neq b + c$, otherwise the cubic curve (7) is reducible.

Suppose that $a_{12} = 0$. Then $n = (-3m)/2$ and we obtain the set of conditions (i) for the existence of a first integral (10) with $\alpha_1 = 3\sqrt{c^2 - 4m} - 4b - 3c$, $\alpha_2 = 3\sqrt{c^2 - 4m} + 4b + 3c$, $\alpha_3 = -2\sqrt{c^2 - 4m}$ and

$$l_{1,2} \equiv 2 + (c \pm \sqrt{c^2 - 4m}\,)x = 0,\ \Phi \equiv 3(x^2 + y^2) + 2(g - c - b)x^3 = 0.$$

Suppose that $a_{12} \neq 0$ and reduce the equation $U_{41} = 0$ by a_{12}^2 from $U_{23} = 0$, then

$$U_{41} \equiv (2b - c)a_{12} + 2(m - n) = 0.$$

If $c = 2b$, then $m = n$, $a_{12} = (24b^2 - 8bg - 6n + 3s)/[4(3b - g)]$ and $U_{23} = 0$ admits the following parametrization $n = [8u(3bv - 2gu)(g - 3b)]/[9(2u - v)^2]$, $s = [8v(3bv-2gu)(g-3b)]/[9(2u-v)^2]$. We get the set of conditions (ii) for the existence of a first integral (10) with $\alpha_1 = 0$, $\alpha_2 = \alpha_3 = 1$ and

$$l_1 \equiv 6u - 3v + (12bu - 4gu)x = 0,\ l_2 \equiv 6u - 3v - (6bv - 4gu)x = 0,$$
$$\Phi \equiv 2(3b - g)(vx^2 + 2uy^2)x + 3(2u - v)(x^2 + y^2) = 0.$$

If $c \neq 2b$, then $U_{41} = 0$ yields $a_{12} = 2(m - n)/(c - 2b)$. The equation $U_{23} = 0$ admits the following parametrization $m = [((u - 2v)c + 2bu)(c - 2b)u]/[4(u - v)^2]$, $n = [((u - 2v)c + 2bu)(c - 2b)v]/[4(u - v)^2]$. We obtain the set of conditions (iii) for the existence of a first integral (10) with $\alpha_1 = 0$, $\alpha_2 = u - 2v$, $\alpha_3 = -u$ and

$$l_1 \equiv 2u - 2v + (2bu + cu - 2cv)x = 0,\ l_2 \equiv 2u - 2v + (cu - 2bu)x = 0,$$
$$\Phi \equiv x[((2b - c + 4g)u + (4b - 2c - 4g)v)x^2 + 3(2bu + cu - 2cv)y^2]+$$
$$+6(u - v)(x^2 + y^2) = 0.$$

II. Let $d \neq 0$, then $U_{05} = 0$ yields $a_{12} = -b$. We express s, n, m from the equations $U_{32} = 0$, $U_{14} = 0$, $U_{31} = 0$, respectively. Then we calculate the resultant of the polynomials U_{23} and U_{41} with respect to g and obtain that (11) is compatible if and only if $c = -3b$ and $g = b(b^2 - d^2)/(2d^2)$. In this case we get the conditions (iv) for the existence of a first integral (10) with $\alpha_1 = \alpha_2 = 1$, $\alpha_3 = -1$ and

$l_{1,2} \equiv 2 - (3b \pm i\sqrt{3b^2 + 12d^2})x = 0$, $\Phi \equiv 3d^2(x^2 + y^2) + (bx + dy)^2(bx - 2dy) = 0$.

In each of the cases (i)–(iv), the system (1) has a Darboux first integral of the form (10) and therefore the origin is a center for (1). □

4. A bundle of two invariant straight lines and one invariant cubic

In this section we find the center conditions for cubic differential system (1) having a bundle of two invariant straight lines and one invariant cubic.

Lemma 4.1. *The cubic differential system (1) with a bundle of two invariant straight lines (3) and one invariant cubic (7) has a Darboux first integral of the form (10) if and only if one of the sets of conditions* (v)–(xi) *is satisfied:*

(v) $a = 1,\ d = -2,\ f = -1,\ k = g,\ l = -b,\ p = b,\ q = -g,\ r = s = 0,\ m = (3c^2 - 4b^2 - 4bc - 16)/16,\ n = -m$;

(vi) $d = 2a - 3,\ f = -3/2,\ g = 2(1 - a)(b + c),\ k = (1 - a)(2b + c),\ l = -b,\ m = (9a - 4b^2 - 2bc + 2c^2 - 9)/9,\ n = (18 - 18a + 2b^2 + bc - c^2)/9,\ p = (2b - c)/2,\ q = 2(a - 1)(b + c),\ r = 1/2,\ s = q(2b - c)/9$;

(vii) $d = 2(a - 1),\ f = -2,\ l = -b,\ k = (2g - 2c + 4b + 3ac - 6ab)/6,\ m = (c^2 - 4b^2)/4,\ n = 1 - 2a,\ q = (ac - 2ab - 2g)/2,\ p = b - c,\ r = 1,\ s = (4bc + 4cg - 8bg - 4b^2 - c^2)/12,\ (c - 2b - 4g)^2 - 3(2b + c)(a - 1)(c - 2b - 4g) - 36a(a - 1)^2 = 0$;

(viii) $a = (1 - v^2)/2,\ c = (3v^2 - 2bv - 1)/v,\ d = -1 - v^2,\ f = -2,\ l = -b,\ g = (3v^4 + 6v^2 - 4bv - 1)/(4v),\ k = v(1 + 2bv - v^2)/2,\ r = 1,\ m = (9v^4 - 12bv^3 - 6v^2 + 4bv + 1)/(4v^2),\ p = b - c,\ n = v^2 = a_2^2,\ q = v(2bv - 1 - 3v^2)/2,\ s = (3v^4 - 4bv^3 + 2v^2 - 4bv - 1)/4$;

(ix) $a = 1,\ b = l = s = 0,\ d = f - 1,\ k = g,\ q = -2g,\ n = 2r,\ r = -f - 1,\ c = g + 4a_2,\ m = 4ga_2 + f + 1,\ p = 4(f + 1)a_2 - g,\ a_2^2 = 1/3$;

(x) $a = 3(1 - a_1^2)/2,\ b = l = 0,\ c = 4a_1 + a_2,\ d = 2a - 5,\ g = (3a_1^3 - a_1 + 2a_2)/2,\ k = aa_2,\ n = 4 - 2a,\ f = -2,\ r = 1,\ m = (3a_1^2 + 8a_1a_2 - 3)/2,\ p = -c,\ q = (a_1 - 3a_1^3 - 6a_1^2a_2 - 2a_2)/2,\ s = a_1a_2(3a_1^2 - 1)/2$;

(xi) $a = (3 - 2a_1a_2 - a_1^2)/2,\ b = l = 0,\ c = 3a_1 + 2a_2,\ d = 2a - 5,\ f = -2,\ g = a_2(3a_1^2 + 1)/2,\ k = (a_1 + 2a_2 - a_1^3 - 2a_1a_2^2)/2,\ m = (3a_1^2 + 6a_1a_2 + 2a_2^2 - 3)/2,\ p = -c,\ n = -d - 1,\ r = 1,\ q = -a_2(7a_1^2 + 2a_1a_2 + 1)/2,\ s = a_1a_2(a_1^2 + 2a_1a_2 - 1)/2$.

Proof. Let the cubic system (1) have two invariant straight lines $l_1 = 0$, $l_2 = 0$ of the form (3) and one invariant cubic $\Phi = 0$ of the form (7) passing through a singular point $(0, 1)$, i.e. $a_{03} = -1$. In this case the system (1) will have a Darboux first integral of the form (10) if and only if the identity (8) holds. Identifying the coefficients of the monomials x^iy^j in (8), we obtain a system of fifteen equations

$$\{U_{ij} = 0,\ \ i + j = 3,\ 4,\ 5\} \tag{12}$$

for the unknowns a_{30}, a_{21}, a_{12}, a_1, a_2, α_1, α_2, α_3 and the coefficients of system (1).

When $i + j = 3$, the equations of (12) yield

$$\begin{array}{l} a_{21} = (2a + 2d - 2f - 3)/3,\ a_{30} = (3a_{12} - 2b - 2c + 2g)/3, \\ \alpha_1 = \alpha_3((4a - 2d + 2f + 3)a_2 + 3(2b - a_{12}))/(3a_1 - 3a_2), \\ \alpha_2 = \alpha_3((-4a + 2d - 2f - 3)a_1 + 3(a_{12} - 2b))/(3a_1 - 3a_2). \end{array} \tag{13}$$

Substituting (13) in $U_{04} = 0$, we get $U_{04} \equiv b(d - 2a + 2f + 6) = 0$. We have to consider two possibilities: $\{d = 2a - 2f - 6\}$, $\{d \neq 2a - 2f - 6,\ b = 0\}$.

I. Let $d = 2a - 2f - 6$, then $U_{04} \equiv 0$ and

$$e_1 = U_{13} + U_{14} \equiv (a_{12} - a_1)(a_{12} - a_2)(f + 2) = 0.$$

1. Assume that $a_{12} = a_1$. Then $U_{13} \equiv (a_1 + a_2 - c)b = 0$.

1.1. If $c = a_1 + a_2$, then $U_{22} \equiv (a - f - 2)(a_1 - 2a_2 - 2b) = 0$. In case $f = a - 2$ and $a_2 = g - b - a_1$, the cubic curve (7) is reducible. In case $f = a - 2$, $a_2 \neq g - b - a_1$ and $a = 1$, $a_1 = 3a_2 + 2b$, we get the set of conditions (v) for the existence of a first integral (10), with $\alpha_1 = 0$, $\alpha_2 = -3$, $\alpha_3 = 1$ and

$$l_1 \equiv (2b + 3c)x - 4y + 4 = 0,\ l_2 \equiv (c - 2b)x - 4y + 4 = 0,$$
$$\Phi \equiv (c - 2b + 8g)x^3 + (6b + 9c)xy^2 + 12(1 - y)(x^2 + y^2) = 0.$$

Suppose that $f \neq a - 2$, then $U_{22} = 0$ yields $a_1 = 2(a_2 + b)$. In this case $U_{23} \equiv (2f + 3)b = 0$. If $f = -3/2$, then $g = 6(1 - a)(a_2 + b)$. We obtain the set of conditions (vi) for the existence of a first integral (10) with $\alpha_1 = 0$, $\alpha_2 = -2$, $\alpha_3 = 1$ and

$$l_1 \equiv 2(b + c)x - 3y + 3 = 0,\ l_2 \equiv (c - 2b)x - 3y + 3 = 0,$$
$$\Phi \equiv 4(b + c)(1 - a)x^3 + 6(a - 1)x^2y + 2(b + c)xy^2 - 3y^3 + 3(x^2 + y^2) = 0.$$

If $f \neq -3/2$ and $b = 0$, then $a = 1$, $f = a_2^2 - 1$ and the system (12) is not consistent.

1.2. Suppose that $c \neq a_1 + a_2$, then $U_{13} = 0$ yields $b = 0$. In this case

$$e_2 = U_{23} + U_{22} \equiv (a - f - 2)(a_1 - 2a_2)(f + 2) = 0.$$

1.2.1. If $f = a - 2$, then the equation $U_{23} = 0$ implies $a = a_1a_2$ and $U_{40} = 0$ becomes $U_{40} \equiv (a_1a_2 - 1)(c + 2g - 3a_1 - 3a_2) = 0$. When $a_1a_2 = 1$, we get the set of conditions (ix) with $f = -1$. When $a_1a_2 \neq 1$ and $a_1 = (c + 2g - 3a_2)/3$, then $c = 4g$, $a_2 = 0$. We get the conditions (vii) with $a = b = 0, c = 4g$.

1.2.2. If $f \neq a - 2$ and $f = -2$, then $e_2 \equiv 0$. We express g from $U_{40} = 0$ and a from $U_{31} = 0$. When $a_1 = c/2$, we obtain the set of conditions (vii) with $b = 0$. When $a_1 \neq c/2$, we express c from $U_{50} = 0$ and a from $U_{22} = 0$, then $U_{31} \neq 0$.

1.2.3. If $f \neq a - 2$, $f \neq -2$ and $a_1 = 2a_2$, then $e_2 \equiv 0$. We express a from $U_{22} = 0$ and g from $U_{40} = 0$. In this case the system of equations (12) has no solutions.

2. Assume that $a_{12} \neq a_1$ and $a_{12} = a_2$. This case is symmetric to the case 1 ($a_{12} = a_1$), replacing a_2 by a_1. We get the sets of conditions (v) and (vi).

3. Assume that $a_{12} \neq a_1$, $a_{12} \neq a_2$ and let $f = -2$, then $e_1 \equiv 0$. In this case the equations $U_{40} = 0$ and $U_{13} = 0$ yield $g = [3(a_1 + a_2)(1 - a) + 3a(c - a_{12}) + 2b - c]/2$, $c = [(a_1 + a_2)(2b - a_{12}) + a_1a_2 + a_{12}^2]/(2b)$ and $U_{50} \equiv g_1g_2g_3 = 0$, where

$$g_1 = a_{12} - a_2 - 2b,\ g_2 = a_{12} - a_1 - 2b$$
$$g_3 = ((3a_1 + 3a_2 + 2b)a_{12} - 3a_{12} - 3a_1a_2)a + (a_{12} - a_1)(a_{12} - a_2).$$

3.1. Suppose that $g_1 = 0$, then $a_{12} = a_2 + 2b$. In this case $a_1^2 - a_1a_2 - 2ba_1 - a = 0$ and we have the set of conditions (vii) for the existence of a first integral (10) with $\alpha_1 = 0$, $\alpha_2 = -1$, $\alpha_3 = 1$ and

$$l_1 \equiv (c - 2b - 4g)x + 6(a - 1)(1 - y) = 0,\ l_2 \equiv (c - 2b)x - 2y + 2 = 0,$$
$$\Phi \equiv (2b - c + 4g)x^3 + 6(2a - 1)x^2y + 3(2b + c)xy^2 - 6y^3 + 6(x^2 + y^2) = 0.$$

3.2. Suppose that $g_1 \neq 0$ and $g_2 = 0$. Then $a_{12} = a_1 + 2b$ and $a = a_2(a_2 - a_1 - 2b)$. This case is symmetric to the case 3.1, if we replace a_2 by a_1. We obtain the set of conditions (vii).

3.3. Suppose that $g_1 \neq 0$, $g_2 \neq 0$ and $g_3 = 0$. Then we express a from $g_3 = 0$. The system of equations $\{U_{31} = 0,\ U_{22} = 0\}$ has real solutions if and only if $a_{12} = (3a_2^2 - 1)/(2a_2)$ and $a_1 = (3a_2^2 - 4ba_2 - 1)/(2a_2)$. We get the set of conditions (viii) for the existence of a first integral (10) with $a_2 = v$, $\alpha_1 = -1$, $\alpha_2 = 0$, $\alpha_3 = 1$, and

$$l_1 \equiv (3v^2 - 4bv - 1)x + 2v(1 - y) = 0,\ l_2 \equiv vx - y + 1 = 0,$$
$$\Phi \equiv v^2(v^2 + 1)x^3 + 2v(1 - v^2y)x^2 + (3v^2 - 1)xy^2 + 2vy^2(1 - y) = 0.$$

II. Let $d \neq 2a - 2f - 6$ and $b = 0$. Then $U_{04} \equiv 0$ and

$$e_3 = U_{13} + U_{14} \equiv (a_{12} - a_1)(a_{12} - a_2)(f + 2) = 0.$$

1. Assume $a_{12} = a_1$, then $U_{13} = 0$ implies $d = f - a_1a_2$. We express c from $U_{22} = 0$. In this case $U_{23} \equiv (a - a_1a_2)(a_1 - 2a_2)(f + 2) = 0$.

1.1. When $a = a_1a_2$, we have $U_{40} \equiv (a_1a_2 - 1)(2a_1 + 2a_2 - 3g) = 0$. If $a_1 = 1/a_2$, then we obtain the set of conditions (ix) for the existence of a first integral (10) with $\alpha_1 = 0,\ \alpha_2 = -3,\ \alpha_3 = 1$ and $l_1 \equiv x - a_2y + a_2 = 0,\ l_2 \equiv a_2x - y + 1 = 0$,

$$\Phi \equiv 9a_2(1 - y)(x^2 + y^2) + x(x^2 + 9y^2) = 0.$$

If $a_1a_2 - 1 \neq 0$ and $g = 2(a_1 + a_2)/3$, then the cubic curve is reducible.

1.2. When $a \neq a_1a_2$ and $f = -2$, we express g from $U_{40} = 0$. In this case we have

$$e_4 = U_{32} + U_{31} \equiv (2a + a_1a_2 - 3)(a_1 - 2a_2)^2a_1a_2 = 0.$$

If $a_1 = 2a_2$, then $a = 2a_2^2 + 3/2$ and $U_{31} \neq 0$. If $a_1 = 0$, then $a = 3/2$ and this case is contained in (x) $(a_1 = 0,\ a_2 = c)$. If $a_2 = 0$, then $a = 3/2$. This case is contained in (xi), if we replace a_1 by a_2 and then put $a_2 = 0, a_1 = c/2$.

If $a = (3 - a_1a_2)/2$ and $a_1 = -a_2$, then this case is contained in (x) $(a_2 = -c)$. If $a = (3 - a_1a_2)/2$ and $a_1 = 3a_2$, then this case is contained in (x), if we replace a_1 by a_2 and then put $a_2 = c/7$.

1.3. When $a \neq a_1a_2,\ f \neq -2$ and $a_1 = 2a_2$, then $U_{40} \equiv 0$ if $a = 1$ or $a_2 = g/2$. In both cases the system (12) has no solutions.

2. Assume that $a_{12} \neq a_1$ and let $a_{12} = a_2$. This case is symmetric to the case 1 $(a_{12} = a_1)$, if we replace a_2 by a_1. We get the set of conditions (ix).

3. Assume that $a_{12} \neq a_1,\ a_{12} \neq a_2$ and let $f = -2$. Then $e_3 \equiv 0$. In this case we express $g,\ c$ and a from the equations $U_{22} = 0,\ U_{23} = 0$ and $U_{13} = 0$, respectively. Then $U_{40} \equiv h_1h_2 = 0$, where

$$h_1 = a_{12}(a_1 + a_2 - a_{12}) - 3a_1a_2 - 2d - 4,\ h_2 = (d + 2)(2a_1 + 2a_2 - a_{12}) + 2a_1a_2a_{12}.$$

3.1. Let $h_1 = 0$, then $U_{31} \equiv (3a_1 - a_{12})(3a_2 - a_{12})(2a_1 + a_2 - a_{12})(a_1 + 2a_2 - a_{12}) = 0$. If $a_{12} = 3a_1$, then we determine the set of conditions (x) for the existence of a first integral (10) with $\alpha_1 = -3,\ \alpha_2 = 0,\ \alpha_3 = 1$ and

$$l_1 \equiv a_1x - y + 1 = 0,\ l_2 \equiv a_2x - y + 1 = 0,\ \Phi \equiv (a_1x - y)^3 + (x^2 + y^2) = 0.$$

If $a_{12} = 2a_1 + a_2$, then we obtain the conditions (xi) for the existence of a first integral (10), with $\alpha_1 = -2,\ \alpha_2 = -1,\ \alpha_3 = 1$ and

$$l_1 \equiv a_1x - y + 1 = 0,\ l_2 \equiv a_2x - y + 1 = 0,$$
$$\Phi \equiv a_1^2a_2x^3 - (a_1^2 + 2a_1a_2)x^2y + (2a_1 + a_2)xy^2 - y^3 + x^2 + y^2 = 0.$$

If $a_{12} = 3a_2$ or $a_{12} = a_1 + 2a_2$, then we get the symmetric conditions to (x) and (xi).

3.2. Let $h_1 \neq 0$ and $h_2 = 0$. In this case the system (12) has no real solutions.

In each of the cases (v)–(xi), the system (1) has a Darboux first integral of the form (10) and therefore the origin is a center for (1). □

5. Two invariant straight lines and one invariant cubic in generic position

In this section we find the center conditions for cubic differential system (1) having two invariant straight lines and one invariant cubic in generic position.

Lemma 5.1. *The cubic system (1) with two invariant straight lines (3) and one invariant cubic (7) in generic position has a Darboux first integral of the form (10) if and only if one of the sets of conditions* (xii)–(xiv) *holds:*

(xii) $a = [3h^2 + u(3-2h)(u+2b)]/(2h^2)$, $g = [(3b+2u)(2hu^2 - 3u^2 - h^2)]/(2h^3)$, $d = 2a - 5$, $m = (28b^2h^2 - 8b^2h^3 - 54b^2 + 8bch^3 - 12bch^2 - 18bch - 2c^2h^3 - c^2h^2 - 3h^2)/(2h^2)$, $p = 4b - 2bh + ch - c$, $n = [(4h-3)u^2 + bu(7h-6) + h^2]/h^2$, $q = [(3b+2u)((2h-3)(2b-u)u + h^2)]/(2h^3)$, $l = -b$, $s = [u(3b+2u)((2b+u)(3-2h)u + h^2)]/(2h^4)$, $k = [(6b^2 + 8bu + 3u^2)(2h-3)u - 3h^2(2b+u)]/(2h^3)$, $r = 1 - h$, $h = f + 2$, $u = 2bh - ch - 6b$;

(xiii) $a = [(3f+5)^2(f+2) + b^2(3f+4)^2]/[(3f+5)^2(f+2)]$, $c = [b(6f^2+11f+2)]/[(3f+5)(f+2)]$, $d = [2b^2(3f+4)^3 - (f+2)(5f+7)(3f+5)^2]/[(f+2)(3f+4)(3f+5)^2]$, $g = b[3b^2(3f+4)^2 - (2f+3)(3f+5)^2]/[(f+2)(3f+5)^3]$, $k = -b[b^2(3f+4)^3 + (f+2)(2f+3)(3f+5)^2]/[(f+2)^2(3f+5)^3]$, $l = -b$, $m = -[b^2(3f+4)^2(9f^2+22f+12) + 3(f+1)(f+2)^2(2f+3)(3f+5)]/[(f+2)^2(3f+4)^2(3f+5)]$, $n = -[b^2(3f+4)^2(27f^2+80f+60) - 2(f+1)(f+2)(2f+3)(3f+5)^2]/[(f+2)(3f+4)^2(3f+5)^2]$, $p = -[b(9f^2+22f+12)]/[(3f+5)(f+2)]$, $q = -b[b^2(3f+4)^2(27f^2+85f+66) - (f+1)(f+2)(2f+3)(3f+5)^2]/[(3f+5)^3(3f+4)(f+2)^2]$, $r = -f-1$, $s = -b^2[b^2(3f+4)^2(9f+14) + (f+2)(2f+3)(3f+5)^2]/[(f+2)^2(3f+5)^4]$;

(xiv) $a = 3c^2 + 1$, $b = l = 0$, $d = 2(9c^2-2)/3$, $f = (-5)/3$, $g = c(9c^2+1)$, $k = g$, $m = (-2)/3$, $n = (4 - 45c^2)/9$, $p = -c$, $q = 2c(-9c^2-1)/3$, $r = 2/3$, $s = cg$.

Proof. Let the cubic system (1) have two invariant straight lines $l_1 = 0$, $l_2 = 0$ of the form (3) and one invariant cubic $\Phi = 0$ of the form (7) in generic position ($a_{03} \neq -1$). In this case the system (1) will have a Darboux first integral of the form (10) if and only if the identity (8) holds. Identifying the coefficients of the monomials x^iy^j in (8), we obtain a system of fifteen equations

$$\{U_{ij} = 0, \ \ i + j = 3, 4, 5\} \tag{14}$$

for the unknowns $a_{30}, a_{21}, a_{12}, a_{03}, a_1, a_2, \alpha_1, \alpha_2, \alpha_3$ and the coefficients of system (1).

When $i+j = 3$, the equations of (14) yield $d = (3a_{21} - 3a_{03} - 2a + 2f)/2$, $g = (3a_{30} - 3a_{12} + 2b + 2c)/2$, $\alpha_1 = \alpha_3(a_{21} - 2a) - \alpha_2$, $\alpha_2 = \alpha_3(a_{12} - 2b + a_1(a_{21} - 2a))/(a_1 - a_2)$.

Substituting this in $U_{04} = 0$, we get $U_{04} \equiv b(a_{21} - a_{03} - 2a + 2f + 4) - a_{12}(a_{03} + 1) = 0$. We consider two cases: $\{b \neq 0\}$ and $\{b = 0\}$.

I. Let $b \neq 0$. In this case we express f from $U_{04} = 0$ and c from $U_{13} = 0$.

1. Assume that $a_{12} = 0$, then $U_{14} \equiv a_{21}(a_{21} - 2a + 2) = 0$. When $a_{21} = 0$, the system (14) is not consistent. If $a_{21} \neq 0$ and $a_{21} = 2(a-1)$, then the equations of (14) imply $a = (b^2+4)/4$, $a_{30} = b^3/4$, $a_1 = (4 - 3b^2 - 4ba_2)/(4a_2 + 4b)$ and $4a_2^2 + 8ba_2 + 5b^2 + 4 = 0$. This subcase is contained in conditions (xiii) ($f = -1$).

2. If $a_{12} \neq 0$, then express a from $U_{05} = 0$. We calculate the resultant of the polynomials U_{14} and U_{40} with respect to a_1 and obtain $Res(U_{14}, U_{40}, a_1) = 8b^4\alpha_3^2 g_1 g_2 g_3$, where

$g_1 = (a_{12} - 2b)a_{03} + a_{12}$, $g_2 = (a_{12} + b)a_{03}a_2 + (a_2 + b)a_{12}$, $g_3 = a_{12}^3(a_{03}a_{30} + ba_{03} + a_{30} + b) + ba_{12}^2(a_{03}a_{30} - ba_{03} - ba_{21} + a_{30}) + 2b^2a_{12}a_{03}(a_{30} - b) + 2b^3a_{03}a_{30}$.

Assume that $g_1 = 0$. Then $a_{03} = a_{12}/(2b - a_{12})$. We express a_{21} from $U_{14} = U_{40} = 0$, a_{30} from $U_{22} = 0$ and obtain that $U_{31} \equiv f_1f_2f_3f_4 = 0$, where

$f_1 = a_{12} - 3a_1 - 2b$, $f_2 = a_{12} - 3a_2 - 2b$, $f_3 = a_{12} - a_2 - 2a_1 - 2b$, $f_4 = a_{12} - a_1 - 2a_2 - 2b$.

If $f_1 = 0$ or $f_2 = 0$, then the right-hand sides of (1) have a common factor.

If $f_3 = 0$ or $f_4 = 0$, then we get the set of conditions (xii) for the existence of a first integral (10) with $\alpha_1 = 2$, $\alpha_2 = 1$, $\alpha_3 = -1$ and

$$l_1 \equiv ux + h(1-y) = 0, \ l_2 \equiv (3b+2u)x + h(y-1) = 0,$$
$$\Phi \equiv (2h-3)(hy - ux)^2(3bx + 2ux + hy) + 3h^3(x^2+y^2) = 0.$$

Assume that $g_1 \neq 0$ and let $g_2 = 0$. Then $a_{03} = [-a_{12}(a_2 + b)]/(a_2(a_{12} + b)]$. We express a_{21} from $U_{14} = 0$ and a_{30} from $U_{40} = 0$. In this case we have $U_{50} \equiv e_1e_2e_3 = 0$, where $e_1 = (ba_{12} - a_1a_2)(a_{12} - 2b) + a_1(a_{12}^2 + 2b^2) - 3ba_2a_{12}$, $e_2 = (3a_{12}^2 - 7a_{12}a_2 - 4ba_{12} - 4ba_2 - 4b^2)(a_{12}^2 - a_{12}a_2 + 2ba_2 + 2b^2) - 2(a_{12} - a_2)(a_{12} + b)^2a_{12}a_2^2$, $e_3 = (a_{12}^2 - a_{12}a_2 + 2ba_2 + 2b^2)(a_{12} - 3a_2 - 5b) - 2(a_{12} - a_2 - b)(a_{12} + b)^2a_2^2$.

If $e_1 = 0$, then $a_1 = [ba_{12}(3a_2 + 2b - a_{12})]/[(a_{12} - a_2)a_{12} + 2b(a_2 + b)]$ and the equation $U_{31} = 0$ admits the following parametrization $a_{12} = wu$, $a_2 = wv$, $b = wz$, $w^2 = (u^2 - uv + 2vz + 2z^2)^2/[2(2z^2 + 2vz + uv - u^2)(u + z)^2v^2]$.
In this case the system of equations (14) is not consistent.

If $e_1 \neq 0$ and $e_2 = 0$, then the equation $e_2 = 0$ admits the following parametrization $a_{12} = wu$, $a_2 = wv$, $b = wz$, $w^2 = [(3u^2 - 7uv - 4uz - 4vz - 4z^2)(u^2 - uv + 2vz + 2z^2)]/[2(u - v)(u + z)^2uv^2]$. In this case we express a_1 from $U_{41} = 0$ and calculate the resultant of the polynomials U_{31} and U_{22} with respect to z. We find that $Res(U_{31}, U_{22}, z) \neq 0$ and the system of equations (14) is not consistent.

If $e_1e_2 \neq 0$ and $e_3 = 0$, then the equation $e_3 = 0$ admits the following parametrization $a_{12} = wu$, $a_2 = wv$, $b = wz$, $w^2 = [(u^2 - uv + 2vz + 2z^2)(u - 3v - 5z)]/[2(u - v - z)(u + z)^2v^2]$. The system of equations (14) is not consistent.

Assume that $g_1g_2 \neq 0$ and let $g_3 = 0$. Then express a_{21} from $g_3 = 0$ and a_1 from $U_{14} = 0$. If $a_{12} = -b$ or $a_{12} = 2b$ or $a_{30} = 0$, the system of equations (14) is not consistent.

Suppose that $(a_{12} + b)(a_{12} - 2b)a_{30} \neq 0$. We reduce the equations of (14) by a_2^2 from $U_{31} = 0$ and express a_{30} from $U_{50} = 0$. Then the equation $U_{22} = 0$ yields $a_{03} = a_{12}^2/[(2b - a_{12})(b + a_{12})]$. In this case we obtain the set of conditions (xiii) for the existence of a first integral of the form (10) with $\alpha_1 = \alpha_2 = 1$, $\alpha_3 = -1$ and

$$l_{1,2} \equiv (9bf^2 + 27bf + 20b \pm \sqrt{\Delta})x + (3f + 5)(3f + 4)(f + 2)(y - 1) = 0,$$
$$\Phi \equiv 2(3bfx + 4bx + 3f^2y + 11fy + 10y)(3bfx + 4bx + 3f^2y + 8fy + 5y)^2 + \\ +(3f + 5)^3(3f + 4)(f + 2)(x^2 + y^2) = 0,$$

where $\Delta = (-2f - 3)(b^2(3f + 4)^2 + (f + 2)^2(3f + 5)^2)$.

II. Let $b = 0$, then $a_{12} = 0$, $a_{30} = 2(g - c)/3$, $a_{21} = (3a_{03} + 2d - 2f + 2a)/3$ and

$$U_{13} \equiv 3a_{03}^2 + 3a_1a_2a_{03} + (4f - 4a - 4d + 3)a_{03} + 2(d - 2a + 2f + 6)a_1a_2 + \\ + 4a(f - d - 1) + 2(d - f)(d + 2f + 4) = 0,$$
$$U_{14} \equiv 3a_{03}^2(a_1a_2 + d + 1) + a_1a_2a_{03}(7f - 4a + 2d + 18) + \\ + a_{03}(2d^2 - 2df + 11d - 8f - 4ad - 4a + 3) - 4(a + d - f)(f + 1) = 0.$$

1. Assume that $a_{03} = 0$, then $U_{14} \equiv (a + d - f)(f + 1) = 0$. If $f = a + d$, the cubic (7) is reducible. If $a + d - f \neq 0$ and $f = -1$, then the system (14) is not consistent.

2. Assume that $a_{03} \neq 0$ and let $f = -1$. In this case $U_{14} \equiv i_1i_2 = 0$, where

$$i_1 = 4a - 2d - 3a_{03} - 11,\ i_2 = a_1a_2 + d + 1.$$

If $i_1 = 0$ and $c = a_1 + a_2$, then we get the set of conditions (xii) ($b = 0$, $f = -1$).

If $i_1 = 0$ and $c \neq a_1 + a_2$, then $a = 1$ and the system (14) is not consistent.

Suppose that $i_1 \neq 0$ and let $i_2 = 0$, then $d = -a_1a_2 - 1$. We express a from $U_{13} = 0$ and c from $U_{23} = 0$. If $a_2 = -a_1$ and $a_{03} = (a_1^2 - 3)/3$, then the right-hand sides of (1) have o common factor. If $a_2 = -a_1$ and $3a_{03}^2 - 7a_{03}a_1^2 - 4a_1^2 - a_{03} - 4 = 0$, then the cubic (7) is reducible. When $a_2 \neq -a_1$, the system (14) is not consistent.

3. Suppose that $(f + 1)a_{03} \neq 0$. If $d = -2$ or $a_2 = 0$, then the cubic system (1) has no an irreducible invariant cubic.

4. Suppose that $(f+1)(d+2)a_2a_{03} \neq 0$. We calculate the resultant of the polynomials U_{14} and U_{13} with respect to a_1 and obtain that $Res(U_{14}, U_{13}, a_1) = \alpha_3^2 a_2 r_1 r_2$, where

$$r_1 = 3a_{03}^2 + a_{03}(7f - 4a - 7d + 3) + 4(f - a - d),$$
$$r_2 = 3a_{03}^2 + a_{03}(4f - 4a + 2d + 15) + 2(2af + 2a - df - d - 2f^2 - 8f - 6).$$

Let $r_1 = 0$ and express a from $r_1 = 0$. Then $U_{13} = 0, U_{14} = 0$ imply $a_1 = (f-d)/a_2$. Reducing the equations of (14) by a_2^2 from $U_{22} = 0$ we find that $U_{23} \equiv h_1h_2h_3 = 0$, where $h_1 = c - g$, $h_2 = 3a_{03}^2 + a_{03}(f - d + 5) + 2$,

$$h_3 = 3(d+2)a_{03}^2 + a_{03}(f^2 - 3df - d - 2f + 2) - 2(df + d + 2f + 2).$$

If $h_1 = 0$, then the right-hand sides of (1) have a common factor. If $h_1 \neq 0$ and $h_2 = 0$, then $d = (3a_{03}^2 + (f+5)a_{03} + 2)/a_{03}$ and $U_{40} \equiv (3a_{03}+4)(3a_{03}+1)(2c-5g) = 0$.

When $a_{03} = -4/3$ or $a_{03} = -1/3$, the right-hand sides of (1) have a common factor. When $g = (2c)/5$, the system (14) is not consistent.

Assume that $h_1h_2 \neq 0$ and let $h_3 = 0$. Then express d, g from the equations $h_3 = 0$, $U_{40} = 0$ and reduce the equations of (14) by c^2 from $U_{50} = 0$. If $a_{03} = (2f+1)/3$, then we obtain the set of conditions (xii) ($b = 0$, $f = (-3c^2 - 2)/(2c^2 + 1)$).

Let $a_{03} \neq (2f+1)/3$ and denote $j_1 = 9a_{03} + 8$, $j_2 = a_{03} + 2$, $j_3 = 3a_{03} - 4$, $j_4 = 3a_{03} + 4$, $j_5 = 3a_{03} + 1$, $j_6 = 675a_{03}^3 + 1530a_{03}^2 + 1116a_{03} + 256$.

If $j_1j_2\cdots j_6 = 0$, then the system (14) is not consistent. Assume that $j_1j_2\cdots j_6 \neq 0$ and calculate the resultant of the polynomials U_{41} and U_{31} with respect to a_1. We find that $Res(U_{41}, U_{31}, a_1) \neq 0$ and the system (14) is not consistent.

Assume that $r_1 \neq 0$ and let $r_2 = 0$. We express a from $r_2 = 0$, a_1 from $U_{13} = 0$ and g from $U_{22} = 0$. Then $U_{23} \equiv s_1s_2$=0, where

$$s_1 = 3(d+2)a_{03}^2 - ((f+2)(a_2 - c)a_2 + (d+2)(2f-1))a_{03} - 2(f+1)(d+2),$$
$$s_2 = 3a_{03}^2 - (4f+3)a_{03} + 2(f+1)^2.$$

Suppose that $s_1 = 0$ and express d from $s_1 = 0$. In this case $U_{40} \equiv c(3a_{03} - 2f - 1) = 0$. If $a_{03} = (2f+1)/3$, then we get the set of conditions (xii) ($b = 0$). If $a_{03} \neq (2f+1)/3$ and $c = 0$, then we obtain the conditions (xiii) ($b = 0$).

Suppose that $s_1 \neq 0$ and let $s_2 = 0$. Then the equation $s_2 = 0$ admits the parametrization $a_{03} = (-2)/(u^2+2)$, $f = (-u^2 - u - 4)/(u^2 + 2)$.

We reduce the equations of (14) by a_2^2 from $U_{40} = 0$. If $d = u(5-2u)/(u^2+2)$, then $U_{50} = 0$ and the system (14) is not consistent. If $d \neq u(5-2u)/(u^2+2)$, then reduce the equations of (14) by c^2 from $U_{50} = 0$. In this case we get the set of conditions (xiv) for the existence of a first integral (10) with $\alpha_1 = \alpha_2 = 1$, $\alpha_3 = -1$ and

$$l_1 \equiv 1 - a_2x - y = 0, \; l_2 \equiv 1 + a_2x - y = 0,$$
$$\Phi \equiv 2(3cx - y)(3cx + 2y)^2 + 9(x^2 + y^2) = 0,$$

where $3a_2^2 = 9c^2 + 1$.

In each of the cases (xii)–(xiv), the system (1) has a Darboux first integral of the form (10) and therefore the origin is a center for (1). □

6. Proof of the Main Theorem

The proof of the main result, Theorem 1.1, follows directly from Lemmas 3.1, 4.1 and 5.1. The existence of a center for system (1), in Cases (i) – (xiv), is equivalent to the existence of the Darboux first integrals of the form (10) defined in a neighborhood of the origin [17].

References

[1] V.V. Amel'kin, N.A. Lukashevich, A.P. Sadovskii, *Nonlinear Oscillations in Second-order Systems*, Belorussian State University, Minsk, 1982 (in Russian).

[2] J. Chavarriga, J. Giné, Integrability of cubic systems with degenerate infinity, *Differential Equations Dynam. Systems* **6** (1998), 425-438.

[3] D. Cozma, A. Şubă, Partial integrals and the first focal value in the problem of centre, *Nonlinear Differ. Equ. and Appl.* **2** (1995), 21–34.

[4] D. Cozma, A. Şubă, The solution of the problem of center for cubic differential systems with four invariant straight lines, *Scientific Annals of the "Al.I.Cuza" University, Mathematics* **44** (1998), 517–530.

[5] D. Cozma, The problem of the center for cubic systems with two parallel invariant straight lines and one invariant conic, *Nonlinear Differ. Equ. and Appl.* **16** (2009), 213–234.

[6] D. Cozma, Center problem for cubic systems with a bundle of two invariant straight lines and one invariant conic, *Bul. Acad. Ştiinţe Repub. Moldova, Mat.* **68** (2012), no. 1, 32–49.

[7] D. Cozma, Darboux integrability and rational reversibility in cubic systems with two invariant straight lines, *Electronic Journal of Differential Equations* **2013** (2013), no. 23, 1–19.

[8] D. Cozma, *Integrability of Cubic Systems with Invariant Straight Lines and Invariant Conics*, Ştiinţa, Chişinău, 2013.

[9] D. Cozma, A. Dascalescu, Integrability conditions for a class of cubic differential systems with a bundle of two invariant straight lines and one invariant cubic, *Bul. Acad. Ştiinţe Repub. Moldova, Mat.* **86** (2018), no. 1, 120–138.

[10] J. Gine, J. Llibre, C. Valls, The cubic polynomial differential systems with two circles as algebraic limit cycles, *Adv. Nonlinear Stud.* 2017, 1–11.

[11] M. Han, V. Romanovski, X. Zhang, Integrability of a family of 2-dim cubic systems with degenerate infinity, *Rom. Journ. Phys.* **61** (2016), no. 1–2, 157–166.

[12] N.G. Lloyd, J.M. Pearson, A cubic differential system with nine limit cycles, *Journal of Applied Analysis and Computation* **2** (2012), no. 3, 293–304.

[13] M.N. Popa, V.V. Pricop, Applications of algebraic methods in solving the center-focus problem, *Bul. Acad. Ştiinţe Repub. Moldova, Mat.* **71** (2013), no. 1, 45–71.

[14] A.P. Sadovskii, Solution of the center and focus problem for a cubic system of nonlinear oscillations, *Differential Equations* **33** (1997), no. 2, 236–244.

[15] A.P. Sadovskii, T.V. Shcheglova, Solution of the center-focus problem for a nine-parameter cubic system, *Differential Equations* **47** (2011), no. 2, 208–223.

[16] A. Şubă, D. Cozma, Solution of the problem of center for cubic differential systems with three invariant straight lines in generic position, *Qualitative Theory of Dynamical Systems* **6** (2005), 45–58.

[17] X. Zhang, *Integrability of Dynamical Systems: Algebra and Analysis*, Springer Nature Singapure, Singapure, 2017.

(Anatoli Dascalescu) Department of Mathematical Analysis and Differential Equations, Tiraspol State University, 5 Gh. Iablocichin str., Chişinău, MD2069, Republic of Moldova

E-mail address: anatol.dascalescu@gmail.com