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Existence and multiplicity of weak solutions for a Neumann boundary value problem with the Sturm-Liouville equation

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Abstract. In this paper, variational methods and critical point theorems are employed to establish the existence of multiple solutions to a Neumann boundary value problem for the Sturm-Liouville equation, under appropriate growth conditions on the nonlinearity.

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1. Introduction

The aim of the present paper is to establish multiple solutions for the following Neumann boundary value problem

$$\begin{cases} -\left(pu'\right)' + ru' + qu = \lambda g\left(x, u\right) \\ u'\left(0\right) = u'\left(1\right) = 0 \end{cases} \tag{1.1}$$

where $p \in C^1\left([0,1]\right), q, r \in C^0\left([0,1]\right)$, with p and q positive functions, λ is a positive parameter and g is nonnegative continuous function.

Multiple solutions for Neumann problems have been obtained in several papers, for instance, in [1, 12]. In these latest years problems of Sturm-Liouville type have been studied by a numerous of authors. For some recent works, we refer the readers to [6, 9, 10, 14]. Precisely [6] deals with equations of Sturm-Liouville-type having nonlinearities on the right-hand side being possibly discontinuous, and in [9] authors obtained an existence result for a class of mixed boundary value problems for second-order differential equations. Motivated by the fact that such kind of problems are used to describe a long class of physical phenomena, here we study the problem (1.1) when the nonlinearity f has the subcritical growth. We use the variational method and critical point theorems established in [3, 7, 8] to obtain the existence of at least one, two and three weak solutions whenever the parameter λ belongs to a precise positive interval. For basic notation and definitions, we refer the reader to [3, 4, 5].

The rest of this paper is organized as follows. In section 2 we present some preliminary results. Our main results and their proofs, on multiplicity of solutions to Neumann problem for the equation

$$-\left(pu'\right)' + qu = \lambda f\left(x, u\right)$$

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are given in section 3 and the last section is dedicated to consequences of results in third section and multiple solutions to main complete problem (1.1) are obtained.

2. Preliminaries

Consider the Sturm-Liouville boundary value problem

$$\begin{cases} -\left(pu'\right)' + qu = \lambda f\left(x, u\right) \\ u'\left(0\right) = u'\left(1\right) = 0 \end{cases} \tag{2.1}$$

where $p \in C^1([0,1]), q \in C^0([0,1])$, with p and q positive functions, λ is a positive parameter and f is nonnegative continuous function such that
$(\mathcal{F}_1)$ there exist three positive constants $a_1, a_2, s < 2$, with the property

$$|f(x,t)| \le a_1 + a_2 |t|^{s-1}$$

for all $(x,t) \in [0,1] \times \mathbb{R}$.
Take $X = W^{1,2}([0,1])$ equipped with the norm

$$\|u\|_X = \left(\int_0^1 \left(p(t) |u'(t)|^2 + q(t) |u(t)|^2 \right) dt \right)^{\frac{1}{2}}.$$

Put

$$\Phi(u) = \frac{1}{2} \int_0^1 \left(p(t) |u'(t)|^2 + q(t) |u(t)|^2 \right) dt,$$

and

$$F(x,\xi) := \int_0^{\xi} f(x,t)\, dt$$

for each $(x,\xi) \in \Omega \times \mathbb{R}$. It is known that $\Phi \in C^1(X,\mathbb{R})$ and the assumption $(\mathcal{F}_1)$ guarantees that the functional Ψ defined by

$$\Psi(u) = \int_0^1 F(x, u(x))\, dx, \quad \forall u \in X$$

belongs to $C^1(X,\mathbb{R})$.

Let us recall some basic definitions and theorems that we need in the forthcoming analysis.

Definition 2.1. $u \in X$ is said to be a weak solution of the problem (2.1) if

$$\int_0^1 pu'v'dx + \int_0^1 quvdx = \lambda \int_0^1 f(x,u)\, vdx,$$

for every $u, v \in X$.

The corresponding energy functional of problem (2.1) is defined by

$$I_\lambda(u) := \Phi(u) - \lambda \Psi(u)$$

and we observe that, for each $\lambda > 0$, the critical points u of I_λ are the weak solutions of (2.1).

Definition 2.2. Let Φ and Ψ be two continuously Gateaux differentiable functionals defined on a real Banach space X and fix $r \in \mathbb{R}$. The functional $I = \Phi - \Psi$ is said to verify the Palais-Smale condition cut off upper at r (in short $(P.S.)^{[r]}$) if any sequence $\{u_n\}_{n\in\mathbb{N}}$ in X such that
(α) $\{I(u_n)\}$ is bounded;
(β) $\lim_{n\to+\infty} \|I'(u_n)\|_{X^*} = 0$;
(γ) $\Phi(u_n) < r$ for each $n \in \mathbb{N}$;
has a convergent subsequence.

Theorem 2.1. *(See theorem 2.3 of* [7]*) Let X be a real Banach space, $\Phi, \Psi : X \to \mathbb{R}$ be two continuously Gateaux differentiable functionals such that $\inf_{x\in X} \Phi(0) = \Psi(0) = 0$. Assume that there exist $r > 0$ and $\bar{x} \in X$, with $0 < \Phi(\bar{x}) < r$, such that :*
$(\mathcal{H}_1)$ $\frac{\sup_{\Phi(x)\le r} \Psi(x)}{r} < \frac{\Psi(\bar{x})}{\Phi(\bar{x})}$,
$(\mathcal{H}_2)$ *for each $\lambda \in \left]\frac{\Phi(\bar{x})}{\Psi(\bar{x})}, \frac{r}{\sup_{\Phi(x)\le r} \Psi(x)}\right[$ the functional $I_\lambda := \Phi - \lambda\Psi$ satisfies $(P.S.)^{[r]}$ condition.*
Then, for each $\lambda \in \Lambda_r := \left]\frac{\Phi(\bar{x})}{\Psi(\bar{x})}, \frac{r}{\sup_{\Phi(x)\le r} \Psi(x)}\right[$, there is $x_{0,\lambda} \in \Phi^{-1}(]0, r[)$ such that $I'_\lambda(x_{0,\lambda}) \equiv \vartheta_{X^*}$ and $I_\lambda(x_{0,\lambda}) \le I_\lambda(x)$ for all $x \in \Phi^{-1}(]0, r[)$.

Theorem 2.2. *(See theorem 3.2 of* [7]*) Let X be a real Banach space, $\Phi, \Psi : X \to \mathbb{R}$ be two continuously Gateaux differentiable functionals such that Φ is bounded from below and $\Phi(0) = \Psi(0) = 0$. Fix $r > 0$ and assume that, for each*

$$\lambda \in \left]0, \frac{r}{\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u)}\right[,$$

the functional $I_\lambda := \Phi - \lambda\Psi$ satisfies $(P.S.)$ condition and it is unbounded from below. Then, for each

$$\lambda \in \left]0, \frac{r}{\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u)}\right[,$$

the functional I_λ admits two distinct critical points.

Theorem 2.3. *(See theorem 3.6 of* [8]*) Let X be a reflexive real Banach space, $\Phi : X \to \mathbb{R}$ be a coercive continuously Gateaux differentiable and sequentially weakly lower semi-continuous functional whose Gateaux derivative admits a continuous inverse on X^*, $\Psi : X \to \mathbb{R}$ be a continuously differentiable functional whose Gateaux derivative is compact such that*

$$\inf_{x\in X} \Phi(0) = \Psi(0) = 0.$$

Assume that there exist $r > 0$ and $\bar{x} \in X$, with $r < \Phi(\bar{x})$, such that:
$(\mathcal{H}_1)$ $\frac{\sup_{\Phi(x)\le r} \Psi(x)}{r} < \frac{\Psi(\bar{x})}{\Phi(\bar{x})}$,
$(\mathcal{H}_2)$ *for each $\lambda \in \Lambda_r = \left]\frac{\Phi(\bar{x})}{\Psi(\bar{x})}, \frac{r}{\sup_{\Phi(x)\le r} \Psi(x)}\right[$ the functional $\Phi - \lambda\Psi$ is coercive.*
Then, for each $\lambda \in \Lambda_r$, the functional $\Phi - \lambda\Psi$ has at least three distinct critical points in X.

3. Main results

In this section we deal with the existence of at least one, two and three weak solution for the problem (2.1). The following results are obtained by applying Theorems 2.1, 2.2 and 2.3.

Theorem 3.1. *Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be a continues function satisfying $(\mathcal{F}_1)$. Moreover, assume that*

$(\mathcal{F}_2)$ $\quad \limsup_{t \to 0^+} \frac{\inf_{x\in\Omega} F(x,t)}{t^2} = +\infty.$

Then for

$$\lambda^* = \frac{1}{\sqrt{2}a_1k_1 + \sqrt{2}^s \frac{a_2}{s} (k_s)^s}$$

where k_1 and k_s denote respectively the constants of the embeddings $X \hookrightarrow L^1([0,1])$ and $X \hookrightarrow L^s([0,1])$, the problem (2.1) admits at least one nontrivial weak solution for each $\lambda \in]0, \lambda^*[$.

Proof. Our aim is to apply Theorem 2.1 in the case $r = 1$, for the space $X := W^{1,2}([0,1])$, with the usual norm and to the functionals $\Phi, \Psi : X \to \mathbb{R}$ defined as

$$\Phi(u) = \frac{1}{2}\int_0^1 \left(p(t)\left|u'(t)\right|^2 + q(t)\left|u(t)\right|^2\right) dt$$

and

$$\Psi(u) = \int_0^1 F(x, u(x))\, dx.$$

The functional Φ is in $C^1(X, \mathbb{R})$. Moreover, by condition $(\mathcal{F}_1)$ and the compact embedding $W_0^{1,2}(\Omega) \hookrightarrow L^s(\Omega)$, Ψ is in $C^1(X,\mathbb{R})$ and has compact derivative (see Lemma 2.5 of [2]). This ensures that the functional $I_\lambda := \Phi - \lambda\Psi$ verifies $(P.S.)^{[r]}$ condition for each $r > 0$ (see Proposition 2.1 of [3]) and so condition $(\mathcal{H}_2)$ of Theorem 2.1 is verified.

Fixed $\lambda \in]0, \lambda^*[$, using $(\mathcal{F}_2)$ there exists

$$0 < \delta_\lambda < \min\left\{1, \left(\frac{2}{\|q\|_\infty}\right)^{\frac{1}{2}}\right\} \tag{3.1}$$

such that

$$\frac{2\inf_{x\in[0,1]} F(x,\delta_\lambda)}{\delta_\lambda^2 \|q\|_\infty} > \frac{1}{\lambda}. \tag{3.2}$$

We denote by u_λ the function in X defined by

$$u_\lambda(x) = \delta_\lambda \qquad \forall x \in [0,1].$$

Then

$$\Phi(u_\lambda) = \frac{1}{2}\|u_\lambda\|^2 = \frac{1}{2}\left(\int_0^1 p(u_\lambda')^2 + \int_0^1 q(u_\lambda)^2\right) = \frac{1}{2}\int_0^1 qu_\lambda^2 \le \frac{1}{2}\|q\|_\infty |\delta_\lambda|^2 < 1,$$

so $\Phi(u_\lambda) < 1$.

Moreover

$$\Psi(u_\lambda) = \int_0^1 F(x,\delta_\lambda)\, \mathrm{d}x \ge \inf_{x\in\Omega} F(x,\delta_\lambda).$$

For each $u \in \Phi^{-1}[-\infty, 1]$, we have

$$\|u\| = (2\Phi(u))^{\frac{1}{2}} < \sqrt{2}, \tag{3.3}$$

using $(\mathcal{F}_1)$ which implies

$$|F(x,t)| \leq a_1 |t| + \frac{a_2}{s} |t|^s, \tag{3.4}$$

(3.3) and (3.4) imply that for each $u \in \Phi^{-1}[-\infty, 1]$;

$$\Psi(u) \leq a_1 \int |u| + \frac{a_2}{s} \int |u|^s \leq a_1 k_1 \|u\| + \frac{a_2}{s} k_s^s \|u\|^s \leq \sqrt{2} a_1 k_1 + \sqrt{2}^s \frac{a_2}{s} k_s^s = \frac{1}{\lambda^*}.$$

From (3.2) we get

$$\sup \Psi(u) \leq \sqrt{2} a_1 k_1 + \sqrt{2}^s \frac{a_2}{s} k_s^s = \frac{1}{\lambda^*} < \frac{1}{\lambda} < \frac{2 \inf_{x \in [0,1]} F(x, \delta_\lambda)}{\delta_\lambda^2 \|q\|_\infty} < \frac{\Psi(u_\lambda)}{\Phi(u_\lambda)}.$$

So condition $(\mathcal{H}_1)$ of Theorem 2.1 is verified. Since $\lambda \in \left] \frac{\phi(u_\lambda)}{\psi(u_\lambda)}, \frac{1}{\sup_{\Phi(u) \leq 1} \Psi(u)} \right[$, Theorem 2.1 guarantees the existence of a local minimum point u_{λ_1} for the functional I_λ such that

$$0 < \Phi(u_{\lambda_1}) < 1$$

and so u_{λ_1} is a nontrivial weak solution of the problem (2.1). □

Theorem 3.2. *Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be a continues function satisfying $(\mathcal{F}_1)$. Moreover, assume that*
$(\mathcal{F}_3)$ *there exist $\mu > 2$ and $r > 0$ such that*

$$0 < \mu F(x,t) \leq t f(x,t)$$

for each $x \in [0,1]$ and $|t| \geq r$.
Then for each $\lambda \in]0, \lambda^*[$ the problem (2.1) admits at least two distinct weak solutions (λ^* is the constant introduced in the statement of Theorem 3.1).

Proof. We apply Theorem 2.2 in the case $r = 1$ for the space X. and the functional $\Phi, \Psi : X \to \mathbb{R}$ defined in the proof of Theorem 3.1.
Using $(\mathcal{F}_3)$, it is easy to see that, there exist $a_3, a_4 > 0$ such that

$$F(t) \geq a_3 |t|^\mu - a_4$$

for each $(x,t) \in \Omega \times \mathbb{R}$. So

$$I_\lambda(t\tilde{u}) = \Phi(t\tilde{u}) - \lambda \Psi(t\tilde{u}) \leq \frac{1}{2} t^2 \|\tilde{u}\|^2 - \lambda a_3 t^\mu \int |\tilde{u}|^\mu \, \mathrm{d}x + \lambda a_4.$$

for $t > 1$ and nonzero $\tilde{u} \in X$. Since $\mu > 2$, so I_λ is unbounded from below.
By standard computation the functional I_λ verifies $(P.S.)$ condition (see [13] and Lemma 3.2 of [2]) and so all hypotheses of Theorem 2.2 are verified. Therefore, for each $\lambda \in]0, \lambda^*[$, the functional I_λ admits two distinct critical points which are weak solutions of problem (2.1). □

Theorem 3.3. *Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be a continues function satisfying $(\mathcal{F}_1)$. Moreover, assume that*
$(\mathcal{F}_4)$ *there exist two positive constants a, s with $s < 2$ such that*

$$F(x,t) \leq a(1 + |t|^s)$$

for all $(x,t) \in [0,1] \times \mathbb{R}$.
$(\mathcal{F}_5)$ *there exist* $r > 0$ *and* $\delta > 0$ *with* $r < \frac{\delta^2}{2} \inf_{x\in\Omega} q(x)$ *such that*

$$\overline{\omega_r} := \frac{1}{r}\left\{\sqrt{2r}a_1k_1 + \sqrt{2r}^s\frac{a_2}{s}k_s^s\right\} < \frac{2\inf_{x\in[0,1]} F(x,\delta)}{\|q\|_\infty |\delta|^2};$$

$(\mathcal{F}_6)$ $F(x,t) \geq 0$ *for each* $(x,t) \in [0,1] \times \mathbb{R}^+$.
Then, for every $\lambda \in \Lambda_{r,\delta} := \left] \frac{\|q\|_\infty |\delta|^2}{2\inf_{x\in\Omega} F(x,\delta)}, \frac{1}{\overline{\omega_r}} \right[$, *the problem* (2.1) *admits at least three weak solutions.*

Proof. Our aim is to apply Theorem 2.3 to the space $X := W^{1,2}([0,1])$, with the usual norm and to the functionals $\Phi, \Psi : X \to \mathbb{R}$ defined before It can be checked that, the functional Φ and Ψ satisfy the regularity assumptions requested in Theorem 2.3.
Now, let $\bar{u} \in X$ defined by

$$\bar{u}(x) = \delta.$$

then

$$\Phi(\bar{u}) = \frac{1}{2}\|\bar{u}\|^2 = \frac{1}{2}\int_0^1 q\delta^2 dx \leq \frac{1}{2}\|q\|_\infty |\delta|^2,$$

and from $(\mathcal{F}_6)$

$$\Psi(\bar{u}) = \int_0^1 F(x,\delta)\,dx \geq \inf_{x\in[0,1]} F(x,\delta),$$

and so

$$\frac{\Psi(\bar{u})}{\Phi(\bar{u})} > \frac{2\inf_{x\in[0,1]} F(x,\delta)}{\|q\|_\infty |\delta|^2}. \tag{3.5}$$

From $r < \frac{\delta^2}{2}\inf_{x\in[0,1]} q(x)$, one has

$$\Phi(\bar{u}) = \frac{1}{2}\int_0^1 q\delta^2 dx > \frac{1}{2}\delta^2 \inf q > r.$$

then

$$\|u\| = (2\Phi(u))^{\frac{1}{2}} \leq \sqrt{2r} \tag{3.6}$$

and

$$\int_0^1 |u(x)|^q\,dx = (|u|_{l^q})^q \leq (k_q\|u\|)^q$$

for $u \in \Phi^{-1}(]-\infty, r[)$.
So, the compact embedding $X \hookrightarrow L^1([0,1])$, $(\mathcal{F}_1)$ and (3.6) imply that, for each $u \in \Phi^{-1}(]-\infty, r[)$, we have

$$\begin{aligned}\Psi(u) = \int_0^1 F(x,u)\,dx &\leq \int_0^1 \left(a_1u + \frac{a_2}{s}|u|^s\right)dx \\ &= a_1\int_0^1 u(x)\,dx + \frac{a_2}{s}\int_0^1 |u(x)|^s\,dx \\ &\leq a_1k_1\|u\| + \frac{a_2}{s}k_s^s\|u\|^s \leq \sqrt{2r}a_1k_1 + \sqrt{2r}^s\frac{a_2}{s}k_s^s\end{aligned}$$

and so

$$\frac{\sup_{\Phi(u)\leq r}\Psi(u)}{r} \leq \frac{1}{r}\left\{\sqrt{2r}a_1k_1 + \sqrt{2r}^s\frac{a_2}{s}k_s^s\right\}.$$

From $(\mathcal{F}_5)$ and (3.5) one has

$$\frac{\sup_{\Phi(u)\le r}\Psi(u)}{r}<\frac{\Psi(\bar u)}{\Phi(\bar u)}$$

and so condition $(\mathcal{H}_1)$ of Theorem 2.3 is verified.

Now we prove that, for each $\lambda>0$, I_λ is coercive.

By arguments similar to those used before, we obtain

$$\int_0^1 |u(x)|^\gamma\,dx\le (k_\gamma\|u\|)^\gamma$$

and so, for each $u\in X$, we get

$$\Psi(u)=\int_0^1 F(x,u(x))\,dx\le\int_0^1 c\left(1+|u|^\gamma\right)dx\le c\left(1+(k_\gamma\|u\|)^\gamma\right).$$

This leads to

$$I_\lambda(u)\ge\frac{1}{2}\|u\|^2-\lambda c\left[1+(k_\gamma)^\gamma\|u\|^\gamma\right]$$

and, since $\gamma<2$, coercivity of I_λ is obtained. Taking into account that

$$\Lambda_{r,\delta}\subseteq\left]\frac{\Phi(\bar u)}{\Psi(\bar u)},\frac{r}{\sup_{\Phi(u)\le r}\Psi(u)}\right[,$$

Theorem 2.3 ensures that, for each $\lambda\in\Lambda_{r,\delta}$, the functional I_λ admits at least three critical points in X which are weak solutions of the problem (2.1). $\square$

4. Multiple solutions for complete problem

In this section we consider the problem (1.1), where $g:[0,1]\times\mathbb{R}\to\mathbb{R}$ is nonnegative continuous function, $p\in C^1([0,1])$, $q,r\in C^0([0,1])$, with p and q positive functions and λ is a positive parameter. Put

$$G(x,\xi):=\int_0^\xi g(x,t)\,dt$$

for each $(x,\xi)\in\Omega\times\mathbb{R}$, satisfying

$(\mathcal{F}_1')$ there exist three positive constants $a_1,a_2,s<2$, with the property

$$|g(x,t)|\le a_1+a_2|t|^{s-1}$$

for all $(x,t)\in[0,1]\times\mathbb{R}$.

Here, we give the following results of solutions for the Neumann problem (1.1). Let R is a primitive of $\frac{r}{p}$. Since the solutions of the problem

$$\begin{cases}-\left(e^{-R}pu'\right)'+e^{-R}qu=\lambda e^{-R}g(x,u)\\ u'(0)=u'(1)=0\end{cases}$$

are solutions of the problem (1.1), respectively from Theorems 2.1, 2.2 and 2.3 the conclusions obtain.

Theorem 4.1. *Let $g:[0,1]\times\mathbb{R}\to\mathbb{R}$ be a continuous function satisfying $(\mathcal{F}_1')$. Moreover, assume that*

($\mathcal{F}_2'$) $\limsup_{t\to 0^+} \frac{\inf_{x\in[0,1]} e^{-R}G(x,t)}{t^2} = +\infty$.
Then for

$$\lambda^* = \frac{1}{\sqrt{2}a_1k_1 + \sqrt{2}^s \frac{a_2}{s}(k_s)^s}$$

where k_1 *and* k_s *denote respectively the constants of the embeddings* $X \hookrightarrow L^1([0,1])$ *and* $X \hookrightarrow L^s([0,1])$*, the problem* (1.1) *admits at least one nontrivial weak solution for each* $\lambda \in]0,\lambda^*[$.

Theorem 4.2. *Let* $g : [0,1]\times\mathbb{R} \to \mathbb{R}$ *be a continues function satisfying* ($\mathcal{F}_1'$)*. Moreover, assume that*
($\mathcal{F}_3$) *there exist* $\mu > 2$ *and* $r > 0$ *such that*

$$0 < \mu G(x,t) \le tf(x,t)$$

for each $x \in \Omega$ *and* $|t| \ge r$.
Then for each $\lambda \in]0,\lambda^*[$ *the problem* (1.1) *admits at least two distinct weak solutions* (λ^* *is the constant introduced in the statement of Theorem 3.1).*

Theorem 4.3. *Let* $g : [0,1]\times\mathbb{R} \to \mathbb{R}$ *be a continues function satisfying* ($\mathcal{F}_1'$)*. Moreover, assume that*
($\mathcal{F}_4'$) *there exist two positive constants* a, s *with* $s < 2$ *such that*

$$G(x,t) \le a(1 + |t|^s)$$

for all $(x,t) \in [0,1]\times\mathbb{R}$.
($\mathcal{F}_5'$) *there exist* $r > 0$ *and* $\delta > 0$ *with* $r < \frac{\delta^2}{2}\inf_{x\in[0,1]} e^{-R}q(x)$ *such that*

$$\overline{\omega_r} := \frac{1}{r}\left\{\sqrt{2r}a_1k_1 + \sqrt{2r}^s\frac{a_2}{s}k_s^s\right\} < \frac{2\inf_{x\in[0,1]} e^{-R}G(x,\delta)}{\|e^{-R}q\|_\infty |\delta|^2}.$$

Then, for every $\lambda \in \Lambda_{r,\delta} := \left]\frac{\|e^{-R}q\|_\infty|\delta|^2}{2\inf_{x\in[0,1]} e^{-R}G(x,\delta)}, \frac{1}{\overline{\omega_r}}\right[$*, the problem* (1.1) *admits at least three weak solutions.*

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