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Existence and multiplicity of solutions for anisotropic elliptic equations with variable exponent

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Abstract. In this article we study the following nonlinear anisotropic elliptic equations

$$(P)\begin{cases} -\sum_{i=1}^{N}\partial_{x_i}a_i(x,\partial_{x_i}u)+b(x)|u|^{P_+^+-2}u=\lambda(x)f(x,u)+\mu(x)g(x,u) & in \quad \Omega,\\ u=0 \quad on \quad \partial\Omega. \end{cases}$$

We set up that the problem (P) admits at least two weak solutions under suitable conditions.

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1. Introduction

Let $\Omega\subset\mathbb{R}^N (N\geq 3)$ be a bounded domain with smooth boundary. In this paper we will study the existence and the multiplicity of weak solutions of the anisotropic problem:

$$(P)\begin{cases} -\sum_{i=1}^{N}\partial_{x_i}a_i(x,\partial_{x_i}u)+b(x)|u|^{P_+^+-2}u=\lambda(x)f(x,u)+\mu(x)g(x,u) & \text{in} \quad \Omega,\\ u=0 \quad \text{on} \quad \partial\Omega, \end{cases}$$

where $\lambda\not\equiv 0$ and $\mu\not\equiv 0$, $b\in L^\infty(\Omega)$, $f,g:\Omega\times\mathbb{R}\to\mathbb{R}$, $a_i:\Omega\times\mathbb{R}\to\mathbb{R}$ are Carathéodory functions fulfilling some natural hypotheses. The anisotropic differential operator $\sum_{i=1}^{N}\partial_{x_i}a_i(x,\partial_{x_i}u)$ is a $\overrightarrow{p}(.)$-Laplace type operator, where $\overrightarrow{p}(x)=(p_1(x),p_2(x),...,p_N(x))$ and $P_+^+=\max_{i\in\{1,2,\dots,N\}}\sup_{\Omega}p_i(x)$ for $i=1,...,N$, we assume that p_i is a continuous function on $\overline{\Omega}$. We denote by $a_i(x,\eta)$ the continuous derivative with respect to η of the mapping $A_i:\Omega\times\mathbb{R}\to\mathbb{R}$, $A_i=A_i(x,\eta)$. We make the following assumptions on the mapping A_i:

(A_0) $A_i(x,0)=0$ for a.e. $x\in\Omega$.

(A_1) There exists a positive constant $\overline{c}_i$ such that a_i satisfies the growth condition

$$|a_i(x,\eta)|\leq\overline{c}_i(1+|\eta|^{p_i(x)-1}),$$

for all $x\in\Omega$ and $\eta\in\mathbb{R}$.

(A_2) The inequalities

$$|\eta|^{p_i(x)}\leq a_i(x,\eta)\eta\leq p_i(x)A_i(x,\eta),$$

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for all $x \in \Omega$ and $\eta \in \mathbb{R}$.
(A_3) assume that a_i is strictly monotone, that is,

$$(a_i(x,\eta) - a_i(x,\xi))(\eta - \xi) > 0,$$

for all $x \in \Omega$ and $\eta, \xi \in \mathbb{R}$, with $\eta \neq \xi$.

Examples

1) If we take $a_i(x,\eta) = |\eta|^{p_i(x)-2}\eta$ for all $i \in \{1, ..., N\}$, we have $A_i(x,\eta) = \frac{1}{p_i(x)}|\eta|^{p_i(x)}$ for all $i \in \{1, ..., N\}$. Obviously, (A_0) - (A_3) are verified, and we obtain the $\overrightarrow{p}(x)$ - Laplace operator

$$\triangle_{\overrightarrow{p}(x)}(u) = \sum_{i=1}^{N} \partial_{x_i}(|\partial_{x_i}u|^{p_i(x)-2}\partial_{x_i}u).$$

2) If we take $a_i(x,\eta) = (1+\eta^2)^{\frac{p_i(x)-2}{2}}\eta$ for all $i \in \{1, ..., N\}$, we have $A_i(x,\eta) = \frac{1}{p_i(x)}[(1+|\eta|^2)^{\frac{p_i(x)}{2}} - 1]$ for all $i \in \{1, ..., N\}$, then (A_0) - (A_4) are verified, and we find the anisotropic variable exponent mean curvature operator

$$\sum_{i=1}^{N} \partial_{x_i}(1+|\partial_{x_i}u|^2)^{\frac{p_i(x)-2}{2}}\partial_{x_i}u).$$

And when $p_i(x) = p(x)$ for all $i = 1, 2, 3..., N$, we obtain the pseudo $p(.)-$Laplace operator which is the natural generalization of pseudo $p-$ Laplace operator, as $p > 1$, moreover it is an isotropic operator. As the $p(.)-$Laplace operator isn't homogeneous, then it has more difficulties than $p-$Laplace operator. In order to overcome these difficulties, we link the classical techniques with those came out recently when treating the problems with variable exponents.

This work is a generalization of the article [1] where the authors considered problem

$$(S)\begin{cases} -\sum_{i=1}^{N} \partial_{x_i}a_i(x,\partial_{x_i}u) + b(x)|u|^{P_+^+-2}u = f(x,u) & \text{in} \quad \Omega \\ u = 0 \quad \text{on} \quad \partial\Omega, \end{cases}$$

where $f(x,u) = \lambda(|u|^{q(x)-2}u + |u|^{\gamma(x)-2}u)$, in which the parameter λ is positive and $q(x)$, $\gamma(x)$ are continuous functions on $\overline{\Omega}$, and they obtained the existence of two nontrivial weak solutions. Their arguments are based on the mountain pass theorem and Ekeland's variational principle [7].

Many other authors studied the same problems in a different cases. For example, in [14], the authors considered (S), without $b(x)|u|^{P_+^+-2}u$, where $f = \lambda|u|^{q(x)-2}u$, and proved that problem (S) has a continuous spectrum, however in [13], the authors demonstrate that when λ depends on the variable x, problem (S) has two nontrivial weak solutions, using the mountain-pass theorem of Ambrosetti and Rabinowitz [3] and the Ekeland's variational principle, but in [4], Boureanu proved that problem (S) has a sequence of weak solutions by means of the symmetric mountain-pass theorem.

Given $\Omega \subset \mathbb{R}^N$, we set

$$C_+(\overline{\Omega}) = \{h \in C(\overline{\Omega}) | \min_{x\in\overline{\Omega}} h(x) > 1\}.$$

For any $h \in C_+(\overline{\Omega})$, we define

$$h^+ = \sup_{x\in\overline{\Omega}} h(x) \quad \text{and} \quad h^- = \inf_{x\in\overline{\Omega}} h(x).$$

Let $p \in C_+(\overline{\Omega})$, then $L^{p(x)}(\Omega)$ is called variable exponent Lebesgue space which is defined as follow

$L^{p(x)}(\Omega) = \{u : u \,\text{is a measurable real-valued function such that } \int_\Omega |u(x)|^{p(x)}\,dx < \infty\}$,

endowed with the *Luxemburg norm*

$$|u|_{L^{p(x)}(\Omega)} = |u|_{p(x)} = \inf\{\mu > 0 : \int_\Omega \left|\frac{u(x)}{\mu}\right|^{p(x)} dx \leq 1\}$$

is a reflexive and separable Banach space (see [16]).

We say that p is logarithmic Hölder continuous if

$$|p(x) - p(y)| \leq -\frac{M}{log(|x-y|)} \qquad \forall x,\, y \in \Omega \text{ such that } |x-y| \leq 1/2. \tag{1}$$

The variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ is defined by

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : \nabla u \in [L^{p(x)}(\Omega)]^N\}.$$

For all $u \in W^{1,p(x)}(\Omega)$, we have $\|u\|_{1,p(x)} = |u|_{p(x)} + |\nabla u|_{p(x)}$. If p satisfies (1), the space $W_0^{1,p(x)}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$ under the norm $\|u\|_{1,p(x)}$. For $u \in W_0^{1,p(x)}(\Omega)$, we can define an equivalent norm $\|u\|_{p(x)} = |\nabla u|_{p(x)}$.

Now, we introduce a natural generalization of the function space $W_0^{1,p(x)}(\Omega)$, which will allow us to study the problem (P), which is called anisotropic variable exponent Sobolev space $W_0^{1,\overrightarrow{p}(x)}(\Omega)$. If $\overrightarrow{p} : \overline{\Omega} \to \mathbb{R}^N$; $\overrightarrow{p}(x) = (p_1(x), p_2(x), ..., p_N(x))$, and for each $i \in \{1, 2, ..., N\}$, we have $p_i \in C_+(\overline{\Omega})$, and satisfy (1), the anisotropic variable exponent Sobolev space $W_0^{1,\overrightarrow{p}(x)}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ under the norm

$$\|u\| = \|u\|_{\overrightarrow{p}(.)} = \sum_{i=1}^N |\partial_{x_i} u|_{p_i(.)},$$

and it is a reflexive Banach space [8, 14]. Henceforth, we put $W_0^{1,\overrightarrow{p}(x)}(\Omega) = X$.

In order to study the problem (P) we have to introduce the vectors $\overrightarrow{P}_+, \overrightarrow{P}_- \in \mathbb{R}^N$ which are defined in the following way

$$\overrightarrow{P}_+ = (p_1^+, p_2^+, ..., p_N^+),\ \overrightarrow{P}_- = (p_1^-, p_2^-, ..., p_N^-),$$

and the positive real numbers P_+^+, P_-^+, P_-^- as the following

$$P_+^+ = \max\{p_1^+, ..., p_N^+\},\ P_-^+ = \max\{p_1^-, ..., p_N^-\},\ P_-^- = \min\{p_1^-, ..., p_N^-\}.$$

Throughout this paper, we assume that

$$\sum_{i=1}^N \frac{1}{p_i^-} > 1, \tag{2}$$

Define $P_-^*, P_{-,\infty} \in \mathbb{R}^+$ by

$$P_-^* = \frac{N}{\sum_{i=1}^N \frac{1}{p_i^-} - 1},\ P_{-,\infty} = \max\{P_-^+, P_-^*\}.$$

throughout this paper, we have $P_+^+ < P_{-,\infty} = \max\{P_-^+, P_-^*\} = P_-^*$.

We assume that the Caratheodory functions $f, g : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfy the conditions:

(f_0) $|f(x,t)| \leq c|t|^{\alpha(x)-1}$ a.e. $x \in \Omega$ and for all $t \in \mathbb{R}$, where $c > 0$ is a constant, $\alpha \in C_+(\Omega)$ such that $\alpha^+ = \sup_{x\in\overline{\Omega}} \alpha(x) < P_-^- < P_+^+ \leq s(x) < P_{-,\infty}$, $\forall x \in \Omega$ and $\lambda \in L^{\frac{s(x)}{s(x)-\alpha(x)}}(\Omega)$, with $s(x) \in C_+(\overline{\Omega})$ and $\alpha(x) + 1 \leq s(x) < P_{-,\infty}$, $\forall x \in \Omega$.

(f_1) There exists $\delta > 0$ such that for a.e. $x \in \Omega$ we have $F(x,t) \geq h_0(x)t^{\alpha_0}$ when $0 < t \leq \delta$, $0 < \alpha_0 < P_-^-, h_0(x) \geq 0\ \forall x \in \Omega, \not\equiv 0$, and $h_0 \in C(\Omega, \mathbb{R})$, with $F(x,t) = \int_0^t f(x,s)\,ds$.

(g_0) $|g(x,t)| < d|t|^{\gamma(x)-1}$, a.e. $x \in \Omega$ and for all $t \in \mathbb{R}$ where $d > 0$ is a constant, and $\gamma \in C_+(\overline{\Omega})$ satisfying $P_+^+ < \gamma^- < \gamma^+ < P_{-,\infty}$, and $\mu \in L^{\frac{r(x)}{r(x)-\gamma(x)}}(\Omega)$, with $r(x) \in C_+(\overline{\Omega})$, and $\gamma(x) + 1 \leq r(x) < P_{-,\infty}\ \forall x \in \Omega$.

(g_1) There exist two constants $\theta > P_+^+$, and $M > 0$ such that $0 < \theta G(x,t) \leq tg(x,t)$, a.e. $x \in \Omega$, and for all $|t| \geq M$, with $G(x,t) = \int_0^t g(x,s)\,ds$.

And assume that

(B) $b \in L^\infty(\Omega)$ and there exists $b_0 > 0$ such that $b(x) \geq b_0$ for all $x \in \Omega$.

The main result of this paper is as follows.

Theorem 1.1. *Suppose f and g satisfy the hypotheses $(f_0) - (f_1)$, $(g_0) - (g_1)$, and assume (B), $(A_0) - (A_3)$ and $\lambda(x), \mu(x) > 0$ a.e. $x \in \Omega$. In addition assume that $p_i(.)$ satisfies (1) for each $i \in \{1, ..., N\}$ and $\overrightarrow{p}(.)$ satisfies (2). Then there exists $\lambda^* > 0$ such that for any function λ which satisfy $|\lambda|_{\frac{s(x)}{s(x)-\alpha(x)}} \in (0, \lambda^*)$, (P) has two nontrivial weak solutions.*

Remark 1.1. For $f(x,u) = |u|^{\alpha(x)-2}u$, $g(x,u) = |u|^{\gamma(x)-2}u$, $\lambda(x) = \mu(x) = \mu = \lambda \in \mathbb{R}$ for all $x \in \Omega$, *with* $\alpha(x) < P_-^- < P_+^+ < P_{-,\infty}$, $\forall x \in \Omega$, and $P_+^+ < \gamma^- < \gamma^+ < P_{-,\infty}$, we obtain the result in the article [1].

This article contains two sections. We will begin to present some basic preliminary results and lemmas. In section 2, we will give the proof of our main results.

2. Preliminaries

We recall some important definitions and properties of the Lebesgue and Sobolev spaces with variable exponent $L^{p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$, where Ω is a bounded domain in $\mathbb{R}^N$.

Proposition 2.1. (*see* [6, 11, 10])

(1) *The space $(L^{p(x)}(\Omega), |u|_{p(x)})$ is a separable, uniformly convex Banach space and its dual space is $L^{q(x)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{q(x)} = 1$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{q(x)}(\Omega)$, we have*

$$\left|\int_\Omega uv\,dx\right| \leq \left(\frac{1}{p^-} + \frac{1}{q^-}\right) |u|_{p(x)}|v|_{q(x)} \leq 2|u|_{p(x)}|v|_{q(x)}.$$

(2) *If $p_1(x)$, $p_2(x) \in C_+(\overline{\Omega})$, $p_1(x) \leq p_2(x)$, $\forall x \in \overline{\Omega}$, then $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$ and the embedding is continuous.*

Remark. If $\frac{1}{p(x)}+\frac{1}{q(x)}+\frac{1}{r(x)}=1$, then for any $u\in L^{p(x)}(\Omega)$, $v\in L^{q(x)}(\Omega)$, $w\in L^{r(x)}(\Omega)$, we have

$$\left|\int_\Omega uvw\,dx\right|\le\left(\frac{1}{p^-}+\frac{1}{q^-}+\frac{1}{r^-}\right)|u|_{p(x)}|v|_{q(x)}|w|_{r(x)}\le 3|u|_{p(x)}|v|_{q(x)}|w|_{r(x)}. \qquad (3)$$

Lemma 2.2. (*see*[2]) *Let $q,s\in C_+(\overline{\Omega})$ with $q(x)\le s(x)$ for all $x\in\overline{\Omega}$, and $u\in L^{s(x)}(\Omega)$. Then, $|u|^{q(x)}\in L^{\frac{s(x)}{q(x)}}(\Omega)$ and*

$$\left||u|^{q(x)}\right|_{\frac{s(x)}{q(x)}}\le |u|^{q^+}_{s(x)}+|u|^{q^-}_{s(x)},$$

or there exists a number $\tilde{q}\in[q^-,q^+]$ such that

$$\left||u|^{q(x)}\right|_{\frac{s(x)}{q(x)}}=|u|^{\tilde{q}}_{s(x)}.$$

Proposition 2.3. (*see*[9]) *Denote $\rho_{p(x)}(u)=\int_\Omega|u(x)|^{p(x)}\,dx$. Then for $u\in L^{p(x)}(\Omega)$, $(u_n)\subset L^{p(x)}(\Omega)$ we have*
(1) $|u|_{p(x)}<1(=1;>1)\Leftrightarrow\rho_{p(x)}(u)<1(=1;>1)$,
(2) $|u|_{p(x)}>1\Rightarrow|u|^{p^-}_{p(x)}\le\rho_{p(x)}(u)\le|u|^{p^+}_{p(x)}$,
(3) $|u|_{p(x)}<1\Rightarrow|u|^{p^+}_{p(x)}\le\rho_{p(x)}(u)\le|u|^{p^-}_{p(x)}$,
(4) $|u|_{p(x)}\to 0(\to\infty)\Leftrightarrow\rho_{p(x)}(u)\to 0(\to\infty)$,
(5) $\lim_{n\to\infty}|u_n-u|_{p(x)}=0\Leftrightarrow\lim_{n\to\infty}\rho_{p(x)}(u_n-u)=0$.

We recall now some results which concerning the embedding theorem.

Proposition 2.4. (*see*[14]) *Suppose that $\Omega\subset\mathbb{R}^N(N>3)$ is a bounded domain with smooth boundary and relation (2) is fulfilled.*
(1) *For any $q\in C(\overline{\Omega})$ verifying*

$$1<q(x)<P_{-,\infty}\ \ \forall x\in\overline{\Omega},$$

the embedding

$$W_0^{1,\overrightarrow{p}(x)}(\Omega)\hookrightarrow L^{q(x)}(\Omega)$$

is continuous and compact.
(2) *Assume that $P_-^->N$, then the embedding*

$$W_0^{1,\overrightarrow{p}(x)}(\Omega)\hookrightarrow C(\overline{\Omega})$$

is continuous and compact.

Under the conditions (A_i), $i=0,1,2,3$. We have the proposition below which is useful.

Proposition 2.5. (cf.[12, 4]) *Let*

$$\mathcal{A}_i(u)=\int_\Omega A_i(x,\partial_{x_i}u)\,dx.$$

For $i\in\{1,2,...,N\}$, we have:
- *$\mathcal{A}_i$ is well defined on X,*

- *the functional* $\mathcal{A}_i \in C^1(X, \mathbb{R})$ *and*
$$\langle \mathcal{A}_i'(u), \varphi\rangle = \int_\Omega a_i(x, \partial_{x_i} u)\partial_{x_i}\varphi \, dx,$$
for all $u, \varphi \in X$.
- $\mathcal{A}_i$ *is weakly lower semi-continuous.*
- *Let*
$$\mathcal{A}(u) = \int_\Omega \sum_{i=1}^N A_i(x, \partial_{x_i} u)\, dx,$$
then $\mathcal{A}'$ *is an operator of type* (S_+) (cf. [4]).

Here we focus on the mountain-pass theorem of Ambrosetti and Rabinowitz and Ekeland's variational principle.

Theorem 2.6 (Montain - Pass Theorem)**.** *(*see[3]*) Let* X *be a real Banach space and* $\phi \in C^1(X, \mathbb{R})$ *a functional satisfies:*
(1) ϕ *satisfying the Palais-Smale condition (that is, any sequence* $(\{u_n\})_n \subset X$ *such that* $(\phi(u_n))_n$ *is bounded and* $\phi'(u_n) \to 0$, *admits a convergent subsequence).*
(2) *There exists* $r > 0$ *such that one can find* $e \in X, \|e\| \geq r$ *with*
$$\max(\phi(0), \phi(e)) < \inf_{\|u\|=r} \phi(u) =: \beta.$$
Then ϕ *possesses a critical value* $c \geq \beta$ *given by*
$$c := \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} \phi(\gamma(t)),$$
where $\Gamma = \{\gamma \in C([0, 1], X) \mid \gamma(0) = 0$ *and* $\gamma(1) = e\}$.

Theorem 2.7 (Ekeland's variational principle)**.** (see[7]) *Let* (X, d) *be a complete metric space, and let* $\phi : X \to \mathbb{R} \cup +\infty$ *be a lower semicontinuous functional on* X *that is bounded below and not identically equal to* $+\infty$. *Fix* $\varepsilon > 0$ *and a point* $u \in X$ *such that*
$$\phi(u) \leq \inf_{x\in X} \phi(x) + \varepsilon.$$
Then, for every $\lambda > 0$, *there exists a point* $v \in X$ *such that*
(1) $\phi(v) \leq \phi(u)$,
(2) $d(u, v) \leq \lambda$,
(3) $\phi(w) > \phi(v) - \dfrac{\varepsilon}{\lambda} d(v, w)$, *for all* $w \neq v$.

3. Existence of two solutions

We are interested to prove the existence of weak solutions. Let's prove lemma which can help us to define the weak solution.

Lemma 3.1. *Let* $B(u) = \int_\Omega [\lambda(x)F(x, u) + \mu(x)G(x, u)]\, dx$. *Suppose that* (f_0) *and* (g_0) *are satisfied, then we have*
(1) B *is well defined and* $B \in C^1(X, \mathbb{R})$.

(2) B, B' *are weakly-strongly continuous.*

Proof. (1) the embeddings $X \hookrightarrow L^{s(x)}(\Omega)$ and $X \hookrightarrow L^{s(x)}(\Omega)$ are compact, then there exist two constants $c_1 > 0$ and $d_1 > 0$ such that $|u|_{s(x)} \leq c_1\|u\|$, and $|u|_{r(x)} \leq d_1\|u\|$. Using (f_0) and (g_0), lemma 2.2 and proposition 2.4, we have

$$\begin{aligned}\int_\Omega |\lambda(x)F(x,u) + \mu(x)G(x,u)|\, dx &\leq \int_\Omega |\lambda(x)||F(x,u)|\, dx + \int_\Omega |\mu(x)||G(x,u)|\, dx \\ &\leq 2\frac{c}{\alpha^-}|\lambda(x)|_{\frac{s(x)}{s(x)-\alpha(x)}}|u^{\alpha(x)}|_{\frac{s(x)}{\alpha(x)}} + 2\frac{d}{\gamma^-}|\mu(x)|_{\frac{r(x)}{r(x)-\gamma(x)}}|u^{\gamma(x)}|_{\frac{s(x)}{\gamma(x)}} \\ &\leq 2c_1\frac{c}{\alpha^-}|\lambda(x)|_{\frac{s(x)}{s(x)-\alpha(x)}}\|u\|^{\tilde{\alpha}} + 2d_1\frac{d}{\gamma^-}|\mu(x)|_{\frac{r(x)}{r(x)-\gamma(x)}}\|u\|^{\tilde{\gamma}}.\end{aligned}$$

Finally, B is well defined. We still have to show that $B \in C^1(X,\mathbb{R})$. The Mean-value theorem gives us

$$\begin{aligned}DB(u,\varphi) &= \lim_{t\to 0}\frac{B(u+t\varphi) - B(u)}{t} \\ &= \lim_{t\to 0}\int_\Omega \frac{\lambda(x)F(x,u+t\varphi) - \lambda(x)F(x,u) + \mu(x)G(x,u+t\varphi) - \mu(x)G(x,u)}{t}\, dx \\ &= \lim_{t\to 0}\int_\Omega [\lambda(x)f(x,u+t\theta\varphi) + \mu(x)g(x,u+t\theta\varphi)]\,\varphi(x)\, dx,\end{aligned}$$

where $0 \leq \theta \leq 1$. From (f_0), (g_0), Young's inequality, and the convexity of the function $h(a) = |a|^p$ with $p \geq 1$, for $|t| \leq 1$, we have

$$\begin{aligned}&[\lambda(x)f(x,u+t\theta\varphi) + \mu(x)g(x,u+t\theta\varphi)]\,\varphi(x) \\ &\quad\leq c|\lambda(x)||u+t\theta\varphi|^{\alpha(x)-1}|\varphi(x)| + d|\mu(x)||u+t\theta\varphi|^{\gamma(x)-1}|\varphi(x)|, \\ &\quad\leq \frac{c(s(x)-\alpha(x))}{s(x)}|\lambda(x)|^{\frac{s(x)}{s(x)-\alpha(x)}} + \frac{c(\alpha(x)-1)}{s(x)}[|u+t\theta\varphi|^{\alpha(x)-1}]^{\frac{s(x)}{\alpha(x)-1}} \\ &\qquad + c|\varphi(x)|^{s(x)} + \frac{d(r(x)-\gamma(x))}{r(x)}|\mu(x)|^{\frac{r(x)}{r(x)-\gamma(x)}} \\ &\qquad + \frac{d(\gamma(x)-1)}{r(x)}[|u+t\theta\varphi|^{\gamma(x)-1}]^{\frac{r(x)}{\gamma(x)-1}} + d|\varphi(x)|^{r(x)}, \\ &\quad\leq \frac{c(s(x)-\alpha(x))}{s(x)}|\lambda(x)|^{\frac{s(x)}{s(x)-\alpha(x)}} + \frac{c(\alpha(x)-1)}{s(x)}2^{s(x)-1}[|u|^{s(x)} + |\varphi|^{s(x)}] \\ &\qquad + c|\varphi(x)|^{s(x)} + \frac{d(r(x)-\gamma(x))}{r(x)}|\mu(x)|^{\frac{r(x)}{r(x)-\gamma(x)}} \\ &\qquad + \frac{d(\gamma(x)-1)}{r(x)}2^{r(x)-1}[|u|^{r(x)} + |\varphi|^{r(x)}] + d|\varphi(x)|^{r(x)}.\end{aligned}$$

The last right expression is independent on t and it is in $L^1(\Omega)$, then by the Lebesgue dominated convergence theorem, we have

$$DB(u,\varphi) = \int_\Omega [\lambda(x)f(x,u) + \mu(x)g(x,u)]\,\varphi(x)\, dx. \tag{4}$$

So using the fact that $N_f : L^{s(x)}(\Omega) \to L^{\frac{s(x)}{\alpha(x)-1}}(\Omega)$; $u \mapsto f(x,u)$ and $N_g : L^{r(x)}(\Omega) \to L^{\frac{r(x)}{\gamma(x)-1}}(\Omega)$; $u \mapsto g(x,u)$ are continuous bounded operators. Then, by (f_0), (g_0), and

Proposition 2.1, we obtain

$$\begin{aligned} DB(u,\varphi) &= \int_\Omega [\lambda(x)f(x,u)+\mu(x)g(x,u)]\,\varphi(x)\,dx \\ &\le \int_\Omega c|\lambda(x)||f(x,u)||\varphi(x)|\,dx + \int_\Omega d|\mu(x)||g(x,u)||\varphi(x)|\,dx \\ &\le 3c|\lambda|_{\frac{s(x)}{s(x)-\alpha(x)}} |f(x,u)|_{\frac{s(x)}{\alpha(x)-1}} |\varphi(x)|_{s(x)} \\ &\quad + 3d|\mu|_{\frac{r(x)}{r(x)-\gamma(x)}} |g(x,u)|_{\frac{r(x)}{\gamma(x)-1}} |\varphi(x)|_{r(x)}. \end{aligned}$$

So $DB(u,\varphi)$, as a function of φ, is a continuous linear functional on X, then it is the *Gâteaux* differential of B. Let's prove that it's continuous, then for u, v, $\varphi \in X$, from (4), we get

$$\begin{aligned} |\langle DB(u)-DB(v),\varphi\rangle| &\le 3|\lambda|_{\frac{s(x)}{s(x)-\alpha(x)}} |f(x,u)-f(x,v)|_{\frac{s(x)}{\alpha(x)-1}} |\varphi|_{s(x)} \\ &\quad + 3|\mu|_{\frac{r(x)}{r(x)-\gamma(x)}} |g(x,u)-g(x,v)|_{\frac{r(x)}{\gamma(x)-1}} |\varphi|_{r(x)} \\ &\le K_1|f(x,u)-f(x,v)|_{\frac{s(x)}{\alpha(x)-1}} \|\varphi\| + K_2|g(x,u)-g(x,v)|_{\frac{r(x)}{\gamma(x)-1}} \|\varphi\|, \end{aligned}$$

where $K_1>0$ and $K_2>0$ are constants. Then,

$$\|DB(u)-DB(v)\|_{X^*} \le K_1|f(x,u)-f(x,v)|_{\frac{s(x)}{\alpha(x)-1}} + K_2|g(x,u)-g(x,v)|_{\frac{r(x)}{\gamma(x)-1}}.$$

Thus, $DB(u)$ is continuous, so B is Frèchet differentiable and $B \in C^1(X,\mathbb{R})$ with

$$\langle B^{'}(u),\varphi\rangle = \int_\Omega [\lambda(x)f(x,u)+\mu(x)g(x,u)]\,\varphi(x)\,dx.$$

(2) Suppose by contradiction that there exists a sequence $(u_n) \subset X$ such that $u_n \rightharpoonup u$ and $B(u_n) \nrightarrow B(u)$, then there exists ε_0 and subsequence still denoted (u_n) such that:

$$0 < \varepsilon_0 \le |B(u_n)-B(u)|.$$

For $0<\theta_n<1$, and by finite increment theorem we have

$$0 < \varepsilon_0 \le |\langle B^{'}(u_n+\theta_n(u_n-u)), u_n-u\rangle|.$$

Put $w_n = u_n+\theta_n(u_n-u)$. As $B^{'}(u)(w) = \int_\Omega [\lambda(x)f(x,u)w+\mu(x)g(x,u)w]\,dx$, using (3), proposition 2.4, (f_0) and (g_0) we obtain

$$\begin{aligned} |\langle B^{'}(w_n),(u_n-u)\rangle| &= \int_\Omega |\lambda(x)f(x,w_n)(u_n-u)+\mu(x)g(x,w_n)(u_n-u)|\,dx \\ &\le \int_\Omega \{|\lambda(x)||u_n-u||f(x,w_n)| + |\mu(x)||u_n-u||g(x,w_n)|\}\,dx \\ &\le \int_\Omega c|\lambda(x)||w_n|^{\alpha(x)-1}|u_n-u|\,dx + \int_\Omega d|\mu(x)||w_n|^{\gamma(x)-1}|u_n-u|\,dx \\ &\le 3c|\lambda|_{\frac{s(x)}{s(x)-\alpha(x)}} ||w_n|^{\alpha(x)-1}|_{\frac{s(x)}{\alpha(x)-1}} |u_n-u|_{s(x)} \\ &\quad + 3d|\mu|_{\frac{r(x)}{r(x)-\gamma(x)}} ||w_n|^{\gamma(x)-1}|_{\frac{r(x)}{\gamma(x)-1}} |u_n-u|_{r(x)}. \end{aligned}$$

Since $\lim\limits_{n\to+\infty}|w_n|_{s(x)} \ne \infty$ and $\lim\limits_{n\to+\infty}|w_n|_{r(x)} \ne \infty$, then by the proposition 2.3, we deduce that

$$\lim_{n\to+\infty} ||w_n|^{\alpha(x)-1}|_{\frac{s(x)}{\alpha(x)-1}} \ne \infty \text{ and } \lim_{n\to+\infty} ||w_n|^{\gamma(x)-1}|_{\frac{r(x)}{\gamma(x)-1}} \ne \infty.$$

So, as the embeddings $X \hookrightarrow L^{s(x)}(\Omega)$ and $X \hookrightarrow L^{r(x}(\Omega)$ are compact, then the last expression on the right goes to 0 as $n \to +\infty$. Finally, B is weakly-strongly continuous.

Let's prove that $B^{'}$ is also weakly-strongly continuous. We know that

$$\langle B^{'}(u), v\rangle = \int_\Omega \lambda(x) f(x,u) v\,dx + \int_\Omega \mu(x) g(x,u) v\,dx,$$

where $v \in X$. For $u_n \rightharpoonup u$, then (u_n) is bounded, using relation (3), we have

$$\begin{aligned}|\langle B^{'}(u_n) - B^{'}(u), v\rangle| &\le \int_\Omega |\lambda(x)||(f(x,u_n) - f(x,u))v|\,dx \\ &\quad + \int_\Omega |\mu(x)||(g(x,u_n) - g(x,u))v|\,dx \\ &\le 3|\lambda|_{\frac{s(x)}{s(x)-\alpha(x)}} |f(x,u_n) - f(x,u)|_{\frac{s(x)}{\alpha(x)-1}} |v|_{s(x)} \\ &\quad + 3|\mu|_{\frac{r(x)}{r(x)-\gamma(x)}} |g(x,u_n) - g(x,u)|_{\frac{r(x)}{\gamma(x)-1}} |v|_{r(x)}.\end{aligned}$$

The compact embedding $X \hookrightarrow L^{s(x)}(\Omega)$ (respectively $X \hookrightarrow L^{r(x)}(\Omega)$) guarantees the existence of subsequence (u_n) which converges to u in $L^{s(x)}(\Omega)$ (respectively $L^{r(x)}(\Omega)$). So, using the continuity of N_f and N_g, we deduce easily that $B^{'}$ is weakly-strongly continuous. $\square$

Let us define now the functional ϕ associated with the problem (P) : $\phi : X \longrightarrow \mathbb{R}$

$$\phi(u) = \int_\Omega \left[\sum_{i=1}^N A_i(x, \partial_{x_i} u) + \frac{b(x)}{P_+^+}|u|^{P_+^+} - \lambda(x)F(x,u) - \mu(x)G(x,u)\right] dx.$$

Under assumptions (A_0), (A_1), (f_0) and (g_0), we have ϕ is well defined on X and $\phi \in C^1(X,\mathbb{R})$, so we can define a weak solution as below.

Definition 3.1. A function u is a weak solution of the problem (P) if and only if

$$\int_\Omega \left[\sum_{i=1}^N a_i(x, \partial_{x_i} u)\partial_{x_i}\varphi + b(x)|u|^{P_+^+-2}u\varphi - \lambda(x)f(x,u)\varphi - \mu(x)g(x,u)\varphi\right] dx = 0,$$

for all $\varphi \in X$.

Lemma 3.2. *(see[5]) Let $u \in X$.*

(1) *When $\|u\| < 1$, we have*

$$\sum_{i=1}^N \int_\Omega |\partial_{x_i}(u)|^{p_i(x)}\,dx \ge \frac{\|u\|^{P_+^+}}{N^{P_+^+-1}}.$$

(2) *When $\|u\| > 1$, we have*

$$\sum_{i=1}^N \int_\Omega |\partial_{x_i}(u)|^{p_i(x)}\,dx \ge \frac{\|u\|^{P_-^-}}{N^{P_-^--1}} - N.$$

Lemma 3.3. *The functional Φ satisfies the Palais-Smale condition.*

Proof. Let $\{u_n\}$ be a (PS) sequence, namely, $|\phi(u_n)| \leq R$, and $\phi^{'}(u_n) \to 0$, then when $\|u_n\| \geq 1$, we have by (f_0), (g_0) and (g_1)

$$
\begin{aligned}
&1 + R + \|u_n\| \geq \phi(u_n) - \frac{1}{\theta}\langle \phi^{'}(u_n), u_n\rangle \\
&\geq \int_\Omega \left\{ \sum_{i=1}^N A_i(x, \partial_{x_i} u_n) + \frac{b(x)}{P_+^+}|u_n|^{P_+^+} - \lambda(x)F(x,u_n) - \mu(x)G(x,u_n) \right\} dx \\
&- \frac{1}{\theta}\int_\Omega \left\{ \sum_{i=1}^N a_i(x, \partial_{x_i} u_n)\partial_{x_i} u_n + b(x)|u_n|^{P_+^+} - \lambda(x)f(x,u_n)u_n - \mu(x)g(x,u_n)u_n \right\} dx \\
&\geq \int_\Omega \left\{ \sum_{i=1}^N A_i(x, \partial_{x_i} u_n) - \frac{1}{\theta} a_i(x, \partial_{x_i} u_n)\partial_{x_i} u_n \right\} dx \\
&+ \left(\frac{1}{P_+^+} - \frac{1}{\theta} \right) \int_\Omega b(x)|u_n|^{P_+^+}\, dx + \int_{[|u_n| \geq M]} \mu(x) \left(\frac{1}{\theta} g(x,u_n)u_n - G(x,u_n) \right) dx \\
&+ \int_{[|u_n| < M]} \mu(x) \left(\frac{1}{\theta} g(x,u_n)u_n - G(x,u_n) \right) dx \\
&+ \int_\Omega \lambda(x) \left(\frac{1}{\theta} f(x,u_n)u_n - F(x,u_n) \right) dx \\
&\geq \int_\Omega \left\{ \sum_{i=1}^N A_i(x, \partial_{x_i} u_n) - \frac{1}{\theta} a_i(x, \partial_{x_i} u_n)\partial_{x_i} u_n \right\} dx \\
&- c\left(\frac{1}{\theta} + \frac{1}{\alpha^-} \right) \int_\Omega \lambda(x)|u_n|^{\alpha(x)} + K,
\end{aligned}
$$

where K is constant obtained by using (g_0) and (g_1). From (A_2), for all $x \in \Omega$ and $i \in \{1, ..., N\}$ we have

$$
-\frac{1}{\theta} a_i(x, \partial_{x_i} u_n)\partial_{x_i} u_n \geq -\frac{P_+^+}{\theta} A_i(x, \partial_{x_i} u_n). \tag{5}
$$

On the other hand, we have by the Lemma 2.2 and Proposition 2.1

$$
\begin{aligned}
\int_\Omega \lambda(x)|u_n|^{\alpha(x)} &\leq 2|\lambda|_{\frac{s(x)}{s(x)-\alpha(x)}} \left| |u_n|^{\alpha(x)} \right|_{\frac{s(x)}{\alpha(x)}} \\
&\leq 2|\lambda|_{\frac{s(x)}{s(x)-\alpha(x)}} |u_n|_{s(x)}^{\tilde{\alpha}},
\end{aligned}
$$

where $\tilde{\alpha} \in [\alpha^-, \alpha^+]$.
Since the embedding $X \hookrightarrow L^{s(x)}(\Omega)$ is compact, then there exists a constant $c_1 > 0$ such that

$$
|u_n|_{s(x)} \leq c_1 \|u_n\|,
$$

Then,

$$
\int_\Omega \lambda(x)|u_n|^{\alpha(x)} \leq C|\lambda|_{\frac{s(x)}{s(x)-\alpha(x)}} \|u_n\|^{\tilde{\alpha}}, \tag{6}
$$

where $C > 0$ is a constant. From (5) and (6), we get

$$
1 + R + \|u_n\| \geq \left(1 - \frac{p_+^+}{\theta} \right) \sum_{i=1}^N \int_\Omega A_i(x, \partial_{x_i} u_n)\, dx - C_1|\lambda|_{\frac{s(x)}{s(x)-\alpha(x)}} \|u_n\|^{\tilde{\alpha}} + K,
$$

where $C_1 > 0$, $R > 0$ are constants. Again from (A_2) we have

$$A_i(x, \partial_{x_i} u_n) \geq \frac{1}{p_i(x)} |\partial_{x_i} u_n|^{p_i(x)} \geq \frac{1}{P_+^+} |\partial_{x_i} u_n|^{p_i(x)},$$

for all $x \in \Omega$ and $i \in \{1, ..., N\}$, so

$$1 + R + \|u_n\| \geq \left(\frac{1}{P_+^+} - \frac{1}{\theta} \right) \sum_{i=1}^{N} \int_\Omega |\partial_{x_i} u_n|^{p_i(x)}\, dx - C_1 |\lambda|_{\frac{s(x)}{s(x)-\alpha(x)}} \|u_n\|^{\tilde{\alpha}} + K.$$

Using Lemma 3.2, we get

$$1 + R + \|u_n\| \geq \left(\frac{1}{P_+^+} - \frac{1}{\theta} \right) \left(\frac{\|u_n\|^{P_-^-}}{N^{P_-^- - 1}} - N \right) - C_1 |\lambda|_{\frac{s(x)}{s(x)-\alpha(x)}} \|u_n\|^{\tilde{\alpha}} + K,$$

and consequently u_n is bounded because $\theta > P_+^+$ and $P_-^- > \tilde{\alpha}$. As X is reflexive then there exists a subsequence, still denoted by (u_n) which converges weakly to u_0 in X. Using the fact that $\phi^{'}(u_n) \to 0$, we can deduce that

$$\lim_{n\to\infty} \langle \phi^{'}(u_n), u_n - u_0 \rangle = 0,$$

more precisely,

$$\lim_{n\to\infty} \int_\Omega [\sum_{i=1}^{N} a_i(x, \partial_{x_i} u_n)(\partial_{x_i} u_n - \partial_{x_i} u_0) + b(x)|u_n|^{P_+^+ - 2} u_n (u_n - u_0)$$
$$- \lambda(x) f(x, u_n)(u_n - u_0) - \mu(x) g(x, u_n)(u_n - u_0)]\, dx = 0.$$

Using Hölder inequality we have

$$\int_\Omega \lambda(x) f(x, u_n)(u_n - u_0) \leq 3c |\lambda|_{\frac{s(x)}{s(x)-\alpha(x)}} \left| |u|^{\alpha(x)-1} \right|_{\frac{s(x)}{\alpha(x)-1}} |u_n - u_0|_{s(x)}.$$

$$\int_\Omega \mu(x) g(x, u_n)(u_n - u_0) \leq 3d |\mu|_{\frac{r(x)}{r(x)-\gamma(x)}} \left| |u|^{\gamma(x)-1} \right|_{\frac{r(x)}{\gamma(x)-1}} |u_n - u_0|_{r(x)}.$$

$$\int_\Omega b(x) |u_n|^{P_+^+ - 2} u_n (u_n - u_0) \leq 2 |b|_\infty \left| |u_n|^{P_+^+ - 1} \right|_{\frac{P_+^+}{P_+^+ - 1}} |u_n - u_0|_{P_+^+}.$$

As $s(x)$, $r(x)$ and P_+^+ fulfill Proposition 2.4, thus (u_n) converges strongly to u_0 in $L^{s(x)}(\Omega)$, $L^{r(x)}(\Omega)$ and $L^{P_+^+}(\Omega)$. By these facts the relation above reduces to

$$\lim_{n\to\infty} \int_\Omega \sum_{i=1}^{N} a_i(x, \partial_{x_i} u_n)(\partial_{x_i} u_n - \partial_{x_i} u_0)\, dx = 0.$$

Using Proposition 2.5, we deduce that (u_n) converges strongly to (u_0) in X, that is to say that ϕ satisfies Palais-Smale condition. $\square$

Now we demonstrate the following geometric conditions of Theorem 2.1.

Lemma 3.4. (1) *There exists $\lambda^* > 0$ and δ, $r > 0$ such that for any $|\lambda|_{\frac{s(x)}{s(x)-\alpha(x)}} \in (0, \lambda^*)$, we have $\phi(u) \geq \delta$ for all $u \in X$ with $\|u\| = r$.*

(2) *There exists $v \in X \setminus \{0\}$ such that $\lim_{t\to+\infty} \phi(tv) = -\infty$.*

Proof. 1) We will show that $\phi(u) \geq \delta$ for $\|u\| = r$. For $\|u\| < 1$, using Hölder inequality and Lemmas 2.2 and 3.2, we have

$$\begin{aligned}
\phi(u) &= \int_\Omega \left\{ \sum_{i=1}^{N} A_i(x, \partial_{x_i}(u)) + \frac{b(x)}{P_+^+}|u|^{P_+^+} - \lambda(x)F(x,u) - \mu(x)G(x,u) \right\} dx \\
&\geq \frac{1}{P_+^+} \sum_{i=1}^{N} \int_\Omega |\partial_{x_i}(u)|^{p_i(x)}\, dx + \frac{b_0}{P_+^+} \int_\Omega |u|^{P_+^+}\, dx - c \int_\Omega \frac{\lambda(x)}{\alpha(x)} |u|^{\alpha(x)}\, dx \\
&- d \int_\Omega \frac{\mu(x)}{\gamma(x)} |u|^{\gamma(x)}\, dx, \\
&\geq \frac{1}{P_+^+} \left(\frac{\|u\|^{P_+^+}}{N^{P_+^+-1}} \right) - C_1^{'} |\lambda|_{\frac{s(x)}{s(x)-\alpha(x)}} \|u\|^{\tilde{\alpha}} - C_2^{'} |\mu|_{\frac{r(x)}{r(x)-\gamma(x)}} \|u\|^{\tilde{\gamma}},
\end{aligned}$$

where $C_1^{'} > 0,\ C_2^{'} > 0$ are constants. The assumption (B) gives us

$$\frac{1}{P_+^+} \int_\Omega b(x)|u|^{P_+^+}\, dx \geq \frac{b_0}{P_+^+} |u|^{P_+^+}_{L^{P_+^+}(\Omega)} \geq 0.$$

This implies that

$$\phi(u) \geq \frac{\|u\|^{P_+^+}}{2P_+^+ N^{P_+^+-1}} - C_1^{'} |\lambda|_{\frac{s(x)}{s(x)-\alpha(x)}} \|u\|^{\tilde{\alpha}} + \frac{\|u\|^{P_+^+}}{2P_+^+ N^{P_+^+-1}} - C_2^{'} |\mu|_{\frac{r(x)}{r(x)-\gamma(x)}} \|u\|^{\tilde{\gamma}},$$

so it follows that,

$$\begin{aligned}
\phi(u) \geq \|u\|^{P_+^+} &\left(\frac{1}{2P_+^+ N^{P_+^+-1}} - C_1^{'} |\lambda|_{\frac{s(x)}{s(x)-\alpha(x)}} \|u\|^{\tilde{\alpha}-P_+^+} \right) \\
&+ \|u\|^{P_+^+} \left(\frac{1}{2P_+^+ N^{P_+^+-1}} - C_2^{'} |\mu|_{\frac{r(x)}{r(x)-\gamma(x)}} \|u\|^{\tilde{\gamma}-P_+^+} \right).
\end{aligned} \tag{7}$$

Since the function $h : [0,1] \to \mathbb{R}$ defined by

$$h(t) = \frac{1}{2P_+^+ N^{P_+^+-1}} - C_2^{'} |\mu|_{\frac{r(x)}{r(x)-\gamma(x)}} t^{\tilde{\gamma}-P_+^+}.$$

is positive in neighborhood of the origin because $\tilde{\gamma} > P_+^+$, it follows that there exists $r \in (0,1)$ such that $h(r) > 0$. On the other hand, for $\|u\| = r$ let us define

$$\lambda^* = min\{1, \frac{1}{4C_1^{'} P_+^+ N^{P_+^+-1}} r^{P_+^+-\tilde{\alpha}}\},$$

then for any $|\lambda|_{\frac{s(x)}{s(x)-\alpha(x)}} < \lambda^*$, there exists $\delta = \frac{r^{P_+^+}}{4P_+^+ N^{P_+^+-1}}$ such that for any $u \in X$ with $\|u\| = r$ we have $\phi(u) \geq \delta > 0$.

2) By the condition (g_1), there exists $M_1,\ M_2 > 0$ such that

$$G(x,t) \geq M_1 |t|^\theta - M_2\ ,\ \forall t \in \mathbb{R},\ a.e.\ x \in \Omega$$

Let $v \in C_0^\infty(\Omega)$ and $t > 1$. Using (A_0), (A_1), and (f_0), (g_0) and (g_1), then

$$\begin{aligned}
\phi(tv) &= \int_\Omega \left\{ \sum_{i=1}^N A_i(x, \partial_{x_i}(tv)) + \frac{b(x)}{P_+^+}|tv|^{P_+^+} - \lambda(x)F(x,tv) - \mu(x)G(x,tv) \right\} dx \\
&\leq C\sum_{i=1}^N \int_\Omega \left(|\partial_{x_i}(tv)| + \frac{|\partial_{x_i}(tv)|^{p_i(x)}}{p_i(x)} \right) dx + \frac{t^{P_+^+}}{P_+^+}\int_\Omega b(x)|v|^{P_+^+}\, dx \\
&+ \int_\Omega \lambda(x)|F(x,tv)|\, dx - \int_\Omega \mu(x)G(x,tv)\, dx, \\
&\leq Ct^{P_+^+}\sum_{i=1}^N \int_\Omega \left(|\partial_{x_i} v| + \frac{|\partial_{x_i} v|^{p_i(x)}}{P_-^-} \right) dx + \frac{t^{P_+^+}}{P_+^+}\int_\Omega b(x)|v|^{P_+^+}\, dx \\
&+ \frac{c}{\alpha^-}t^{\alpha^+}\int_\Omega \lambda(x)|v|^{\alpha(x)}\, dx - M_1 t^\theta \int_\Omega \mu(x)|v|^\theta\, dx - M_2\int_\Omega \mu(x)\, dx.
\end{aligned}$$

As $b(x)|v|^{P_+^+}$ and $\mu(x)|v|^\theta$ are positive, and $\theta > P_+^+ > \alpha^+$, then $\lim_{t\to+\infty} \phi(tv) = -\infty$. $\square$

Then, for a such λ^* we can say that ϕ satisfies the conditions of Theorem 2.1 (mountain pass theorem), finally ϕ has a nontrivial critical point u_0 with $\phi(u_0) = c_2$ and thus a nontrivial weak solution of problem (P).

In order to prove that there exists a second weak solution, we need the following lemma.

Lemma 3.5. *There exists $\psi \in X$, $\psi \geq 0$, $\psi \not\equiv 0$ such that $\phi(t\psi) < 0$ for all $t > 0$ small enough.*

Proof. Let $\psi \in C_0^\infty(\Omega)$, $\psi \geq 0$, $\psi \not\equiv 0$, and without loss of generality, we may assume that $\sup_{x\in\Omega} |\psi(x)| = 1$, and $t \in (0, \delta)$. Using (A_0), (A_1), and (f_1) we obtain

$$\begin{aligned}
\phi(t\psi) &= \int_\Omega \left\{ \sum_{i=1}^N A_i(x, \partial_{x_i}(t\psi)) + \frac{b(x)}{P_+^+}|t\psi|^{P_+^+} - \lambda(x)F(x,t\psi) - \mu(x)G(x,t\psi) \right\} dx \\
&\leq C\sum_{i=1}^N \int_\Omega \left(|\partial_{x_i}(t\psi)| + \frac{|\partial_{x_i}(t\psi)|^{p_i(x)}}{p_i(x)} \right) dx + \frac{t^{P_+^+}}{P_+^+}\int_\Omega b(x)|\psi|^{P_+^+}\, dx \\
&- \int_\Omega \lambda(x)F(x,t\psi)\, dx - \int_\Omega \mu(x)G(x,t\psi)\, dx, \\
&\leq Ct^{P_-^-}\sum_{i=1}^N \int_\Omega \left(|\partial_{x_i}\psi| + \frac{|\partial_{x_i}\psi|^{p_i(x)}}{P_-^-} \right) dx + \frac{t^{P_+^+}}{P_+^+}\int_\Omega b(x)|\psi|^{P_+^+}\, dx \\
&- t^{\alpha_0}\int_\Omega \lambda(x)h_0(x)|\psi|^{\alpha_0}\, dx < 0,
\end{aligned}$$

for all $t < \rho^{\frac{1}{P_+^+ - \alpha_0}}$, with

$$0 < \rho < \min\left\{1, \frac{\int_\Omega \lambda(x)h_0(x)|\psi|^{\alpha_0}\,dx}{C\sum_{i=1}^N \int_\Omega \left(|\partial_{x_i}\psi| + \frac{|\partial_{x_i}\psi|^{p_i(x)}}{P_-^-}\right)dx + \frac{1}{P_+^+}\int_\Omega b(x)|\psi|^{P_+^+}\,dx}\right\}.$$

□

Let λ^* be as in Lemma 3.4 and assume that $|\lambda|_{\frac{s(x)}{s(x)-\alpha(x)}} < \lambda^*$, giving a ball $B_r(0) = \{\varphi \in X; \|\varphi\| < r\}$, it follows that

$$\inf_{\partial B_r(0)} \phi(u) > 0.$$

On the other hand, from Lemma 3.5 there exists $\psi \in X$ such that

$$\phi(t\psi) < 0 \ for\ t > 0\ small\ enough.$$

Using the inequality (7), we can see easily that ϕ is bounded below on $B_r(0)$, then for $u \in B_r(0)$ we have

$$-\infty < c_3 = \underline{\inf_{B_r(0)}}\, \phi(u) < 0.$$

Let now $0 < \varepsilon < \inf_{\partial B_r(0)} \phi(u) - \underline{\inf_{B_r(0)}}\,\phi(u)$. Applying Theorem 2.2 (Ekeland variational principle [7]) to the functional $\phi : \overline{B_r(0)} \to \mathbb{R}$, we find $u_\varepsilon \in \overline{B_r(0)}$ such that

$$\begin{aligned}\phi(u_\varepsilon) &< \underline{\inf_{B_r(0)}}\,\phi + \varepsilon,\\ &< \phi(u) + \varepsilon\|u - u_\varepsilon\|,\ u \neq u_\varepsilon.\end{aligned}$$

As

$$\phi(u_\varepsilon) \le \underline{\inf_{B_r(0)}}\,\phi + \varepsilon \le \inf_{B_r(0)} \phi + \varepsilon < \inf_{\partial B_r(0)} \phi.$$

Consequently $u_\varepsilon \in B_r(0)$. Let's define $H : \overline{B_r(0)} \to \mathbb{R}$ by $H(u) = \phi(u) + \varepsilon\|u - u_\varepsilon\|$. It's easy to see that u_ε is a minimum point of H and thus

$$\frac{H(u_\varepsilon + tv) - H(u_\varepsilon)}{t} \ge 0,$$

for a small $t > 0$ and $v \in B_r(0)$. The above relation yields

$$\frac{\phi(u_\varepsilon + tv) - \phi(u_\varepsilon)}{t} + \varepsilon\|v\| \ge 0,$$

letting t goes to 0, it follows that $\langle \phi^{'}(u_\varepsilon), v\rangle + \varepsilon\|v\| > 0$, we deduce that $\|\phi^{'}(u_\varepsilon)\| \le \varepsilon$. We infer that there exists a sequence $(v_n) \subset B_r(0)$ such that

$$\phi(v_n) \to c_3 \ and\ \phi^{'}(v_n) \to 0. \tag{8}$$

It's clear that (v_n) is bounded in X. Thus, there exists $u_1 \in X$ such that, up to a subsequence, (v_n) converges weakly to u_1 in X. From Propositions 2.1, 2.4 , 2.5, and Lemma 2.2 we deduce that $v_n \to u_1$. Therefore, by relation (8)

$$\phi(v_n) = c_3 \ and\ \phi^{'}(v_n) = 0, \tag{9}$$

thereby u_1 is a nontrivial weak solution for (P). Finally, as

$$\phi(u_0) = c_2 > 0 > c_3 = \phi(u_1),$$

then, $u_0 \neq u_1$. Thus, (P) has two nontrivial solutions.

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