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Existence theorems for degenerate Schrödinger equations involving a singular potential and an indefinite sign perturbation

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Abstract. We study a degenerate quasilinear problem involving a singular potential and some bounded weights. The equation is perturbed by a critical nonlinear term and an indefinite sign perturbation involving a real parameter. Under suitable assumptions on the potentials and on the real parameter we use two different critical point techniques, in order to reveal two types of solutions. The proofs rely on variational arguments based on the Mountain-Pass Theorem, Ekeland's Variational Principle and energy estimates.

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1. Introduction

In this paper we are concerned with the study of a class of perturbed quasilinear problems involving a singular potential, a critical nonlinear perturbation term and an indefinite sign perturbation.

Problems of this type are associated with Schrödinger's equation and the existence of some classes of time-independent solutions. We point out that these types of equations have a large spectrum of applications, from the stability of the Stokes waves in the water, the propagation of the electric field in optical fibers, the self-focusing and collapse of Langmuir waves in plasma physics, to the theory of Heisenberg ferromagnets and magnons and the phenomenons involving the high-power ultra-short lasers in matter, or condensed matter theory and dissipative quantum mechanics. To this end we may refer to the following papers: [10], [21].

Due to it's large frequency in natural phenomenons it is normal to study the cases when the equations are perturbed by one or more perturbation terms.

Variational studies of nonlinear Schrödinger equation started with the pioneering work of P. Rabinowitz, who studied the problem:

$$-\Delta u + a(x)u = f(x,u), \quad \text{in } \mathbb{R}^N \ \ (N \geq 3) \tag{1}$$

In this paper we deal with a new class of differential operators, which was introduced by A. Azzollini in [5] and Azzollini *et al.* in [6], with an associated functional framework. Therefore we deal with a problem driven by the following nonhomogeneous operator:

$$\operatorname{div}\left[\phi'(|\nabla u|^2)\nabla u\right],$$

where $\phi \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ has a different behavior when it approaches zero and to infinity.

We point out that this type of behavior corresponds to the capillary surface operator, and we obtain this kind of operator if we take $\phi(\xi) = 2\left(\sqrt{1+\xi}-1\right)$ and we get

$$\operatorname{div}\left[\phi'(|\nabla u|^2)\nabla u\right] = \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right).$$

This type of operators have been intensively studied in the last few years, to this end we refer to the following works: [21], [22], [24].

Furthermore, the mapping $\xi \mapsto \phi(\xi)$ behaves as it follows:

$$\phi(\xi) \simeq \begin{cases} |\xi|^{\frac{p}{2}}, & \text{if } |\xi| \gg 1; \\ |\xi|^{\frac{q}{2}}, & \text{if } |\xi| \ll 1; \end{cases} \tag{2}$$

where $1 < p < q < N$.

To obtain this kind of function we can consider

$$\phi(\xi) = \frac{2}{p}\left[\left(1+\xi^{q/2}\right)^{p/q} - 1\right].$$

Hence our equation is generated by the operator:

$$\operatorname{div}\left[\phi'(|\nabla u|^2)\nabla u\right] = \operatorname{div}\left[(1+|\nabla u|^q)^{(p-q)/q}\,|\nabla u|^{q-2}\nabla u\right].$$

The main goal of this paper is to establish the existence of two different types of solutions. For the first one we apply a version of Mountain-Pass Theorem of A. Ambrosetti and P. Rabinowitz and we show that the weak limit of the Palais-Smale sequence induced by this method is a solution for our problem. The second solution is obtained via Ekeland's Variational Principle.

In the final of the paper we argue the necessary conditions for which our solutions are different, because they realize different energy levels.

2. Basic hypotheses

The main goal of our paper is to study the effects of an unbounded potential and a double perturbation on the equation studied initially by Azzollini *et al.* in [5] and [6]. So we consider the problem

$$-\operatorname{div}\left[\phi'(|\nabla u|^2)\nabla u\right] + h(x)|u|^{\alpha-2}u = j(x)|u|^{r-2}u + k(x)|u|^{p^*-2}u + \mu f(x) \text{ in } \mathbb{R}^N. \tag{3}$$

In our problem α, r, p, q, p^* are real numbers which satisfy the following assumptions:

$$\begin{cases} 1 < p < q < N \\ 1 < \alpha < p^* \frac{q'}{p'} \\ \max\{\alpha, q\} < r < p^* := \frac{Np}{N-p}, \end{cases} \tag{4}$$

Remark 2.1. We note that the exponents p and p' satisfy the condition $\frac{1}{p} + \frac{1}{p'} = 1$, therefore they are conjugate exponents. The same relation holds also for q and q'.

Let us now impose some conditions on our potentials h, j, k.

The potential h is a function which possesses strong singularities. It's main properties are:

(h_1) $h \in L^{\infty}_{loc}(\mathbb{R}^N \setminus \{0\})$ and for every $x \in \mathbb{R}^N$ we have that $h(x) > h_0$, where h_0 is a

strictly positive constant.
(h_2) $\lim_{x\to 0} h(x) = \lim_{|x|\to\infty} h(x) = +\infty$.

Remark 2.2. As an example of this type of potential we may consider the functions:

$$h(x) = \frac{e^{|x|}}{|x|}, \text{ for } x \in \mathbb{R}^N \setminus \{0\},$$

or

$$h(x) = \frac{|x|^{a_1} + |x|^{a_2}}{|x|^b}, \text{ for } x \in \mathbb{R}^N \setminus \{0\} \text{ and } 1 < a_2 < b < a_1.$$

For more details about singular points involved in nonlinear elliptic equations we refer to [13].

The function $j : \mathbb{R}^N \to \mathbb{R}$ satisfy the following conditions:
(j_1) $j \in L^\infty(\mathbb{R}^N)$;
(j_2) there exists a constant $j_0 > 0$ such that $j(x) > j_0$ for all $x \in \mathbb{R}^N$.

The potential associated to the critical nonlinear term verify the following conditions:
(k_1) $k : \mathbb{R}^N \to \mathbb{R}$, $k \in L^\infty(\mathbb{R}^N)$;
(k_2) $k(x) > 0$ for every $x \in \mathbb{R}^N$, $k(0) = \|k\|_\infty$ and $k(x) = k(0)$ as $x \to \infty$.

At this point we state that our nonhomogeneous differential operator is driven by the potential function $\phi \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ which fulfill the following assumptions:
(ϕ_1) $\phi(0) = 0$;
(ϕ_2) There exists a constant $C_1 > 0$ such that for every $\xi \geq 1$, we have

$$\phi(\xi) \geq C_1 \xi^{\frac{p}{2}}$$

and for every $0 \leq \xi \leq 1$ we have

$$\phi(\xi) \geq C_1 \xi^{\frac{q}{2}}.$$

(ϕ_3) There exists a constant $C_2 > 0$, such that for every $\xi \geq 1$ we have

$$\phi(\xi) \leq C_2 \xi^{\frac{p}{2}}$$

and for every $0 \leq \xi \leq 1$ we have

$$\phi(\xi) \leq C_2 \xi^{\frac{q}{2}}.$$

(ϕ_4) There exists $0 < s < 1$ such that

$$\phi'(\xi)\xi \leq \frac{rs}{2}\phi(\xi), \text{ for all } \xi \geq 0.$$

(ϕ_5) The function which drive ξ into $\phi(\xi^2)$ is strictly convex.

We remark that from assumptions (ϕ_4) and (ϕ_5) we obtain that:

$$\phi(\xi) < 2\phi'(\xi)\xi \leq rs\phi(\xi), \text{ for all } t > 0,$$

hence $rs > 1$.

Due to the fact that ϕ' is allowed to approach zero, the problem (3) is degenerate and no ellipticity condition is assured.

We remark that differential equations with degenerate character were intensively studied over the years due to the pioneering paper of M. K. V. Murthy and G. Stampacchia in [17]. For more details about degenerate equations we refer to [11] and [14].

We proceed now to set some properties for the second perturbation term of our problem:
(f_1) $\mu > 0$, $\mu \in (0, \mu_0)$, where $\mu_0 \in \mathbb{R}_+^*$ is a sufficiently small parameter;
(f_2) $f : \mathbb{R}^N \to \mathbb{R}$ and $f \in \left(L^{p^*}(\mathbb{R}^N)\right)' = L^\gamma(\mathbb{R}^N)$, where

$$\gamma = (p^*)' := \frac{p^*}{p^* - 1} = \frac{Np}{N(p-1)+p}.$$

We remark that the effects of a perturbation is also studied in [26] but under the assumption that the perturbation function is strictly positive and the growth of the nonlinear term in the right-hand side of the equation is assumed to be subcritical.

We will describe in what follows the functional framework associated to our problem.

For the ease of notation we set $\|\cdot\|_z$ the Lebesgue norm for all $1 \le z \le \infty$ and $C_c^\infty(\mathbb{R}^N)$ as the space of all C^∞ functions with a compact support.

Definition 2.3. The function space $L^p(\mathbb{R}^N) + L^q(\mathbb{R}^N)$ is defined as the completion of $C_c^\infty(\mathbb{R}^N)$ in the norm

$$\|u\|_{L^p+L^q} := \inf\left\{\|v\|_p + \|w\|_q;\ v \in L^p(\mathbb{R}^N),\ w \in L^q(\mathbb{R}^N),\ u = v + w\right\}.$$

Remark 2.4. M. Badiale, L. Pisani and S. Rolando showed in [7] that the space $L^p(\mathbb{R}^N) + L^q(\mathbb{R}^N)$ has the structure of an Orlicz space.

From now on, for the ease of notation we set:

$$\|u\|_{p,q} = \|u\|_{L^p+L^q}.$$

For more details about Orlicz-Sobolev functional frameworks used in the study of quasilinear equations we may refer to: M. Al-Hawmi, A. Benkirane, H. Hjiaj, A. Touzani [2], M. Avci [3], M. Mihăilescu, V. Rădulescu, D. Repovš [16], V. Rădulescu, D. Repovš [20], I. Stăncuţ, I. Stîrcu [23].

To state our problem in an appropriate functional framework we will use the space defined by N. Chorfi and V. Rădulescu in [10], i.e. :

$$\mathcal{X} := \overline{C_c^\infty(\mathbb{R}^N)}^{\|\cdot\|},$$

where

$$\|u\| := \|\nabla u\|_{p,q} + \left(\int_{\mathbb{R}^N} h(x)|u|^\alpha dx\right)^{\frac{1}{\alpha}}.$$

Remark 2.5 ([10]). The following continuous embedding occurs:

$$\mathcal{X} \hookrightarrow W$$

where W is the reflexive Banach space defined in [6] as the completion of $C_c^\infty(\mathbb{R}^N)$ in the following norm $\|u\| = \|\nabla u\|_{p,q} + \|u\|_\alpha$.

Remark 2.6 ([6]). A crucial property is the fact that the following continuous embedding results take place:

$$W \hookrightarrow L^{p^*}(\mathbb{R}^N)$$

and by interpolation we may say that:

$$W \hookrightarrow L^z(\mathbb{R}^N),$$

for all $z \in [\alpha, p^*]$.

Taking account of Remark 2.6 we can state that there exist some constants $C_z > 0$ such that

$$C_z\|u\| \geq \|u\|_z, \qquad \text{for all } u \in L^z(\mathbb{R}^N) \text{ and } z \in [\alpha, p^*].$$

3. Existence of solutions

Definition 3.1. We say that $u \in \mathcal{X} \setminus \{0\}$ is a weak solution of the problem (3) if

$$\int_{\mathbb{R}^N} \phi'(|\nabla u|^2)\nabla u \nabla\varphi + h(x)|u|^{\alpha-2}u\varphi dx = \int_{\mathbb{R}^N} j(x)|u|^{r-2}u\varphi + k(x)|u|^{p^*-2}u\varphi dx + \mu\int_{\mathbb{R}^N} f(x)\varphi dx,$$

for all $\varphi \in \mathcal{X}$.

Taking account of Definition 3.1, we get that the energy functional corresponding to our problem is $\mathcal{F} : \mathcal{X} \to \mathbb{R}$, such that:

$$\begin{aligned}\mathcal{F}(u) \quad := \quad & \frac{1}{2}\int_{\mathbb{R}^N} \phi(|\nabla u|^2)dx + \frac{1}{\alpha}\int_{\mathbb{R}^N} h(x)|u|^{\alpha}dx - \\ - \quad & \frac{1}{r}\int_{\mathbb{R}^N} j(x)|u|^r dx - \frac{1}{p^*}\int_{\mathbb{R}^N} k(x)|u|^{p^*}dx - \mu\int_{\mathbb{R}^N} f(x)udx.\end{aligned}$$

Using the properties described in [6] and [10], we can say that $\mathcal{F}$ is well defined and $\mathcal{F} \in C^1(X, \mathbb{R})$. In order to reveal the critical points of $\mathcal{F}$ we state that for all $u, \varphi \in \mathcal{X}$, the Gâteaux directional derivative of $\mathcal{F}$ is:

$$\begin{aligned}\mathcal{F}'(u)(\varphi) \quad = \quad & \int_{\mathbb{R}^N} \left[\phi'(|\nabla u|^2)\nabla u\nabla\varphi + h(x)|u|^{\alpha-2}u\varphi\right] dx - \\ - \quad & \int_{\mathbb{R}^N} \left[j(x)|u|^{r-2}u\varphi + k(x)|u|^{p^*-2}u\varphi\right] dx - \mu\int_{\mathbb{R}^N} f(x)\varphi dx.\end{aligned}$$

We can now enunciate our results.

Theorem 3.2. Assuming that the following hypotheses hold:(4), $(\phi_1) - (\phi_5)$, (h_1), (h_2), (j_1), (j_2), (k_1), (k_2), (f_1) and (f_2), then the problem (3) admits a mountain pass type solution.

We remark that unboundedness of the domain is handled in [6] by reducing the study to radially symmetric test functions. Due to the existence of the potentials h, j, k we can not use the radially symmetric framework.

We recall that similar results were obtained by N. Chorfi and V. Rădulescu in [10] by using in the right-hand side of the equation a function which is not a pure power nonlinearity, but it satisfies appropriate technical conditions. We also state that the behavior of a critical perturbation term is studied by M. Cencelj, D. Repovš, Ž. Virk in [9].

In this paper we extend the results obtained by A. Azzollini, P. d'Avenia, A. Pomponio in [6] and N. Chorfi and V. Rădulescu [10], by studying the problem in the critical case when the nonlinear term behaves like $G(x,t) = k(x)|t|^{p^*-2}t$ which includes the critical Sobolev exponent.

Furthermore, by the fact that the second perturbation term $\mu f(x) \neq 0$ for any $x \in \mathbb{R}^N$, we get that 0 is no longer a trivial solution, and it allows us to do a different qualitative analysis on our problem (3). These facts leads us to the second result developed in this paper, which is the fact that we can obtain a new type of solution by applying Ekeland's Variational Principle.

By the fact that $f : \mathbb{R}^N \to \mathbb{R}$ is general function and we did not assume that $f > 0$ we can not say that our solutions are strictly positive or not.

To find the second type of solution we enunciate the following theorem.

Theorem 3.3. If the following assumptions: (4), $(\phi_1) - (\phi_5)$, (h_1), (h_2), (j_1), (j_2), (k_1), (k_2), (f_1) and (f_2) hold, then there exists $\mu_1 > 0$, and $R = R(\mu_1) > 0$ such that for all $\mu \in (0, \mu_1)$, our energy functional $\mathcal{F}$ admits the existence of a Palais-Smale sequence for the level

$$c_2 := \inf_{u \in \overline{B_R}} \mathcal{F}(u),$$

where $c_2 = c_2(R)$ and $\overline{B_R} := \{u \in \mathcal{X} : \ \|u\| \leq R\}$.

Furthermore, we obtain the existence of some $u_2 \in \mathcal{X}$, such that u_2 is a weak solution for the problem (3).

We proceed now to prove our results.

4. Proof of Theorem 3.2

The main tool in the proof of our result will be the following version of mountain pass lemma of A. Abrosetti and P. Rabinowitz from [4] (we also studied ideas developed by H. Brézis and L. Nirenberg in [8]). For more details about mountain pass critical point techniques we refer to [18] and [19].

Theorem 4.1. Let $\mathcal{X}$ be a real Banach space and assume that $J : \mathcal{X} \to \mathbb{R}$ is a C^1-functional that satisfies the following geometric hypotheses:
(i) $J(0) = 0$;
(ii) there exist positive numbers λ and ρ such that $J(u) \geq \lambda$ for all $u \in \mathcal{X}$ with $\|u\| = \rho$;
(iii) there exists $e \in \mathcal{X}$ with $\|e\| > \rho$ such that $J(e) < 0$.

Set

$$\mathcal{P} := \{p \in C([0,1]; \mathcal{X}); \ p(0) = 0, \ p(1) = e\}$$

and

$$c := \inf_{p \in \mathcal{P}} \sup_{t \in [0,1]} J(p(t)).$$

Then there exists a sequence $(u_n) \subset \mathcal{X}$ such that

$$\lim_{n \to \infty} J(u_n) = c \quad \text{and} \quad \lim_{n \to \infty} \|J'(u_n)\|_{\mathcal{X}^*} = 0.$$

Moreover, if J satisfies the Palais-Smale condition at the level c, then c is a critical value of J.

Firstly we have that $\mathcal{F}(0) = 0$. We point out in what follows that there exists a ,,mountain" near the origin.

Lemma 4.2. If hypotheses of Theorem 3.2 hold, then there exist two strictly positive constants λ and ρ such that for all $u \in \mathcal{X}$ with $\|u\| = \rho$

$$\mathcal{F}(u) \geq \lambda > 0.$$

Proof. Due to relation (2) we may say that

$$\frac{1}{2}\int_{\mathbb{R}^N}\phi(|\nabla u|^2)dx \simeq \frac{1}{2}\int_{[|\nabla u|\leq 1]}|\nabla u|^q dx + \frac{1}{2}\int_{[|\nabla u|>1]}|\nabla u|^p dx.$$

Using hypotheses (ϕ_1) and (ϕ_2), for fixed $\rho \in (0,1)$ we obtain that:

$$\begin{aligned}\mathcal{F}(u) \geq\ & \frac{C_1}{2}\int_{[|\nabla u|\leq 1]}|\nabla u|^q dx + \frac{C_1}{2}\int_{[|\nabla u|>1]}|\nabla u|^p dx + \frac{1}{\alpha}\int_{\mathbb{R}^N}h(x)|u|^\alpha dx - \\ & -\frac{1}{r}\int_{\mathbb{R}^N}j(x)|u|^r dx - \frac{1}{p^*}\int_{\mathbb{R}^N}k(x)|u|^{p^*}dx - \mu\int_{\mathbb{R}^N}f(x)udx \\ \geq\ & \frac{C_1}{2}\max\left\{\int_{[|\nabla u|\leq 1]}|\nabla u|^q dx, \int_{[|\nabla u|>1]}|\nabla u|^p dx\right\} + \frac{1}{\alpha}\int_{\mathbb{R}^N}h(x)|u|^\alpha dx - \\ & -\frac{1}{r}\int_{\mathbb{R}^N}j(x)|u|^r dx - \frac{1}{p^*}\int_{\mathbb{R}^N}k(x)|u|^{p^*}dx - \mu\int_{\mathbb{R}^N}f(x)udx \\ \geq\ & \frac{C_1}{2}\|\nabla u\|^q_{p,q} + \frac{1}{\alpha}\int_{\mathbb{R}^N}h(x)|u|^\alpha dx - \frac{1}{r}\int_{\mathbb{R}^N}j(x)|u|^r dx - \frac{1}{p^*}\int_{\mathbb{R}^N}k(x)|u|^{p^*}dx \\ & - \mu\int_{\mathbb{R}^N}f(x)udx \\ \geq\ & C_\alpha\|u\|^{\max\{q,\alpha\}} - \frac{1}{r}\int_{\mathbb{R}^N}j(x)|u|^r dx - \frac{1}{p^*}\int_{\mathbb{R}^N}k(x)|u|^{p^*}dx - \mu\int_{\mathbb{R}^N}f(x)udx,\end{aligned}$$

where C_α represents the minimum quantity between $\frac{C_1}{2}$ and $\frac{1}{\alpha}$.

Moreover, by the behavior of the potentials $j(x)$ and $k(x)$ and taking account of the continuous embeddings $\mathcal{X} \hookrightarrow L^r(\mathbb{R}^N)$ and $\mathcal{X} \hookrightarrow L^{p^*}(\mathbb{R}^N)$ one have that there exists a constant $C_{r,p^*} > 0$ such that

$$\mathcal{F}(u) \geq C_\alpha\|u\|^{\max\{q,\alpha\}} - C_{r,p^*}(\|u\|^r + \|u\|^{p^*}) - \mu\int_{\mathbb{R}^N}f(x)udx.$$

In order to deal with the second perturbation term, we remind that $\mathcal{X} \subset L^{p^*}(\mathbb{R}^N)$ and by Hölder's inequality we have that:

$$\mu\int_{\mathbb{R}^N}f(x)udx \leq \mu\|f\|_\gamma \cdot \|u\|_{p^*}. \tag{5}$$

Using Young's inequality we obtain that:

$$\mu\int_{\mathbb{R}^N}f(x)udx \leq \mu\left(\varepsilon\|u\|^{p^*}_{p^*} + C_\varepsilon\|f\|^\gamma_\gamma\right), \tag{6}$$

where ε and $C_\varepsilon = \varepsilon^{\frac{-1}{p^*-1}}$ are some positive numbers.

Therefore, we have that:

$$\mathcal{F}(u) \geq C_\alpha\|u\|^{\max\{q,\alpha\}} - C_{r,p^*}\|u\|^r - (C_{r,p^*} + \mu\varepsilon C^{p^*}_{p^*})\|u\|^{p^*} - \mu C_\varepsilon\|f\|^\gamma_\gamma.$$

In conclusion, taking $\rho \in (0,1)$ and $\mu \in (0,\mu_0)$ small enough we find a constant $\lambda > 0$ such that

$$\mathcal{F}(u) \geq \lambda > 0.$$

□

Lemma 4.3. Suppose that assumptions of Theorem 3.2 hold true. Then we can find $\vartheta \in \mathcal{X}$ with $\|\vartheta\| > \rho$ (with ρ given by Lemma 4.2) such that

$$\mathcal{F}(\vartheta) < 0.$$

Proof. The presence of the critical nonlinear term $k(x)|u|^{p^*-2}u$ in the right-hand side of the equation make the existence of a ,,valley" over the chain of mountains hard to prove. To this end we will use the hypotheses (k_1) and (k_2) which describe the decay of the potential k near its maximum point in relationship with the critical nonlinear perturbation.

Therefore we choose $\xi \in C_c^\infty(\mathbb{R}^N) \setminus \{0\}$ and $t > 0$. Then one have

$$\begin{aligned}
\mathcal{F}(t\xi) &= \frac{1}{2}\int_{\mathbb{R}^N} \phi(|\nabla t\xi|^2)dx + \frac{1}{\alpha}\int_{\mathbb{R}^N} h(x)|t\xi|^\alpha dx - \frac{1}{r}\int_{\mathbb{R}^N} j(x)|t\xi|^r dx \\
&\quad - \frac{1}{p^*}\int_{\mathbb{R}^N} k(x)|t\xi|^{p^*}dx - \mu\int_{\mathbb{R}^N} f(x)t\xi dx \\
&\le \frac{C_2}{2}\int_{[|\nabla t\xi|\le 1]} |\nabla t\xi|^q dx + \frac{C_2}{2}\int_{[|\nabla t\xi|>1]} |\nabla t\xi|^p dx + \frac{1}{\alpha}\int_{\mathbb{R}^N} h(x)|t\xi|^\alpha dx - \\
&\quad - \frac{1}{r}\int_{\mathbb{R}^N} j(x)|t\xi|^r dx - \frac{1}{p^*}\int_{\mathbb{R}^N} k(x)|t\xi|^{p^*}dx - t\mu\int_{\mathbb{R}^N} f(x)\xi dx \\
&\le \frac{C_2}{2}\left(t^q\int_{\mathbb{R}^N} |\nabla\xi|^q dx + t^p\int_{\mathbb{R}^N} |\nabla\xi|^p dx\right) + \frac{t^\alpha}{\alpha}\int_{\mathbb{R}^N} h(x)|\xi|^\alpha dx - \\
&\quad - \frac{t^r}{r}\int_{\mathbb{R}^N} j(x)|\xi|^r dx - \frac{t^{p^*}}{p^*}\int_{\mathbb{R}^N} k(x)|\xi|^{p^*}dx - t\mu\int_{\mathbb{R}^N} f(x)\xi dx.
\end{aligned}$$

So for $t > 0$ chosen sufficiently large the previous inequality imply the fact that

$$\lim_{t\to\infty} \mathcal{F}(t\xi) = -\infty,$$

and by the fact that ξ is fixed we obtain the existence of a t_0 such that

$$\mathcal{F}(t_0\xi) < 0.$$

□

We pointed out that our energy functional satisfies the geometrical conditions of the mountain pass lemma. We are able now to see if our setting is non-degenerate, i.e., the associated min-max value is positive.

Lemma 4.4. Suppose that Lemma 4.2 and Lemma 4.3 hold true. We set

$$\Omega := \{g \in C([0,1], \mathcal{X});\ g(0) = 0, g(1) = \vartheta\},$$

where ϑ is given by Lemma 4.3 and

$$c_1 := \inf_{g\in\Omega} \max_{t\in[0,1]} \mathcal{F}(g(t)).$$

Then $c_1 > 0$.

Proof. It is trivial to say that $c_1 \ge 0$ because

$$c_1 \ge \inf_{g\in\Omega} \max_{t\in[0,1]} \mathcal{F}(g(t))$$

and $g(0) = 0$ imply the fact that $\mathcal{F}(g(0)) = \mathcal{F}(0) = 0$ and $g(1) = \vartheta$ imply the fact that $\mathcal{F}(g(1)) = \mathcal{F}(\vartheta) < 0$.

Arguing by contradiction we suppose that $c_1 = 0$. So one have that

$$0 = \inf_{g\in\Omega} \max_{t\in[0,1]} \mathcal{F}(g(t)). \tag{7}$$

Now using relation (7) we obtain that $\max_{t\in[0,1]} \mathcal{F}(g(t)) \geq 0$, for all $g \in \Omega$ and the fact that for any $\varepsilon > 0$, there exists $m \in \Omega$ such that

$$\max_{t\in[0,1]} \mathcal{F}(m(t)) < \varepsilon.$$

Using λ given by Lemma 4.2, we fix $0 < \varepsilon < \lambda$. Then we have $m(0) = 0$ and $m(1) = \vartheta$, which yields to

$$\|m(0)\| = 0 \text{ and } \|m(1)\| = \|\vartheta\| > \rho.$$

By the fact that the function $t \mapsto \|m(t)\|$ is continuous we point out that there exists $t_\varepsilon \in [0,1]$ such that

$$\|m(t_\varepsilon)\| = \rho$$

and we get to the fact that

$$\|\mathcal{F}(m(t_\varepsilon))\| = \lambda > \varepsilon$$

which is a contradiction, and so we conclude that

$$c_1 > 0.$$

□

Now, by means of Theorem 4.1 we find a Palais-Smale sequence $(u_n) \subset \mathcal{X}$ for the level c_1 such that

$$\lim_{n\to\infty} \mathcal{F}(u_n) = c_1 + o(1) \text{ and } \lim_{n\to\infty} \|\mathcal{F}'(u_n)\|_{\mathcal{X}^*} = 0. \tag{8}$$

Proposition 4.5. If the assumptions (4), (h_1), (h_2), (j_1), (j_2), (k_1), (k_2), $(\phi_1)-(\phi_5)$, are fulfilled the functional $\mathcal{F}$ satisfies a Palais-Smale type condition.

Proof. As we are on an unbounded domain, the lack of compactness stops us to prove that the Palais-Smale sequence induced by the min-max value c_1 (shortly $(PS)_{c_1}$) converges strongly to some u_1; however we may show that u_1 is the weak limit of the $(PS)_{c_1}$ sequence.

To this end we show that (u_n) is bounded in $\mathcal{X}$. Using relation (8) we find that:

$$\begin{aligned}
&c_1 + O(1) + o(\|u_n\|) = \mathcal{F}(u_n) - \frac{1}{r}\mathcal{F}'(u_n)(u_n) \\
&= \frac{1}{2}\int_{\mathbb{R}^N} \phi(|\nabla u_n|^2)dx + \frac{1}{\alpha}\int_{\mathbb{R}^N} h(x)|u_n|^\alpha dx - \frac{1}{r}\int_{\mathbb{R}^N} j(x)|u_n|^r dx \\
&- \frac{1}{p^*}\int_{\mathbb{R}^N} k(x)|u_n|^{p^*}dx - \mu\int_{\mathbb{R}^N} f(x)u_n dx - \frac{1}{r}\int_{\mathbb{R}^N} \phi'(|\nabla u_n|^2)|\nabla u_n|^2 dx - \\
&- \frac{1}{r}\int_{\mathbb{R}^N} h(x)|u_n|^\alpha dx + \frac{1}{r}\int_{\mathbb{R}^N} j(x)|u_n|^r dx + \frac{1}{r}\int_{\mathbb{R}^N} k(x)|u_n|^{p^*}dx + \frac{\mu}{r}\int_{\mathbb{R}^N} f(x)u_n dx \\
&= \int_{\mathbb{R}^N}\left[\frac{1}{2}\phi(|\nabla u_n|^2) - \frac{1}{r}\phi'(|\nabla u_n|^2)|\nabla u_n|^2\right]dx + \left(\frac{1}{\alpha} - \frac{1}{r}\right)\int_{\mathbb{R}^N} h(x)|u_n|^\alpha dx + \\
&+ \left(\frac{1}{r} - \frac{1}{p^*}\right)\int_{\mathbb{R}^N} k(x)|u_n|^{p^*}dx + \left(\frac{\mu}{r} - \mu\right)\int_{\mathbb{R}^N} f(x)u_n dx.
\end{aligned}$$

Since hypothesis (ϕ_4) implies that for all $\xi \geq 0$

$$\phi'(\xi)\xi \leq \frac{rs}{2}\phi(\xi)$$

we obtain

$$\frac{1}{2}\phi(\xi) - \frac{1}{r}\phi'(\xi)\xi \geq \frac{1-rs}{2}\phi(\xi),$$

where $rs \in (0,1)$.

On the other hand due to the fact that $1 < \alpha < r < p^*$ one have that:

$$\begin{aligned} c_1 + O(1) + o(\|u_n\|) &\geq \frac{1-s}{2}\int_{\mathbb{R}^N}\phi(|\nabla u_n|^2)dx + \left(\frac{1}{\alpha} - \frac{1}{r}\right)\int_{\mathbb{R}^N} h(x)|u_n|^\alpha dx + \\ &+ \left(\frac{1}{r} - \frac{1}{p^*}\right)\int_{\mathbb{R}^N} k(x)|u_n|^{p^*}dx + \left(\frac{\mu}{r} - \mu\right)\int_{\mathbb{R}^N} f(x)u_n dx \\ &\geq \eta\left[\min\{\|\nabla u_n\|_{p,q}^q, \|\nabla u_n\|_{p,q}^p\} + \int_{\mathbb{R}^N} h(x)|u_n|^\alpha dx\right] + \\ &+ \left(\frac{\mu}{r} - \mu\right)\int_{\mathbb{R}^N} f(x)u_n dx, \end{aligned}$$

where η is a strictly positive constant.

To prove that (u_n) is bounded in $\mathcal{X}$ we will use some ideas from [15].

Passing eventually to a subsequence we argue by contradiction and say that $\|u_n\| \to \infty$. Hence using relation (5) and the fact that $\mathcal{X}$ is continuously embedded in $L^{p^*}(\mathbb{R}^N)$ we obtain that:

$$c_1 + O(1) + \|u_n\| \geq \eta\|u_n\|^{\min\{p,q,\alpha\}} - \left(\mu - \frac{\mu}{r}\right) C_{p^*}\|u_n\|\|f\|_\gamma.$$

Dividing by $\|u_n\|$ and passing to the limit, we obtain a contradiction.

So, using the fact that $\mathcal{X}$ is a closed subset of W and following the same arguments as Azzollini *et. al.* in [6] and with the same arguments as in the proof of Theorem 2.3 in [10] we may find (passing eventually to a subsequence) an element $u_1 \in \mathcal{X}$ such that:

$$u_n \rightharpoonup u_1 \quad \text{in} \quad \mathcal{X}$$

and

$$u_n \to u_1 \quad \text{in} \quad L^\beta_{loc}(\mathbb{R}^N), \text{ with } \beta \in [r, p^*]. \tag{9}$$

□

Lemma 4.6. Let $(u_n) \subset \mathcal{X}$ be a $(PS)_c$ sequence of $\mathcal{F}$, for some $c \in \mathbb{R}$. If $u_n \rightharpoonup u$ in $\mathcal{X}$, then $\mathcal{F}'(u) = 0$, i.e., u solves the problem (3) in weak sense.

Proof. Let us fix $\varphi \in C_c^\infty(\mathbb{R}^N)$ and set $M := \operatorname{supp}(\varphi)$.

We define the following functionals:

$$\Phi(u) := \frac{1}{2}\int_M \phi(|\nabla u|^2)dx + \frac{1}{\alpha}\int_M h(x)|u|^\alpha dx$$

and

$$\Psi(u) := \frac{1}{r}\int_M j(x)|u|^r dx + \frac{1}{p^*}\int_M k(x)|u|^{p^*}dx + \mu\int_M f(x)u dx.$$

Considering $(u_n) \subset \mathcal{X}$ a Palais-Smale sequence such that

$$\mathcal{F}(u_n) \to c \quad \text{and} \quad \|\mathcal{F}'(u_n)\|_{\mathcal{X}^*} \to 0 \quad \text{as } n \to \infty, \tag{10}$$

we obtain

$$\Phi'(u_n)(\varphi) - \Psi'(u_n)(\varphi) \to 0 \quad \text{as } n \to \infty. \tag{11}$$

Now using relation (9) and (10) we may say that:

$$\Psi(u_n) \to \Psi(u) \text{ and } \Psi'(u_n)(\varphi) \to \Psi'(u)(\varphi) \quad \text{as } n \to \infty. \tag{12}$$

Now, using arguments from [6], [9], [10], relation (11) and (12) yield us to the fact that

$$\Phi'(u_n)(\varphi) - \Psi'(u)(\varphi) \to 0 \quad \text{as } n \to \infty. \tag{13}$$

By the convexity of the function ϕ we obtain that the application $u \mapsto \Phi$ is convex. Hence

$$\Phi(u_n) \leq \Phi(u) + \Phi'(u_n)(u_n - u) \quad \text{for all } n \in \mathbb{N}. \tag{14}$$

Taking into account that $u_n \rightharpoonup u$ in $\mathcal{X}$ and relation (14) holds, one have that

$$\limsup_{n\to\infty} \Phi(u_n) \leq \Phi(u).$$

Due to the fact that the nonlinear function Φ is continuous and convex, we have that Φ is lower semicontinuous and that yields

$$\Phi(u) \leq \liminf_{n\to\infty} \Phi(u_n),$$

so we can say that

$$\Phi(u_n) \to \Phi(u) \quad \text{as } n \to \infty.$$

Following the ideas developed by Azzollini *et. al.* in [6] and Chorfi and Rădulescu in [10] we find that

$$\nabla u_n \to \nabla u \quad \text{as } n \to \infty \text{ in } L^p(\mathbb{R}^N) + L^q(\mathbb{R}^N)$$

and

$$\int_{\mathbb{R}^N} h(x)|u_n|^\alpha dx \to \int_{\mathbb{R}^N} h(x)|u|^\alpha dx \quad \text{as } n \to \infty.$$

So far we pointed out that for all $\varphi \in M$ we have that:

$$\int_M \phi'(|\nabla u|^2)\nabla u \nabla\varphi + h(x)|u|^{\alpha-2}u\varphi dx = \int_M j(x)|u|^{r-2}u\varphi dx + \int_M k(x)|u|^{p^*-2}u\varphi dx + \mu \int_M f(x)\varphi dx,$$

relation which we can extend using density arguments for all $\varphi \in \mathcal{X}$. Therefore we can observe that the weak limit of the $(PS)_c$ sequence is a solution of the problem (3). □

Conclusion of Theorem 3.2. In order to say that the problem (3) has at least one nontrivial solution, we combine the results of Lemmas 4.2, 4.3, 4.4, 4.6 and Proposition 4.5 with the results of Theorem 4.1 and we find that there exists $u_1 \in \mathcal{X}$ such that, u_1 verifies the problem (3).

As the second perturbation term $\mu f(x) \neq 0$, for any $x \in \mathbb{R}^N$, and for all $\mu \in (0, \mu_0)$ we can observe that the solution $u_1 \not\equiv 0$.

5. Proof of Theorem 3.3

Let $u \in \mathcal{X}$. We have that:

$$\mathcal{F}(u) = \frac{1}{2}\int_{\mathbb{R}^N} \phi(|\nabla u|^2)dx + \frac{1}{\alpha}\int_{\mathbb{R}^N} h(x)|u|^\alpha dx -$$

$$-\frac{1}{r}\int_{\mathbb{R}^N} j(x)|u|^r dx - \frac{1}{p^*}\int_{\mathbb{R}^N} k(x)|u|^{p^*}dx - \mu\int_{\mathbb{R}^N} f(x)u dx$$

$$\geq \frac{1}{2}\int_{\mathbb{R}^N} \phi(|\nabla u|^2)dx + \frac{1}{\alpha}\int_{\mathbb{R}^N} h(x)|u|^\alpha dx - \frac{\|j\|_\infty}{r}\|u\|_r^r - \frac{C_k}{p^*}\|u\|^{p^*} - \mu\|f\|_\gamma\|u\|_{p^*},$$

where $C_k > 0$ is a constant which depends only of $\|k\|_\infty$ and the continuous embedding of the space $\mathcal{X}$ in $L^{p^*}(\mathbb{R}^N)$.

Using Young's inequality we obtain:

$$\begin{aligned}\mathcal{F}(u) \quad &\geq \quad \frac{1}{2}\int_{\mathbb{R}^N} \phi(|\nabla u|^2)dx + \frac{1}{\alpha}\int_{\mathbb{R}^N} h(x)|u|^\alpha dx - \frac{\|j\|_\infty}{r}\|u\|_r^r \\ &- \quad \frac{C_k}{p^*}\|u\|^{p^*} - \mu\left(\varepsilon\|u\|_{p^*}^{p^*} + C_\varepsilon\|f\|_\gamma^\gamma\right),\end{aligned}$$

where $\varepsilon, C_\varepsilon > 0$ were defined in relation (6).

Now taking account of the assumption (ϕ_2) the continuous embeddings of $\mathcal{X}$ in $L^r(\mathbb{R}^N)$, respectively in $L^{p^*}(\mathbb{R}^N)$, we have that

$$\begin{aligned}\mathcal{F}(u) \quad &\geq \quad \frac{C_1}{2}\int_{[|\nabla u|\leq 1]} |\nabla u|^q dx + \frac{C_1}{2}\int_{[|\nabla u|>1]} |\nabla u|^p dx + \frac{1}{\alpha}\int_{\mathbb{R}^N} h(x)|u|^\alpha dx \\ &- \quad \frac{\|j\|_\infty}{r}C_r^r\|u\|^r - \left(\frac{C_k}{p^*} + \mu\varepsilon C_{p^*}^{p^*}\right)\|u\|^{p^*} - \mu C_\varepsilon\|f\|_\gamma^\gamma.\end{aligned}$$

Now taking C_α the same as in the proof of Theorem 3.2 we get that:

$$\mathcal{F}(u) \geq C_\alpha\|u\|^\theta - \frac{\|j\|_\infty}{r}C_r^r\|u\|^r - \left(\frac{C_k}{p^*} + \mu\varepsilon C_{p^*}^{p^*}\right)\|u\|^{p^*} - \mu C_\varepsilon\|f\|_\gamma^\gamma,$$

where $\|u\|^\theta = \begin{cases} \|u\|^{\max\{\alpha,q\}}, & \text{if } \|u\| < 1 \\ \|u\|^{\min\{\alpha,p\}}, & \text{if } \|u\| > 1. \end{cases}$

Taking use of the relation (4), the above estimation gives us the existence of a $\mu_1 > 0$ such that $R = R(\mu_1) > 0$, $\delta = \delta(\mu_1) > 0$:

$$\mathcal{F}(u) \geq -\mu C_\varepsilon\|f\|_\gamma^\gamma, \text{ for all } u \in \overline{B_R} \text{ and } \mu \in (0, \mu_1)$$

and

$$\mathcal{F}(u) \geq \delta > 0, \text{ for all } u \in \partial B_R \text{ and } \mu \in (0, \mu_1).$$

For instance we can take $\|u\| = R \in (0, 1)$ and:

$$R := \left(\frac{\mu_1 C_\varepsilon\|f\|_\gamma^\gamma - C_\alpha}{C_3}\right)^{\frac{1}{r-\theta}}; \quad C_3 := \max\left\{\frac{\|j\|_\infty}{r}C_r^r, \left(\frac{C_k}{p^*} + \mu_0\varepsilon C_{p^*}^{p^*}\right)\right\};$$

$$\mu_1 \in \left(\frac{C_\alpha}{C_\varepsilon\|f\|_\gamma^\gamma}, \frac{C_3 + C_\alpha}{C_\varepsilon\|f\|_\gamma^\gamma}\right);$$

and

$$\delta = \delta(\mu_1) := \mu_1 C_\varepsilon\|f\|_\gamma^\gamma.$$

In what follows we set

$$c_2 = c_2(R) := \inf_{u\in\overline{B_R}} \mathcal{F}(u) = \inf\{\mathcal{F}(u);\ \|u\| \le R\}.$$

Hence

$$c_2 \le \mathcal{F}(0) = 0.$$

Using the metric

$$\operatorname{dist}(x,y) = \|x-y\| \quad \text{ for any } x,y \in \overline{B_R},$$

the set $\overline{B_R}$ becomes a complete metric space.

Using arguments from [6] Proposition 3.3, and [10] section 3.2, we see that $\mathcal{F}$ is lower semicontinuous and bounded from below on $\overline{B_R}$.

Therefore, using Ekeland's Variational Principle ([12], Theorem 1.1), we obtain that for any $n \in \mathbb{N}$, there exists u_n such that

$$c_2 \le \mathcal{F}(u_n) \le c_2 + \frac{1}{n} \tag{15}$$

$$\mathcal{F}(\varrho) \ge \mathcal{F}(u_n) - \frac{1}{n}\|u_n - \varrho\|, \quad \text{ for all } \varrho \in \overline{B_R}. \tag{16}$$

In what follows we point out that $\|u_n\| < R$ for n sufficiently large.

Let us suppose the contrary, then we can say that $\|u_n\| = R$ for infinitely many n, so up to a subsequence we can presume that $\|u_n\| = R$, for all $n \ge 1$. So, we have that $\mathcal{F}(u_n) \ge \delta > 0$. Using (15) and letting $n \to \infty$, we have that $0 \ge c_2 \ge \delta > 0$ which is a contradiction.

We proceed now to show that $\mathcal{F}'(u_n) \to 0$ in $\mathcal{X}^*$. We set $u \in \mathcal{X}$ and $v_n := u_n + tu$, with $\|u\| = 1$. For $n \in \mathbb{N}$ fixed, we obtain that:

$$\|v_n\| \le \|u_n\| + t\|u\| < R, \quad \text{ for } t > 0 \text{ small enough }.$$

Taking account of relation (16) we have that

$$\mathcal{F}(u_n + tu) \ge \mathcal{F}(u_n) - \frac{t}{n}\|u\|,$$

therefore we obtain

$$\frac{\mathcal{F}(u_n + tu) - \mathcal{F}(u_n)}{t} \ge -\frac{1}{n}\|u\| = -\frac{1}{n}.$$

Letting $t \searrow 0$ we deduce that $\mathcal{F}'(u_n)(u) \ge -\frac{1}{n}$.

Analogous, for $t \nearrow 0$, we obtain that

$$|\mathcal{F}'(u_n)(u)| \le \frac{1}{n}\|u\| \le \frac{1}{n}.$$

So, with the same arguments as in [14] Lemma 4, since u has been chosen arbitrarily, one have that

$$\|\mathcal{F}'(u_n)\|_{\mathcal{X}^*} \le \frac{1}{n} \to 0 \quad \text{ as } n \to \infty.$$

Hence, we have obtained the existence of a $(PS)_{c_2}$ sequence, i.e., $(u_n) \subset \mathcal{X}$ with

$$\mathcal{F}(u_n) \to c_2 \qquad \text{and} \qquad \mathcal{F}'(u_n) \to 0 \text{ in } \mathcal{X}^*. \tag{17}$$

By the fact that $\|u_n\| < R$, with R fixed before, we have that (u_n) (passing eventually to a subsequence) converges weakly to a u_2 in $\mathcal{X}$.

To complete the proof of our theorem it only remains to apply Lemma 4.6 and the conclusion follows. $\square$

6. Final remarks

(1) To prove that u_1 and u_2 provided by Theorems 3.2 and 3.3 are different, we need to assume much more stronger hypotheses on the domain or on the behavior of the potentials j and k.

Let us suppose now that there exists $(u_n) \subset \mathcal{X}$ such that (u_n) is a $(PS)_{c_1}$ sequence and moreover $u_n \to u_1$ in $\mathcal{X}$ and $(v_n) \subset \mathcal{X}$ such that (v_n) is a $(PS)_{c_2}$ sequence such that $v_n \to u_2$ in $\mathcal{X}$.

Arguing by contradiction, we suppose that $u_1 = u_2$. Therefore we have:

$$\mathcal{F}(u_1) = \lim_{n\to\infty} \mathcal{F}(u_n) = c_1 > 0 \geq c_2 = \lim_{n\to\infty} \mathcal{F}(v_n) = \mathcal{F}(u_2),$$

which is a contradiction.

(2) Under stronger hypotheses on h, j, for $k \equiv 0$ and $\phi'(|\nabla u|^2) \equiv 1$, in a much simplified functional framework F. Cîrstea and V. Rădulescu proved in [11] with similar critical point techniques that $u_1 \neq u_2$ without the existence of the strong convergence for any Palais-Smale type sequence.

(3) With similar arguments we can treat degenerate singular problems of type:

$$-\mathrm{div}\left[|x|^{-2a}\nabla u\right] = \frac{h(x)}{|x|^{bp}}|u|^{p-2}u + \mu f(x) \text{ in } \mathbb{R}^N$$

where p denotes the Caffarelli-Kohn-Nirenberg critical exponent associated to a, b and N and $f(x)$ belongs to a suitable weighted Sobolev space. To this end we refer to [1], [9], [14], [15].

(4) For more detailed technical conditions on the functions j and k and a slightly stronger assumption (ϕ_2) one can prove that solution u_1 can be extended to a ground state solution for problem (3) in $\mathcal{X}$ and it is always different of the solution provided by Theorem 3.3, without assuming that any Palais-Smale sequence is strongly convergent in $\mathcal{X}$. More details about this type of solution are given in the following works: [5], [6], [25].

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