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Continuous spectrum for a degenerate eigenvalue problem with (p_2, p_2)-growth

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Abstract. In this paper we consider an eigenvalue problem driven by two non-homogeneous differential operators with variable (p_2, p_2)-growth. We establish that for $\lambda_1 > 0$, any $\lambda \in [\lambda_1, \infty)$ is an eigenvalue; moreover, for a positive constant $\lambda_0 \leq \lambda_1$, we find the nonexistence of eigenvalues in $(0, \lambda_0)$. The proof is based on variational arguments and a Caffarelli–Kohn–Nirenberg-type inequality with variable exponent.

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1. Introduction

In this paper our attention is focused on a new class of differential operators recently introduced by Kim and Kim in [7] and extended in [2] by the nonlinear eigenvalue problem

$$-\mathrm{div}(\phi(x, |\nabla u|)\nabla u) - \mathrm{div}(\psi(x, |\nabla u|)\nabla u) = \lambda f(x, u) \text{ in } \Omega, \tag{1}$$

involving non-homogeneous operators of the type $(\phi(x, |\nabla u|)\nabla u)$. In the case $\phi(x, t) = t^{p(x)-2}$ we obtain the $p(x)$-Laplace operator. Problems involving the $p(x)$-Laplacian have been extensively studied in the last decades (see [1, 4, 5, 6, 8, 14, 17, 18]).

Our interest is related to the study of a class of non-autonomous stationary problems, that means problems in which, according to the point, the associated energy density changes its ellipticity and growth properties. The contribution of Kim and Kim serves us to understand problems with possible lack of uniform convexity.

The main purpose of this paper is to study problem (1) in a particular case of $f(x, u)$ under the presence of a nonnegative measurable weighted function in the divergence operator. Degenerate differential operators involving a nonnegative weight who can have zeros at some points or even to be unbounded serve the study of many physical phenomena associated to equilibrium of anisotropic continuous media. By means of variational arguments and a Caffarelli–Kohn–Nirenberg inequality with variable exponent we establish the existence of a continuous spectrum consisting in an unbounded interval and the lack of existence of eigenvalues in a neighbourhood of the origin.

In the next section we make a brief introduction of the variable exponent Sobolev spaces, in order to choose the corresponding space to our problem. In section 3 we

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introduce a set of basic hypotheses and we state the main theorem. The proof of this result is developed in the last section of this paper.

2. Preliminaries

We state in this section some definitions and properties of the variable exponent Lebesgue-Sobolev spaces.

We consider Ω be a bounded domain in $\mathbb{R}^N$ and we define

$$C_+(\Omega) = \left\{ p \in C(\overline{\Omega}) : \min_{x\in\Omega} p(x) > 1 \right\}$$

and for any (Lebesgue) continuous function $p : \overline{\Omega} \to (1, \infty)$, denote

$$p^- = \inf_{x\in\Omega} p(x) \qquad \text{and} \qquad p^+ = \sup_{x\in\Omega} p(x).$$

For all $p \in C_+(\overline{\Omega})$ we define the *variable exponent Lebesgue space*

$$L^{p(x)}(\Omega) = \left\{ u : \Omega \to \mathbb{R} \text{ a measurable function} : \int_\Omega |u|^{p(x)} dx < \infty \right\}.$$

Equipped with the *Luxemburg norm*

$$|u|_{p(x)} = \inf \left\{ \mu > 0 : \int_\Omega \left| \frac{u(x)}{\mu} \right| dx \leq 1 \right\},$$

$L^{p(x)}(\Omega)$ is a Banach space.

If $p(x) = p \equiv$ *constant* for any $x \in \Omega$, then the $L^{p(x)}(\Omega)$ space is reduced to the classic Lebesgue space $L^p(\Omega)$ and the Luxemburg norm becomes the standard norm in $L^p(\Omega)$, $\|u\|_{L^p} = \left(\int_\Omega |u(x)|^p dx\right)^{1/p}$.

For $1 < p^- \leq p^+ < \infty$, $L^{p(x)}(\Omega)$ is a reflexive uniformly convex Banach space, and for any measurable bounded exponent p, the $L^{p(x)}(\Omega)$ space is separable.

If p_1 and p_2 are two variable exponents such that $p_1(x) \leq p_2(x)$ almost everywhere in Ω, with $|\Omega| < \infty$, then there exists a continuous embedding

$$L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$$

whose norm does not exceed $|\Omega| + 1$.

We define the conjugate variable exponent $p' : \overline{\Omega} \to (1, \infty)$, satisfying $\dfrac{1}{p(x)} + \dfrac{1}{p'(x)} = 1$, for every $x \in \overline{\Omega}$.

We denote by $L^{p'(x)}(\Omega)$ the conjugate space of the $L^{p(x)}(\Omega)$.

If $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$ then the *Hölder type inequality* holds:

$$\left| \int_\Omega uv\, dx \right| \leq \left(\frac{1}{p^+} + \frac{1}{p^-} \right) |u|_{p(x)} |v|_{p'(x)} \leq 2 |u|_{p(x)} |v|_{p'(x)}. \tag{2}$$

The *modular* of the $L^{p(x)}(\Omega)$ space, defined by the mapping $\rho_{p(x)} : L^{p(x)}(\Omega) \to \mathbb{R}$,

$$\rho_{p(x)} = \int_\Omega |u(x)|^{p(x)} dx,$$

has an important role in manipulating the generalized Lebesgue spaces.

If $p(x) = p \equiv$ *constant* for every $x \in \Omega$, then the modular $\rho_{p(x)}(u)$ becomes $\|u\|^p_{L^p}$.

If $p(x) \not\equiv constant$ in Ω and $u, u_n \in L^{p(x)}(\Omega)$ then the following relations hold true:

$$|u|_{p(x)} < 1 \Rightarrow \; |u|_{p(x)}^{p^+} \leq \rho_{p(x)}(u) \leq |u|_{p(x)}^{p^-}, \tag{3}$$

$$|u|_{p(x)} > 1 \Rightarrow \; |u|_{p(x)}^{p^-} \leq \rho_{p(x)}(u) \leq |u|_{p(x)}^{p^+}, \tag{4}$$

$$|u|_{p(x)} = 1 \Rightarrow \; \rho_{p(x)}(u) = 1, \tag{5}$$

$$|u_n - u|_{p(x)} \to 0 \Leftrightarrow \; \rho_{p(x)}(u_n - u) \to 0. \tag{6}$$

See [9] for more properties of these variable exponent Lebesgue spaces.

Now, we define the *variable exponent Sobolev space* as

$$W^{1,p(\cdot)}(\Omega) = \left\{ u \in L^{p(\cdot)}(\Omega); |\nabla u| \in L^{p(\cdot)}(\Omega) \right\}$$

equipped with the norm

$$\|u\|_{p(\cdot)} = |u|_{p(\cdot)} + |\nabla u|_{p(\cdot)}$$

which is equivalent with the norm

$$\|u\| = \inf \left\{ \mu > 0; \int_\Omega \left(\left| \frac{\nabla u(x)}{\mu} \right|^{p(x)} + \left| \frac{u(x)}{\mu} \right|^{p(x)} \right) dx \leq 1 \right\},$$

where $|\nabla u|_{p(\cdot)}$ is the Luxemburg norm of $|\nabla u|$.

We define $W_0^{1,p(\cdot)}(\Omega) = \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{p(\cdot)}}$ and we remark that $W_0^{1,p(\cdot)}(\Omega)$ is a separable and reflexive Banach space.

For the density of $C_0^\infty(\Omega)$ in $W_0^{1,p(\cdot)}(\Omega)$ we consider $p \in C_+(\overline{\Omega})$ to be *logarithmic Hölder continuous*, that means, there exists $M > 0$ such that

$$|p(x) - p(y)| \leq \frac{-M}{\log(|x-y|)}, \; \forall x, y \in \Omega,$$

with $|x - y| \leq \frac{1}{2}$.

Also, we remark that if $s \in C_+(\overline{\Omega})$ and $s(x) < p^*(x)$ for all $x \in \overline{\Omega}$, then $W_0^{1,p(\cdot)}(\Omega)$ is compactly embedded in $L^{s(\cdot)}(\Omega)$, where $p^*(x) = \frac{Np(x)}{N-p(x)}$ and $p(x) < N$.

Finally, we define the *modular* of the $W_0^{1,p(\cdot)}(\Omega)$ space by the mapping

$$\varrho_{p(\cdot)} : W_0^{1,p(\cdot)}(\Omega) \to \mathbb{R},$$

$$\varrho_{p(\cdot)}(u) = \int_\Omega \left(|\nabla u|^{p(x)} + |u|^{p(x)} \right) dx.$$

If $(u_n), u \in W_0^{1,p(\cdot)}(\Omega)$, then we have the following relations:

$$\|u\| > 1 \Rightarrow \|u\|^{p^-} \leq \varrho_{p(\cdot)}(u) \leq \|u\|^{p^+}, \tag{7}$$

$$\|u\| < 1 \Rightarrow \|u\|^{p^+} \leq \varrho_{p(\cdot)}(u) \leq \|u\|^{p^-}, \tag{8}$$

$$\|u_n - u\| \to 0 \Leftrightarrow \varrho_{p(\cdot)}(u_n - u) \to 0. \tag{9}$$

For more details about these spaces we refer [3, 9, 12, 15, 13, 16].

3. Continuous spectrum for differential operators with variable (p_1, p_2)-growth

We consider the following eigenvalue problem

$$\begin{cases} -\mathrm{div}\,[a(x)(\phi(x,|\nabla u|)\nabla u + \psi(x,|\nabla u|)\nabla u)] = \lambda |u|^{q(x)-2}u, & x \in \Omega \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{10}$$

where $\Omega \in \mathbb{R}^N (N \geq 3)$ is a bounded domain with smooth boundary, λ is a positive real number and $a : \Omega \to [0, \infty)$ a weighted function which has the property that $a \in L^1_{loc}(\Omega)$.

Problem (10) is based on non-homogeneous operators of the type $(\phi(x, |\nabla u|)\nabla u)$. When $\phi(x, \mu) = \mu^{p(x)-2}$, the operator implicated in (10) is the $p(x)$-Laplacian, that is,

$$\Delta_{p(x)} u = \mathrm{div}(|\nabla u|^{p(x)-2} \nabla u).$$

Throughout this paper we assume that $p_1, p_2, q \in C_+(\bar{\Omega})$ and

$$1 < p_1(x) < q^- \leq q^+ < p_2(x) < p_1^*(x), \tag{11}$$

where $p_1^*(x) := \dfrac{Np_1(x)}{N - p_1(x)}$ if $p_1(x) < N$ and $p_1^*(x) := +\infty$ if $p_1(x) > N$.

We remark that if $p_1(x) < p_2(x)$, for any $x \in \bar{\Omega}$, then $W_0^{1,p_2(x)}(\Omega)$ is continuously embedded in $W_0^{1,p_1(x)}(\Omega)$.

Let consider the functions $\phi, \psi : \Omega \times [0, \infty) \to [0, \infty)$ which fulfill the following assumptions:

(h_1) $\phi(\cdot, \mu)$ and $\psi(\cdot, \mu)$ are two measurable mappings on Ω for any $\mu \geq 0$; further, $\phi(x, \cdot)$ and $\psi(x, \cdot)$ are locally absolutely continuous on $[0, \infty)$ for almost $x \in \Omega$;

(h_2) there exist $\alpha_1 \in L^{p_1'}(\Omega)$, $\alpha_2 \in L^{p_2'}(\Omega)$ functions and $\beta > 0$ such that

$$|\phi(x, |v|)v| \leq \alpha_1(x) + \beta |v|^{p_1(x)-1}, \ \ |\psi(x, |v|)v| \leq \alpha_2(x) + \beta |v|^{p_2(x)-1}$$

for almost all $x \in \Omega$ and for any $v \in \mathbb{R}^N$.

(h_3) there exists a positive constant $c > 0$ such that

$$\phi(x, \mu) \geq c\mu^{p_1(x)-2}, \ \ \phi(x, \mu) + \mu \frac{\partial \phi}{\partial \mu}(x, \mu) \geq c\mu^{p_1(x)-2}$$

and

$$\psi(x, \mu) \geq c\mu^{p_2(x)-2}, \ \ \psi(x, \mu) + \mu \frac{\partial \psi}{\partial \mu}(x, \mu) \geq c\mu^{p_2(x)-2},$$

for almost all $x \in \Omega$ and for all positive μ.

Definition 3.1. A *weak solution* for problem (10) is a function $u \in D_0^{1,p_2(x)}(\Omega) \setminus \{0\}$ such that

$$\int_\Omega a(x) \left[\phi(x, |\nabla u|) + \psi(x, |\nabla u|)\right] \nabla u \nabla v dx - \lambda \int_\Omega |u|^{q(x)-2} uv dx = 0,$$

for all $v \in D_0^{1,p_2(x)}(\Omega) \setminus \{0\}$.

Therefore, $\lambda \in \mathbb{R}$ is an eigenvalue for problem (10), with the corresponding eigenfunction u.

The function space for problem (10) is $D_0^{1,p_2(x)}(\Omega)$, this choice being motivated by hypothesis (11) and the presence of a weight function $a : \Omega \to [0, \infty)$ which satisfies $a \in L^1_{loc}(\Omega)$.

For any $\lambda > 0$ we define $F_\lambda : D_0^{1,p_2(x)}(\Omega) \to \mathbb{R}$ by

$$F_\lambda(u) = \int_\Omega \frac{a(x)}{p_1(x)} |\nabla u|^{p_1(x)} dx + \int_\Omega \frac{a(x)}{p_2(x)} |\nabla u|^{p_2(x)} dx - \lambda \int_\Omega \frac{1}{q(x)} |u|^{q(x)} dx.$$

Then, $F_\lambda \in C^1(D_0^{1,p_2(x)}(\Omega), \mathbb{R})$ and

$$\left\langle F_\lambda^{'}(u), v \right\rangle = \int_\Omega a(x) \left[\phi(x, |\nabla u|) + \psi(x, |\nabla u|)\right] \nabla u \nabla v dx - \lambda \int_\Omega |u|^{q(x)-2} uv dx,$$

for all $u, v \in D_0^{1,p_2(x)}(\Omega)$.

We define the *first Rayleigh quotient* by

$$\lambda_1 := \inf_{u \in D_0^{1,p_2(x)}(\Omega) \setminus \{0\}} \frac{\displaystyle\int_\Omega \frac{a(x)}{p_1(x)} |\nabla u|^{p_1(x)} dx + \int_\Omega \frac{a(x)}{p_2(x)} |\nabla u|^{p_2(x)} dx}{\displaystyle\int_\Omega \frac{1}{q(x)} |u|^{q(x)} dx}.$$

The main result of this paper is given by the following theorem.

Theorem 3.2. Suppose that hypothesis (11) is satisfied. Then $\lambda_1 > 0$. Furthermore, any $\lambda \in [\lambda_1, \infty)$ is an eigenvalue of problem (10). Moreover, there exists a positive constant λ_0 such that $\lambda_0 \le \lambda_1$ and no $\lambda \in (0, \lambda_0)$ is an eigenvalue of problem (10).

Since problem (10) is driven by two operators who are not uniformly elliptic, we use the following results in order to proof Theorem 3.2.

In [11] it is defined $\vec{a} : \Omega \to \mathbb{R}^N$ by

$$\vec{a}(x) = (a_1(x_1), \dots, a_N(x_N))$$

with the assumption that there exists a positive constant a_0 such that

$$\text{div}\vec{a}(x) \ge a_0 > 0, \quad \text{for all } x \in \bar{\Omega} \tag{12}$$

and $p : \bar{\Omega} \to (1, N)$ a function of class C^1 which satisfies

$$\vec{a}(x) \cdot \nabla p(x) = 0, \quad \text{for all } x \in \Omega. \tag{13}$$

Theorem 3.3. Suppose that $\vec{a}(x)$ and $p(x)$ satisfie hypotheses (12) and (13). Then there exists a positive constant C such that

$$\int_\Omega |u(x)|^{p(x)} dx \le C \int_\Omega |\vec{a}(x)|^{p(x)} |\nabla u|^{p(x)} dx,$$

$\forall u \in C_c^1(\bar{\Omega})$.

This result is named the Caffarelli–Kohn–Nirenberg inequality with variable exponent. Throughout this paper we consider $a(x) = |x|^{p_1(x)}$, respectively $a(x) = |x|^{p_2(x)}$. Under these conditions we use a particular form of the Caffarelli–Kohn–Nirenberg inequality stated in [11]: with N, p and a defined as above, there exists a positive constant C such that

$$\int_\Omega |u(x)|^{p(x)} dx \le C \int_\Omega |x|^{p(x)} |\nabla u|^{p(x)} dx, \tag{14}$$

$\forall u \in C_c^1(\bar{\Omega})$.

For simplicity, we note with $\|\cdot\|_1$, respectively $\|\cdot\|_2$ the norm of $W_0^{1,p_1(x)}(\Omega)$, respectively $W_0^{1,p_2(x)}(\Omega)$ and $\|\cdot\|$ the norm of $D_0^{1,p_2(x)}(\Omega) := X$.

4. Proof of Theorem 3.2

We define the energy functionals I_1, J_1, I_2, $J_2 : X \to \mathbb{R}$ by

$$I_1(u) = \int_\Omega \frac{|x|^{p_1(x)}}{p_1(x)} |\nabla u|^{p_1(x)} dx + \int_\Omega \frac{|x|^{p_2(x)}}{p_2(x)} |\nabla u|^{p_2(x)} dx,$$

$$J_1(u) = \int_\Omega \frac{1}{q(x)} |u|^{q(x)} dx,$$

$$I_2(u) = \int_\Omega |x|^{p_1(x)} |\nabla u|^{p_1(x)} dx + \int_\Omega |x|^{p_2(x)} |\nabla u|^{p_2(x)} dx,$$

$$J_2(u) = \int_\Omega |u|^{q(x)} dx.$$

Proposition 4.1. The functionals I_1 and J_1 are well-defined on X and $I_1, J_1 \in C^1(X, \mathbb{R})$, with the Gâteaux derivative given by

$$\left\langle I_1^{'}(u), v \right\rangle = \int_\Omega |x|^{p_1(x)} |\nabla u|^{p_1(x)-2} \nabla u \nabla v dx + \int_\Omega |x|^{p_2(x)} |\nabla u|^{p_2(x)-2} \nabla u \nabla v dx$$

and

$$\left\langle J_1^{'}(u), v \right\rangle = \int_\Omega |u|^{q(x)-2} u v dx,$$

for all $u, v \in X$.

The proof of Proposition 4.1 is based on standard arguments (see, e.g. [10]).

In this context, we proof the following assertions:

(a) $\lambda_1 > 0$;
(b) λ_1 is an eigenvalue of problem (10);
(c) any $\lambda \in (\lambda_1, \infty)$ is an eigenvalue of problem (10);
(d) no $\lambda \in (0, \lambda_0)$ is an eigenvalue of problem (10).

(a): For any $x \in \Omega$, taking into account hypothesis (11), we obtain that for any $u \in X$,

$$2(|u(x)|^{p_1(x)} + |u(x)|^{p_2(x)}) \geq |u(x)|^{q^+} + |u(x)|^{q^-} \tag{15}$$

and

$$|u(x)|^{q^+} + |u(x)|^{q^-} \geq |u(x)|^{q(x)}. \tag{16}$$

Now, integrating the above inequalities we obtain

$$2 \int_\Omega (|u(x)|^{p_1(x)} + |u(x)|^{p_2(x)}) dx \geq \int_\Omega (|u(x)|^{q^+} + |u(x)|^{q^-}) dx \geq \int_\Omega |u(x)|^{q(x)} dx. \tag{17}$$

By (14) and (17) we finally obtain that there exists a positive constant $\tilde{c}$ such that

$$\begin{aligned} \int_\Omega |x|^{p_1(x)} |\nabla u|^{p_1(x)} dx + \int_\Omega |x|^{p_2(x)} |\nabla u|^{p_1(x)} dx &\geq \tilde{c} \int_\Omega (|u(x)|^{p_1(x)} + |u(x)|^{p_2(x)}) dx \\ &\geq \frac{\tilde{c}}{2} \int_\Omega |u(x)|^{q(x)} dx, \end{aligned} \tag{18}$$

for every $u \in X$.

Relation (18) leads us to the second Rayleigh quotient

$$\lambda_0 := \inf_{v \in X\{0\}} \frac{I_2(v)}{J_2(v)} > 0. \tag{19}$$

Hence,

$$I_2(u) \geq \lambda_0 J_2(u), \quad \text{for every } u \in X. \tag{20}$$

From the above inequality follows that

$$p_1^+ I_1(u) \geq I_2(u) \geq \lambda_0 J_2(u) \geq \lambda_0 J_1(u), \quad \forall u \in X \tag{21}$$

and therefore we obtain that $\lambda_1 > 0$.

(b): In order to prove our result, we use the following two lemmas.

Lemma 4.2. The following equalities are satisfied:

$$\lim_{\|u\| \to \infty} \frac{I_1(u)}{J_1(u)} = \infty \tag{22}$$

and

$$\lim_{\|u\| \to 0} \frac{I_1(u)}{J_1(u)} = \infty. \tag{23}$$

Proof. By the continuous embedding of X in $L^{q^\pm}(\Omega)$ (see [15]), we deduce that there exists μ_1 and μ_2 two positive constants such that

$$\|u\| \geq \mu_1 |u|_{q^+}, \quad \text{for any } u \in X \tag{24}$$

and

$$\|u\| \geq \mu_2 |u|_{q^-}, \quad \text{for any } u \in X. \tag{25}$$

Relations (4), (14), (17), (24) and (25) lead to the fact that, for any $u \in X$ with $\|u\| > 1$, it holds

$$\begin{aligned} \frac{I_1(u)}{J_1(u)} &\geq \frac{\dfrac{1}{p_1^+}\displaystyle\int_\Omega |x|^{p_1(x)} |\nabla u|^{p_1(x)} dx + \dfrac{1}{p_2^+}\displaystyle\int_\Omega |x|^{p_2(x)} |\nabla u|^{p_2(x)} dx}{\dfrac{1}{q^-}\displaystyle\int_\Omega |u|^{q(x)} dx} \\ &\geq \frac{\dfrac{1}{p_1^+}\displaystyle\int_\Omega |u|^{p_1(x)} dx + \dfrac{1}{p_2^+}\displaystyle\int_\Omega |u|^{p_2(x)} dx}{\dfrac{1}{q^-}\displaystyle\int_\Omega |u|^{q(x)} dx} \geq \frac{\dfrac{1}{p_1^+}|u|_{p_1(x)}^{p_1^-} + \dfrac{1}{p_2^+}|u|_{p_2(x)}^{p_2^-}}{\dfrac{|u|_{q^+}^{q^+} + |u|_{q^-}^{q^-}}{q^-}}. \end{aligned}$$

Since X is continuously embedded in $W^{1,p_2(x)}(\Omega)$ we obtain that

$$\frac{I_1(u)}{J_1(u)} \geq \frac{\dfrac{1}{p_2^+}\|u\|^{p_2^+}}{\dfrac{\tilde{c_1}\|u\|^{q^+} + \tilde{c_2}\|u\|^{q^-}}{q^-}}. \tag{26}$$

Passing to the limit as $\|u\| \to \infty$ in inequality (26) and considering that $p_2^- > q^+ \geq q^-$ it follows that relation (22) holds.

From (11) we have that $p_2(x) > p_1(x)$, for any $x \in \bar{\Omega}$, hence $W_0^{1,p_2(x)}(\Omega)$ is continuously embedded in $W_0^{1,p_1(x)}(\Omega)$. Therefore, if $\|u\|$ converges to 0 then $\|u\|_1$ converges to 0.

We conclude from the above remarks that for any $u \in X$ with $\|u\| < 1$ small enough, we have $\|u\|_2 < 1$.

Again from (11) we deduce that $W_0^{1,p_1(x)}(\Omega)$ is continuously embedded in $L^{q^\pm}(\Omega)$. Then there exist c_1, $c_2 > 0$ constants such that

$$\|u\|_2 \geq c_1 |u|_{q^+}, \quad \forall u \in W_0^{1,p_1(x)}(\Omega) \tag{27}$$

and

$$\|u\|_2 \geq c_2 |u|_{q^-}, \quad \forall u \in W_0^{1,p_1(x)}(\Omega). \tag{28}$$

Hence, for every $u \in X$ with $\|u\| < 1$ small enough, relations (3), (17), (27) and (28) involve

$$\frac{I_1(u)}{J_1(u)} \geq \frac{\dfrac{1}{p_1^+}\displaystyle\int_\Omega |\nabla u|^{p_1(x)} dx}{\dfrac{|u|_{q^+}^{q^+} + |u|_{q^-}^{q^-}}{q^-}} \geq \frac{\dfrac{\|u\|_2^{p_1^+}}{p_1^+}}{\dfrac{c_1 \|u\|_2^{q^+} + c_2 \|u\|_2^{q^-}}{q^-}}.$$

Because $p_1^+ < q^- \leq q^+$, passing to the limit as $\|u\| \to 0$ (so $\|u\|_1 \to 0$) in the above inequality we obtain that relation (23) is fulfill and then, this complete the proof of Lemma 4.2. $\square$

Lemma 4.3. There exists $u \in X \setminus \{0\}$ such that $\dfrac{I_1(u)}{J_1(u)} = \lambda_1$.

Proof. We consider $(u_n) \in X \setminus \{0\}$ a minimizing sequence for λ_1, this means that

$$\lim_{n\to\infty} \frac{I_1(u_n)}{J_1(u_n)} = \lambda_1 > 0. \tag{29}$$

Relation (22) ensures us that (u_n) is a bounded sequence in X. Because X is reflexive (see [15]) then there exists $u \in X$ such that, up to a subsequence, (u_n) is weakly convergent to u in X.

Let $u \in X$, by (4) and (14) we have that from any $\|u\| > 1$,

$$I_1(u) \geq \frac{1}{p_2^+} \|u\|^{p_2^-}.$$

From (11), the above inequality leads to

$$\lim_{\|u\|\to\infty} I_1(u) = \infty,$$

so, I_1 is coercive.

On the other hand, for any $u \in X$ we have

$$I_1(u) \geq \frac{1}{p_2^+} \min\left\{\|u\|^{p_2^-}, \|u\|^{p_1^-}\right\},$$

hence, we deduce that I_1 is bounded from below. Therefore, it follows that I_1 is weakly lower semi-continuous. Now, we can conclude that

$$\liminf_{n\to\infty} I_1(u_n) \geq I_1(u). \tag{30}$$

From (11) we know that $1 \le q^+ < p_2(x)$, for all $x \in \bar{\Omega}$. Taking into account this hypothesis, the compact embedding theorem for spaces with variable exponent and [15, Remark 2] we obtain that X is compact embedded in $L^{q(x)}(\Omega)$. It follows that (u_n) strongly converges in $L^{q(x)}(\Omega)$. Then, by relation (6) we obtain

$$\lim_{n\to\infty} J_1(u_n) = J_1(u). \tag{31}$$

By (30) and (31) we deduce that if $u \not\equiv 0$ then

$$\frac{I_1(u)}{J_1(u)} = \lambda_1.$$

The last step in proving Lemma 4.3 is to show that u is nontrivial. Suppose that u is trivial. Thus, (u_n) weakly converges to 0 in X and strongly in $L^{q(x)}$, which means that

$$\lim_{n\to\infty} J_1(u_n) = 0. \tag{32}$$

With $\varepsilon \in (0, \lambda_1)$ a fixed point, by relation (29) it follows that for n large enough we have

$$|I_1(u_n) - \lambda_1 J_1(u_n)| < \varepsilon J_1(u_n)$$

and

$$(\lambda_1 - \varepsilon) J_1(u_n) < I_1(u_n) < (\lambda_1 + \varepsilon) J_1(u_n).$$

Passing to the limit in the previous inequalities and considering relation (32) fulfilled we deduce that

$$\lim_{n\to\infty} I_1(u_n) = 0. \tag{33}$$

Taking into account (6), relation (33) implies that (u_n) strongly converges to 0 in X, that means, $\lim_{n\to\infty} \|u_n\| = 0$. This result and relation (23) yield to

$$\lim_{n\to\infty} \frac{I_1(u_n)}{J_1(u_n)} = \infty,$$

which means a contradiction. Hence, $u \not\equiv 0$ and this completes the proof of Lemma 4.3.

□

By Lemma 4.3 we infer that there exists $u \in X \setminus \{0\}$ such that

$$\frac{I_1(u)}{J_1(u)} = \lambda_1 = \inf_{w \in X\setminus\{0\}} \frac{I_1(w)}{J_1(w)}. \tag{34}$$

However, for any $v \in X$ we have

$$\frac{d}{d\varepsilon} \frac{I_1(u+\varepsilon v)}{J_1(u+\varepsilon v)}\Big|_{\varepsilon=0} = 0.$$

We can easly obtain that

$$\int_\Omega \left(|x|^{p_1(x)} \phi(x, |\nabla u|) + |x|^{p_2(x)} \psi(x, |\nabla u|) \right) \nabla u \nabla v dx \cdot J_1(u) - I_1(u) \int_\Omega |u|^{q(x)-2} uv dx = 0, \tag{35}$$

for any $v \in X$.

From relation (35) and taking into account the fact that $I_1(u) = \lambda_1 J_1(u)$ and $J_1(u) \neq 0$ we deduce that λ_1 is an eigenvalue of problem (10). Therefore, hypothesis (b) is verified.

(c): We consider $\lambda \in (\lambda_1, \infty)$ arbitrary but fixed and define $F_\lambda : X \to \mathbb{R}$ by

$$F_\lambda(u) = I_1(u) - \lambda J_1(u).$$

Standard arguments show that $F_\lambda \in C^1(X, \mathbb{R})$ with the Gâteaux derivative

$$\left\langle F_\lambda'(u), v \right\rangle = \left\langle I_1'(u), v \right\rangle - \lambda \left\langle J_1'(u), v \right\rangle, \ \forall u \in X.$$

Then, λ is an eigenvalue of problem (10) if and only if there exists a critical point $u_\lambda \in X \setminus \{0\}$ of F_λ.

Similarly with the proof of relation (22) we obtain that the functional F_λ is coercive, so

$$\lim_{\|u\| \to \infty} F_\lambda(u) = \infty.$$

The fact that the functional F_λ is weakly lower semi-continuous is obtained by similar arguments as in the proof of [15, Lemma 3]. Then, there exists a global minimum point $u_\lambda \in X$ of F_λ which is a critical point of F_λ.

Finally, it remains to prove that u_λ is not trivial. Certainly, whereas $\lambda_1 = \inf_{u \in X \setminus \{0\}} \dfrac{I_1(u)}{J_1(u)}$ and $\lambda > \lambda_1$ we deduce that there exists $v_\lambda \in X$ such that

$$I_1(v_\lambda) < \lambda J_1(v_\lambda)$$

and

$$F_\lambda(v_\lambda) < 0.$$

Then,

$$\inf_X F_\lambda < 0$$

and hence, the hypothesis that u_λ is a nontrivial critical point of F_λ is verified. This concludes (c).

(d): Finally, we prove that no $\lambda \in (0, \lambda_0)$ with λ defined by (19) is an eigenvalue of problem (10).

Suppose that there exists an eigenvalue $\lambda \in (0, \lambda_0)$ of problem (10). Then, there exists $u_\lambda \in X \setminus \{0\}$ such that

$$\left\langle I_1'(u_\lambda), u_\lambda \right\rangle = \lambda \left\langle J_1'(u_\lambda), u_\lambda \right\rangle,$$

that is

$$I_2(u_\lambda) = \lambda J_2(u_\lambda).$$

Since $u_\lambda \in X \setminus \{0\}$ it follows that $J_2(u_\lambda) > 0$. The fact that $\lambda < \lambda_0$ combined with the previous informations yields to

$$I_2(u_\lambda) \geq \lambda_0 J_2(u_\lambda) > \lambda J_2(u_\lambda) = I_2(u_\lambda).$$

By the above inequalities we obtain a contradiction. Hence, there is no eigenvalue $\lambda \in (0, \lambda_0)$ of problem (10).

From (b), (c) and (d) we conclude that $\lambda_0 \leq \lambda_1$ and this completes the proof of Theorem 3.2.

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