Annals of the University of Craiova, Mathematics and Computer Science Series
Volume 47(2), 2020, Pages 276–284
ISSN: 1223-6934

Multiwavelets on local fields of positive characteristic

Mohammad Younus Bhat

Abstract. Multiwavelets have more freedom in their construction and thus can combine more useful properties than the scalar wavelets. Symmetric scaling functions constructed have short support, generate an orthogonal MRA and provide approximation order 2. These properties are very desirable in many applications but cannot be achieved by one scaling function. In this paper we construct multiwavelets on local fields of positive characteristic. We also give their characterization by means of some basic equations in the frequency domain.

2010 Mathematics Subject Classification. 42C40, 42C15, 43A70, 11S85.
Key words and phrases. Multiwavelet, orthonormal, Fourier transform, local field.

1. Introduction

Multiresolution analysis (MRA) is an important mathematical tool since it provides a natural framework for understanding and constructing discrete wavelet systems. Usually it is assumed that an MRA is generated by one scaling function and dilates and translates of only one wavelet $\psi \in L^2(K)$ form a stable basis of $L^2(K)$. Here, we consider a generalization allowing several wavelet functions $\psi_1, \ldots, \psi_r$. The vector generated from this system will be called as a multiwavelet on local fields of positive characteristic.

In recent years there has been a considerable interest in the problem of constructing wavelet bases on various groups, namely, Cantor dyadic groups,locally compact Abelian groups,6 positive half-line R^+, p-adic fields, Vilenkin groups, Heisenberg groups and Lie groups. Benedetto and Benedetto[1] developed a wavelet theory for local fields and related groups. They did not develop the MRA approach, their method is based on the theory of wavelet sets and only allows the construction of wavelet functions whose Fourier transforms are characteristic functions of some sets. Since local fields are essentially of two types: Zero and positive characteristic (excluding the connected local fields $\mathbb{R}$ and $\mathbb{C}$). Examples of local fields of characteristic zero include the p-adic field Q_p whereas local fields of positive characteristic are the Cantor dyadic group and the Vilenkin p-groups. Even though the structures and metrics of local fields of zero and positive characteristics are similar, but their wavelet and MRA theory are quite different. In recent years, local fields have attracted the attention of several mathematicians, and have found innumerable applications not only to number theory but also to representation theory, division algebras, quadratic forms and algebraic geometry. As a result, local fields are now consolidated as part of the standard repertoire of contemporary mathematics. For more details we refer to [1, 8, 9, 10] and the many references therein.

Received February 10, 2019. Revised December 23, 2019. Accepted November 3, 2020.

Multiwavelets have more freedom in their construction and thus can combine more useful properties than the scalar wavelets. Symmetric scaling functions constructed by Geronimo, Hardin, and Massopust[4] have short support, generate an orthogonal MRA and provide approximation order. These properties are very desirable in many applications but cannot be achieved by one scaling function. Multiwavelets is a recent topic of active research in the field of wavelets. There is much research conducted on the construction of orthonormal and biorthogonal multiwavelets [2, 3, 5, 6] and the application of these multiwavelets to signal and image processing is gaining interest as well see [7]. Thus, multiwavelets can be useful for various practical problems

This paper is organized as follows. In Sec. 2, we discuss some preliminary facts about local fields of positive characteristic including the definition of Fourier transform. In Sec. 3, we construct multiwavelets on local fields of positive characteristic. We also give their characterization by means of some basic equations in the frequency domain.

2. Preliminaries on local fields

Let K be a field and a topological space. Then K is called a *local field* if both K^+ and K^* are locally compact Abelian groups, where K^+ and K^* denote the additive and multiplicative groups of K, respectively. If K is any field and is endowed with the discrete topology, then K is a local field. Further, if K is connected, then K is either $\mathbb{R}$ or $\mathbb{C}$. If K is not connected, then it is totally disconnected. Hence by a local field, we mean a field K which is locally compact, non-discrete and totally disconnected. The p-adic fields are examples of local fields. More details are referred to [10]. In the rest of this paper, we use the symbols $\mathbb{N}, \mathbb{N}_0$ and $\mathbb{Z}$ to denote the sets of natural, non-negative integers and integers, respectively.

Let K be a local field. Let dx be the Haar measure on the locally compact Abelian group K^+. If $\alpha \in K$ and $\alpha \neq 0$, then $d(\alpha x)$ is also a Haar measure. Let $d(\alpha x) = |\alpha| dx$. We call $|\alpha|$ the *absolute value* of α. Moreover, the map $x \to |x|$ has the following properties: (a) $|x| = 0$ if and only if $x = 0$; (b) $|xy| = |x||y|$ for all $x, y \in K$; and (c) $|x + y| \leq \max\{|x|, |y|\}$ for all $x, y \in K$. Property (c) is called the *ultrametric inequality.* The set $\mathfrak{D} = \{x \in K : |x| \leq 1\}$ is called the *ring of integers* in K. Define $\mathfrak{B} = \{x \in K : |x| < 1\}$. The set $\mathfrak{B}$ is called the *prime ideal* in K. The prime ideal in K is the unique maximal ideal in $\mathfrak{D}$ and hence as result $\mathfrak{B}$ is both principal and prime. Since the local field K is totally disconnected, so there exist an element of $\mathfrak{B}$ of maximal absolute value. Let $\mathfrak{p}$ be a fixed element of maximum absolute value in $\mathfrak{B}$. Such an element is called a *prime element* of K. Therefore, for such an ideal $\mathfrak{B}$ in $\mathfrak{D}$, we have $\mathfrak{B} = \langle \mathfrak{p} \rangle = \mathfrak{p}\mathfrak{D}$. As it was proved in [10], the set $\mathfrak{D}$ is compact and open. Hence, $\mathfrak{B}$ is compact and open. Therefore, the residue space $\mathfrak{D}/\mathfrak{B}$ is isomorphic to a finite field $GF(q)$, where $q = p^k$ for some prime p and $k \in \mathbb{N}$.

Let $\mathfrak{D}^* = \mathfrak{D} \setminus \mathfrak{B} = \{x \in K : |x| = 1\}$. Then, it can be proved that $\mathfrak{D}^*$ is a group of units in K^* and if $x \neq 0$, then we may write $x = \mathfrak{p}^k x', x' \in \mathfrak{D}^*$. For a proof of this fact we refer to [10]. Moreover, each $\mathfrak{B}^k = \mathfrak{p}^k \mathfrak{D} = \left\{x \in K : |x| < q^{-k}\right\}$ is a compact subgroup of K^+ and usually known as the *fractional ideals* of K^+. Let $\mathcal{U} = \{a_i\}_{i=0}^{q-1}$ be any fixed full set of coset representatives of $\mathfrak{B}$ in $\mathfrak{D}$, then every element $x \in K$ can be expressed uniquely as $x = \sum_{\ell=k}^{\infty} c_\ell \mathfrak{p}^\ell$ with $c_\ell \in \mathcal{U}$. Let χ be a fixed character on K^+ that is trivial on $\mathfrak{D}$ but is non-trivial on $\mathfrak{B}^{-1}$. Therefore, χ is constant on cosets of $\mathfrak{D}$ so if $y \in \mathfrak{B}^k$, then $\chi_y(x) = \chi(yx), x \in K$. Suppose that χ_u is any character on K^+, then clearly the restriction $\chi_u|\mathfrak{D}$ is also a character on $\mathfrak{D}$. Therefore, if $\{u(n) : n \in \mathbb{N}_0\}$ is a complete list of distinct coset representative of $\mathfrak{D}$ in K^+, then, as it was proved in [10], the set $\{\chi_{u(n)} : n \in \mathbb{N}_0\}$ of distinct characters on $\mathfrak{D}$ is a complete orthonormal system on $\mathfrak{D}$.

The Fourier transform $\hat{f}$ of a function $f \in L^1(K) \cap L^2(K)$ is defined by

$$\hat{f}(\xi) = \int_K f(x)\overline{\chi_\xi(x)}dx. \tag{2.1}$$

It is noted that

$$\hat{f}(\xi) = \int_K f(x)\,\overline{\chi_\xi(x)}dx = \int_K f(x)\chi(-\xi x)dx.$$

Furthermore, the properties of Fourier transform on local field K are much similar to those of on the real line. In particular Fourier transform is unitary on $L^2(K)$.

We now impose a natural order on the sequence $\{u(n)\}_{n=0}^{\infty}$. We have $\mathfrak{D}/\mathfrak{B} \cong GF(q)$ where $GF(q)$ is a c-dimensional vector space over the field $GF(p)$. We choose a set $\{1 = \zeta_0, \zeta_1, \zeta_2, \dots, \zeta_{c-1}\} \subset \mathfrak{D}^*$ such that span $\{\zeta_j\}_{j=0}^{c-1} \cong GF(q)$. For $n \in \mathbb{N}_0$ satisfying

$$0 \le n < q,\ n = a_0 + a_1 p + \cdots + a_{c-1}p^{c-1},\ 0 \le a_k < p, \text{ and } k = 0, 1, \dots, c-1,$$

we define

$$u(n) = (a_0 + a_1\zeta_1 + \cdots + a_{c-1}\zeta_{c-1})\,\mathfrak{p}^{-1}. \tag{2.2}$$

Also, for $n = b_0 + b_1 q + b_2 q^2 + \cdots + b_s q^s,\ n \in \mathbb{N}_0,\ 0 \le b_k < q, k = 0, 1, 2, \dots, s$, we set

$$u(n) = u(b_0) + u(b_1)\mathfrak{p}^{-1} + \cdots + u(b_s)\mathfrak{p}^{-s}. \tag{2.3}$$

This defines $u(n)$ for all $n \in \mathbb{N}_0$. In general, it is not true that $u(m+n) = u(m)+u(n)$. But, if $r, k \in \mathbb{N}_0$ and $0 \le s < q^k$, then $u(rq^k + s) = u(r)\mathfrak{p}^{-k} + u(s)$. Further, it is also easy to verify that $u(n) = 0$ if and only if $n = 0$ and $\{u(\ell) + u(k) : k \in \mathbb{N}_0\} = \{u(k) : k \in \mathbb{N}_0\}$ for a fixed $\ell \in \mathbb{N}_0$. Hereafter we use the notation $\chi_n = \chi_{u(n)}$, $n \ge 0$.

Let the local field K be of characteristic $p > 0$ and $\zeta_0, \zeta_1, \zeta_2, \dots, \zeta_{c-1}$ be as above. We define a character χ on K as follows:

$$\chi(\zeta_\mu \mathfrak{p}^{-j}) = \begin{cases} \exp(2\pi i/p), & \mu = 0 \text{ and } j = 1, \\ 1, & \mu = 1, \dots, c-1 \text{ or } j \ne 1. \end{cases} \tag{2.4}$$

3. Multiresolution analysis with multiplicity r

Definition 3.1. A multiresolution analysis (MRA) of of multiplicity r of $L^2(K)$ is a nested sequence of closed subspaces $\{V_j : j \in \mathbb{Z}\}$ of $L^2(K)$ satisfying the following properties:

(a) $V_j \subset V_{j+1}$ for all $j \in \mathbb{Z}$;
(b) $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(K)$;
(c) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$;
(d) $f(x) \in V_j$ if and only if $f(\mathfrak{p}^{-1}x) \in V_{j+1}$ for all $j \in \mathbb{Z}$;
(e) there exist a vector $\Phi = (\varphi_1, \dots, \varphi_r)^T$, of L^2 functions such that $\{\varphi_\nu(x - u(k));\ \nu = 1, \dots, r,\ k \in \mathbb{N}_0\}$ forms an orthonormal basis for V_0.

The vector function $\Phi = (\varphi_1, \ldots, \varphi_r)^T$ is called a *scaling vector* and is said to generate multiresolution analysis. Since $\{\varphi_\nu(\mathfrak{p}^{-1}x - u(k)); \nu = 1, \ldots, r, \ k \in \mathbb{N}_0\}$ is an orthonormal basis for V_1 and $V_0 \subset V_1$, it follows that Φ satisfies a matrix-valued refinement equation of the form

$$\Phi(x) = \sqrt{q} \sum_{k \in \mathbb{N}_0} C_k \, \Phi\big(\mathfrak{p}^{-1}x - u(k)\big), \tag{3.1}$$

for some sequences of $r \times r$ matrices. Equation (3.1) can be rewritten in the frequency domain as

$$\hat{\Phi}(\xi) = H_0(\mathfrak{p}\xi)\hat{\Phi}(\mathfrak{p}\xi), \tag{3.2}$$

where

$$H_0(\xi) = \frac{1}{\sqrt{q}} \sum_{k \in \mathbb{N}_0} C_k \, \overline{\chi_k(\xi)}, \tag{3.3}$$

is an $r \times r$ matrix of integral periodic functions.

By the repeated application of (3.2), we have

$$\hat{\Phi}(\xi) = \left(\prod_{j=1}^{\infty} H_0(\mathfrak{p}^j \xi) \right) \hat{\Phi}(0).$$

If the infinite product $\prod_{j=1}^{\infty} H_0(\mathfrak{p}^j\xi)$ converges, then $\hat{\Phi}(\xi)$ is well defined and we say that $\hat{\Phi}(\xi)$ is generated by $H_0(\xi)$.

Definition 3.2. A vector function $\Psi = (\psi_1, \ldots, \psi_r)^T$ is called a *multiwavelet* associated with the scaling function Φ if for $\nu = 1, \ldots, r$ and $k \in \mathbb{N}_0$,

$$\Big\{\varphi_\nu\big(x - u(k)\big)\Big\} \cup \Big\{\psi_\nu\big(x - u(k)\big)\Big\}$$

is an orthonormal basis of V_1.

Let

$$W_j = \overline{\operatorname{span}}\big\{\psi_\nu\big(x - u(k)\big); \nu = 1, \ldots, r, \ k \in \mathbb{N}_0\big\}.$$

Then

$$V_{j+1} = V_j + W_j, \ j \in \mathbb{Z}. \tag{3.4}$$

Therefore, the properties of the Definition 3.1 imply that

$$V_j = \overline{\operatorname{span}} \bigcup_{j' < j} W_{j'}. \tag{3.5}$$

Since $W_0 \subset V_1$, it follows that Ψ can be expressed as

$$\hat{\Psi}(\xi) = H_\ell(\mathfrak{p}\xi)\hat{\Phi}(\mathfrak{p}\xi), \tag{3.6}$$

where

$$H_\ell(\xi) = \frac{1}{\sqrt{q}} \sum_{k \in \mathbb{N}_0} D_k \, \overline{\chi_k(\xi)}. \tag{3.7}$$

Theorem 3.1. *The necessary and sufficient condition for the system*

$$\big\{\varphi_\nu\big(x - u(k)\big); \nu = 1, \ldots, r, \ k \in \mathbb{N}_0\big\} \tag{3.8}$$

to constitute an orthonormal system is

$$\sum_{k \in \mathbb{N}_0} C_k C_{q\sigma + k}^T = q\delta_{0,\sigma} I_r, \ \sigma \in \mathbb{N}_0.$$

Proof. From the orthonormality of system (3.8), we have

$$\int_K \Phi(x)\Phi^T\big(x-u(k)\big)dx = q\delta_{o,k}I_r,\ k\in\mathbb{N}_0.$$

Using equation (3.1), we have

$$\begin{aligned}
q\int_K \sum_{k\in\mathbb{N}_0} C_k\,\Phi\big(\mathfrak{p}^{-1}x-u(k)\big) &\sum_{\sigma\in\mathbb{N}_0} \Phi^T\big(\mathfrak{p}^{-1}x-\mathfrak{p}^{-1}u(k)-u(\sigma)\big)C_\sigma^T\,dx\\
&= \frac{1}{q}\sum_{k\in\mathbb{N}_0}\sum_{\sigma\in\mathbb{N}_0} C_k\left(\int_K \Phi\big(x-u(k)\big)\Phi^T\big(x-u(qk)-u(\sigma)\big)\right)C_\sigma^T\,dx\\
&= \frac{1}{q}\sum_{k\in\mathbb{N}_0}\sum_{\sigma\in\mathbb{N}_0} C_k\delta_{k,qk+\sigma}C_\sigma^T\,dx\\
&= \frac{1}{q}\sum_{\sigma\in\mathbb{N}_0} C_{qk+\sigma}C_\sigma^T\,dx\\
&= \delta_{0,k}I_r.
\end{aligned}$$

This completes the proof. □

Theorem 3.2. *The necessary conditions for matrix D_k to produce orthogonal multiwavelets are*

$$\begin{aligned}
\sum_{k\in\mathbb{N}_0} D_k D_{q\sigma+k}^T &= \delta_{0,\sigma}I_r,\\
\sum_{k\in\mathbb{N}_0} C_k D_{q\sigma+k}^T &= \mathbf{0},\quad \sigma\in\mathbb{N}_0.
\end{aligned}$$

Proof. Corresponding to $\Phi(x)$, we have $\Psi = (\psi_1,\ldots,\psi_r)^T$. Thus, to generate an orthonormal basis of W_0, the translates of $\Psi\big(x-u(k)\big)$ should satisfy

$$\int_K \Psi(x)\Psi^T\big(x-u(k)\big)dx = q\delta_{o,k}I_r,\ k\in\mathbb{N}_0. \tag{3.9}$$

On the other hand, since $V_{-1} = V_0\oplus W_0$, so

$$\int_K \Phi(x)\Psi^T\big(x-u(k)\big)dx = \mathbf{0},\ k\in\mathbb{N}_0. \tag{3.10}$$

Using equation (3.6) and Theorem 3.1, the identities (3.9) and (3.10) give the desired result. This completes the proof. □

Let $\{\varphi_\nu : 1\le\nu\le r\}$ be functions in $L^2(K)$ such that the system (3.8) is an orthonormal. Let $V=\overline{\operatorname{span}}\big\{\varphi_\nu\big(\mathfrak{p}^{-1}x-u(k)\big);\nu=1,\ldots,r,\ k\in\mathbb{N}_0\big\}$. For $1\le\nu,\mu\le r$ and $0\le\ell\le q-1$, suppose there exists a sequence $\big\{h^\ell_{\nu\mu k}\big\}\in l^2(\mathbb{N}_0)$. Then equation (3.6) can be readdressed as

$$\psi^\ell_\nu(x) = q^{1/2}\sum_{\nu=1}^{r}\sum_{k\in\mathbb{N}_0} h^\ell_{\nu\mu k}\varphi_\nu\big(\mathfrak{p}^{-1}x-u(k)\big). \tag{3.11}$$

Taking Fourier transform on both sides of (3.11), we have

$$\begin{aligned}\hat{\psi}^{\ell}_{\nu}(\xi) &= \frac{1}{q}\sum_{\nu=1}^{r}\sum_{k\in\mathbb{N}_0} h^{\ell}_{\nu\mu k}\hat{\varphi}_{\nu}(\mathfrak{p}\xi)\overline{\chi_k(\mathfrak{p}\xi)}\\ &= \sum_{\nu=1}^{r} h^{\ell}_{\nu\mu}(\mathfrak{p}\xi)\hat{\varphi}_{\nu}(\mathfrak{p}\xi), \qquad (3.12)\end{aligned}$$

where

$$h^{\ell}_{\nu\mu}(\xi) = \frac{1}{q}\sum_{k\in\mathbb{N}_0} h^{\ell}_{\nu\mu k}\overline{\chi_k(\xi)},\ 1\le \nu,\mu\le r,\ 0\le \ell\le q-1, \qquad (3.13)$$

is an integral periodic functions in $L^2(\mathfrak{D})$. For $0\le \ell\le q-1$, we define the $r\times r$ matrices as

$$H_{\ell}(\xi) = \left[h^{\ell}_{\nu\mu k}(\xi)\right]^{r}_{\nu,\mu=1}. \qquad (3.14)$$

We now give a characterization of the orthonormality of the system defined by (3.8).

Theorem 3.3. *The system (3.8) is orthonormal if and only if*

$$\sum_{k\in\mathbb{N}_0}\hat{\varphi}_{\nu}\big(\xi+u(k)\big)\overline{\hat{\varphi}_{\mu}}\big(\xi+u(k)\big) = \delta_{\nu,\mu}, \quad 1\le \nu,\mu\le r.$$

Proof. Suppose that the system (3.8) is orthonormal. Since for $1\le \nu,\mu\le r$ and $t,s\in\mathbb{N}_0$,

$$\big\langle \varphi_{\nu}\big(x-u(s)\big), \varphi_{\mu}\big(x-u(t)\big)\big\rangle = \big\langle \varphi_{\nu}, \varphi_{\mu}\big(x-u(t-s)\big)\big\rangle,$$

Hence

$$\begin{aligned}\delta_{\nu,\mu}\delta_{0,s} &= \big\langle \varphi_{\nu}, \varphi_{\mu}\big(x-u(s)\big)\big\rangle\\ &= \Big\langle \hat{\varphi}_{\nu}, \big(\varphi_{\mu}\big(x-u(s)\big)\big)^{\wedge}\Big\rangle\\ &= \int_K \hat{\varphi}_{\nu}(\xi)\overline{\hat{\varphi}_{\mu}(\xi)\overline{\chi_s(\xi)}}\,d\xi\\ &= \int_{\mathfrak{D}}\left(\sum_{k\in\mathbb{N}_0}\hat{\varphi}_{\nu}\big(\xi+u(k)\big)\overline{\hat{\varphi}_{\mu}\big(\xi+u(k)\big)}\right)\chi_s(\xi)\,d\xi,\end{aligned}$$

where we have splited the integral over every coset of $\mathfrak{D}$ in K. Therefore the $\mathbb{N}_0-$periodic functions

$$G_{\nu,\mu}(\xi) = \sum_{k\in\mathbb{N}_0}\hat{\varphi}_{\nu}\big(\xi+u(k)\big)\overline{\hat{\varphi}_{\mu}\big(\xi+u(k)\big)},$$

has Fourier coefficients $\hat{G}_{\nu,\mu}\big(x-u(s)\big) = \delta_{\nu,\mu}\delta_{0,s}$ for $s\in\mathbb{N}_0$, which imply that $G_{\nu,\mu} = \delta_{\nu,\mu}$. This proves the necessary part. The sufficient part can be proved by reversing the above steps. This completes the proof. □

Theorem 3.4. *Let $\{\varphi_{\nu} : 1\le \nu\le r\}$ be functions in $L^2(K)$ such that the system $\big\{\varphi_{\nu}\big(\mathfrak{p}^{-1}x-u(k)\big); \nu=1,\dots,r,\ k\in\mathbb{N}_0\big\}$ is an orthonormal. Let V be its closed linear span. Further, let ψ^{ℓ}_{ν} and H_{ℓ} be defined by equations (3.11) and (3.13) respectively. Then the system*

$$\big\{\psi^{\ell}_{\nu}\big(x-u(k)\big); \nu=1,\dots,r,\ 0\le \ell\le q-1,\ k\in\mathbb{N}_0\big\} \qquad (3.15)$$

is an orthonormal system if and only if

$$\sum_{t=1}^{q-1} H_\ell\big(\xi+\mathfrak{p}u(t)\big)\overline{H_m}\big(\xi+\mathfrak{p}u(t)\big)=\delta_{\ell,m}I_r,\quad 0\le \ell,m\le q-1. \tag{3.16}$$

Proof. Using equation (3.12) and Theorem 3.3, we have for $0\le \ell,m\le q-1$, $1\le \nu,\mu\le r$ and $s\in\mathbb{N}_0$,

$$\begin{aligned}
\big\langle \psi_\mu^\ell,\psi_\nu^m\big(x-u(s)\big)\big\rangle &= \Big\langle \hat{\psi}_\mu^\ell,\big(\psi_\nu^m\big(x-u(s)\big)\big)^\wedge\Big\rangle\\
&= \int_K \hat{\psi}_\mu^\ell(\xi)\overline{\hat{\psi}_\nu^m(\xi)\overline{\chi_s(\xi)}}d\xi\\
&= \int_K \sum_{\sigma=1}^{r}\sum_{\gamma=1}^{r} h_{\mu\sigma}^\ell(\mathfrak{p}\xi)\overline{h_{\nu\gamma}^m(\mathfrak{p}\xi)}\hat{\varphi}_\sigma(\mathfrak{p}\xi)\overline{\hat{\varphi}_\gamma(\mathfrak{p}\xi)}\chi_s(\xi)d\xi\\
&= \int_{\mathfrak{D}} \sum_{k\in\mathbb{N}_0}\sum_{\sigma=1}^{r}\sum_{\gamma=1}^{r} h_{\mu\sigma}^\ell\left(\mathfrak{p}\big(\xi+u(k)\big)\right)\overline{h_{\nu\gamma}^m\left(\mathfrak{p}\big(\xi+u(k)\big)\right)}\\
&\quad\times\hat{\varphi}_\sigma\left(\mathfrak{p}\big(\xi+u(k)\big)\right)\overline{\hat{\varphi}_\gamma\left(\mathfrak{p}\big(\xi+u(k)\big)\right)}\chi_s\big(\xi+u(k)\big)d\xi\\
&= \int_{\mathfrak{D}} \sum_{t=1}^{q-1}\sum_{\sigma=1}^{r}\sum_{\gamma=1}^{r} h_{\mu\sigma}^\ell\left(\mathfrak{p}\big(\xi+u(t)\big)\right)\overline{h_{\nu\gamma}^m\left(\mathfrak{p}\big(\xi+u(t)\big)\right)}\\
&\quad\times\sum_{k\in\mathbb{N}_0}\hat{\varphi}_\sigma\left(\mathfrak{p}\big(\xi+u(k)\big)+u(t)\right)\overline{\hat{\varphi}_\gamma\left(\mathfrak{p}\big(\xi+u(k)\big)+u(t)\right)}\chi_s(\xi)d\xi\\
&= \int_{\mathfrak{D}} \sum_{t=1}^{q-1}\sum_{\sigma=1}^{r}\sum_{\gamma=1}^{r} h_{\mu\sigma}^\ell\left(\mathfrak{p}\big(\xi+u(t)\big)\right)\overline{h_{\nu\gamma}^m\left(\mathfrak{p}\big(\xi+u(t)\big)\right)}\delta_{\sigma,\gamma}\chi_s(\xi)d\xi\\
&= \int_{\mathfrak{D}} \sum_{t=1}^{q-1}\sum_{\sigma=1}^{r} h_{\mu\sigma}^\ell\left(\mathfrak{p}\big(\xi+u(t)\big)\right)\overline{h_{\nu\sigma}^m\left(\mathfrak{p}\big(\xi+u(t)\big)\right)}\chi_s(\xi)d\xi
\end{aligned}$$

Since the system $\{\chi_s(\xi);\xi\in K\}$ is an orthonormal system and $h_{\mu\sigma}^\ell$ and $h_{\nu\sigma}^m$ are integral periodic functions in $L^2(\mathfrak{D})$, we have

$$\big\langle \psi_\mu^\ell,\psi_\nu^m\big(x-u(s)\big)\big\rangle=\delta_{\ell,m}\delta_{\mu,\nu}\delta_{0,s}. \tag{3.17}$$

From equation (3.17), we can say that the system

$$\big\{\psi_\nu^\ell\big(x-u(k)\big);\nu=1,\dots,r,\ 0\le\ell\le q-1,\ k\in\mathbb{N}_0\big\}$$

will be an orthonormal system if and only if

$$\sum_{t=1}^{q-1} H_\ell\big(\xi+\mathfrak{p}u(t)\big)\overline{H_m}\big(\xi+\mathfrak{p}u(t)\big)=\delta_{\ell,m}I_r,\quad 0\le \ell,m\le q-1,$$

where I_r is an identity matrix of order r. This completes the proof. □

Theorem 3.5. *Under that assumptions of Theorem 3.4, if the system defined by (3.16) is orthonormal, then it will form an orthonormal basis of V.*

Proof. Suppose that the system (3.16) is orthonormal. Let $\psi\in V$. Then there exists a sequence $\{a_{\nu s}:1\le\nu\le r,\ s\in\mathbb{N}_0\}\in l^2(\mathbb{N}_0)$ such that

$$\psi(x)=q^{1/2}\sum_{\mu=1}^{r}\sum_{s\in\mathbb{N}_0} a_{\mu s}\varphi_\mu\big(\mathfrak{p}x+u(s)\big).$$

Assume that for all ν, ℓ, k, $\psi \perp \psi_\nu^\ell\big(x-u(k)\big)$. We claim that $\psi = 0$. For $0 \le \ell \le q-1$, $1 \le \nu \le r$ and $k \in \mathbb{N}_0$, we have

$$\begin{aligned}
0 &= \big\langle \psi_\nu^\ell\big(x-u(k)\big), \psi \big\rangle \\
&= \left\langle \psi_\nu^\ell\big(x-u(k)\big), q^{1/2}\sum_{\mu=1}^{r}\sum_{s\in\mathbb{N}_0} a_{\mu s}\varphi_\mu\big(\mathfrak{p}x+u(s)\big) \right\rangle \\
&= \left\langle \big(\psi_\nu^\ell\big(x-u(k)\big)\big)^\wedge, \left(q^{1/2}\sum_{\mu=1}^{r}\sum_{s\in\mathbb{N}_0} a_{\mu s}\varphi_\mu\big(\mathfrak{p}x+u(s)\big)\right)^\wedge \right\rangle \\
&= \frac{1}{q}\int_K \hat{\psi}_\nu^\ell(\xi)\overline{\chi_k(\xi)}\sum_{\mu=1}^{r}\sum_{s\in\mathbb{N}_0}\overline{a_{\mu s}}\,\chi_s(\mathfrak{p}\xi)\,\overline{\hat{\varphi}_\mu(\mathfrak{p}\xi)}d\xi \\
&= \frac{1}{q}\int_K \sum_{\sigma=1}^{r} h_{\nu\sigma}^\ell(\mathfrak{p}\xi)\hat{\varphi}_\sigma(\mathfrak{p}\xi)\overline{\chi_k(\xi)}\sum_{\mu=1}^{r}\sum_{s\in\mathbb{N}_0}\overline{a_{\mu s}}\,\chi_s(\mathfrak{p}\xi)\,\overline{\hat{\varphi}_\mu(\mathfrak{p}\xi)}d\xi \\
&= \frac{1}{q}\int_K \sum_{\sigma=1}^{r} h_{\nu\sigma}^\ell(\xi)\hat{\varphi}_\sigma(\xi)\sum_{\mu=1}^{r}\sum_{s\in\mathbb{N}_0}\overline{a_{\mu s}}\,\overline{\chi_k(\mathfrak{p}^{-1}\xi)}\chi_s(\xi)\,\overline{\hat{\varphi}_\mu(\xi)}d\xi \\
&= \frac{1}{q}\int_{\mathfrak{p}(\mathfrak{D}+u(k))}\sum_{t\in\mathbb{N}_0}\sum_{\sigma=1}^{r} h_{\nu\sigma}^\ell\big(\xi+u(t)\big)\hat{\varphi}_\sigma\big(\xi+u(t)\big) \\
&\qquad\times\sum_{\mu=1}^{r}\sum_{s\in\mathbb{N}_0}\overline{a_{\mu s}}\,\overline{\chi_k\left(\mathfrak{p}^{-1}\big(\xi+u(t)\big)\right)}\chi_s\big(\xi+u(t)\big)\,\overline{\hat{\varphi}_\mu\big(\xi+u(t)\big)}d\xi \\
&= \frac{1}{q}\int_{\mathfrak{p}(\mathfrak{D}+u(k))}\sum_{s\in\mathbb{N}_0}\sum_{\sigma=1}^{r}\sum_{\mu=1}^{r} h_{\nu\sigma}^\ell(\xi)\overline{a_{\mu s}}\left(\sum_{t\in\mathbb{N}_0}\hat{\varphi}_\sigma\big(\xi+u(t)\big)\overline{\hat{\varphi}_\mu\big(\xi+u(t)\big)}\right) \\
&\qquad\times\overline{\chi_k(\mathfrak{p}^{-1}\xi)}\chi_s(\xi)d\xi \\
&= \frac{1}{q}\int_{\mathfrak{p}(\mathfrak{D}+u(k))}\sum_{s\in\mathbb{N}_0}\sum_{\sigma=1}^{r} h_{\nu\sigma}^\ell(\xi)\overline{a_{\sigma s}}\,\overline{\chi_k(\mathfrak{p}^{-1}\xi)}\chi_s(\xi)d\xi \\
&= \frac{1}{q}\sum_{t'\in\mathbb{N}_0}\int_{\mathfrak{p}\mathfrak{D}}\sum_{s\in\mathbb{N}_0}\sum_{\sigma=1}^{r} h_{\nu\sigma}^\ell\big(\xi+\mathfrak{p}u(t')\big)\overline{a_{\sigma s}}\,\overline{\chi_k\left(\mathfrak{p}^{-1}\big(\xi+\mathfrak{p}u(t')\big)\right)}\chi_s\big(\xi+\mathfrak{p}u(t')\big)d\xi \\
&= \frac{1}{q}\int_{\mathfrak{p}\mathfrak{D}}\left(\sum_{t'\in\mathbb{N}_0}\sum_{s\in\mathbb{N}_0}\sum_{\sigma=1}^{r} h_{\nu\sigma}^\ell\big(\xi+\mathfrak{p}u(t')\big)\overline{a_{\sigma s}}\chi_s\big(\xi+\mathfrak{p}u(t')\big)\right)\overline{\chi_k(\mathfrak{p}^{-1}\xi)}d\xi.
\end{aligned}$$

Since $\{q^{1/2}\overline{\chi_k(\mathfrak{p}^{-1}\xi)} : k \in \mathbb{N}_0\}$ is an orthonormal basis of $L^2(\mathfrak{p}\mathfrak{D})$, the above system reduces to

$$\sum_{t'\in\mathbb{N}_0}\sum_{s\in\mathbb{N}_0}\sum_{\sigma=1}^{r} h_{\nu\sigma}^\ell\big(\xi+\mathfrak{p}u(t')\big)\overline{a_{\sigma s}}\chi_s\big(\xi+\mathfrak{p}u(t')\big) = 0 \; a.e. \quad \forall\,\nu,\ell \tag{3.18}$$

For $\sigma = 1,\dots,r$, let

$$A_\sigma(\xi) = \sum_{s\in\mathbb{N}_0} a_{\sigma s}\overline{\chi_s(\xi)} \tag{3.19}$$

Using equation (3.19) in (3.18), we have

$$\sum_{t'\in\mathbb{N}_0}\sum_{s\in\mathbb{N}_0} h_{\nu\sigma}^\ell\big(\xi+\mathfrak{p}u(t')\big)\overline{A_\sigma\big(\xi+\mathfrak{p}u(t')\big)} = 0 \; a.e. \quad \forall\,\nu,\ell \tag{3.20}$$

From equation (3.15), it is evident that for $1 \le \nu \le r$, $0 \le \ell \le q-1$ and for a.e $\xi \in K$, the vectors

$$\Big(h^{\ell}_{\nu\sigma}\big(\xi + \mathfrak{p}u(t')\big) : 1 \le \sigma \le r,\, t' \in \mathbb{N}_0\Big) \tag{3.21}$$

considered in the qr-dimensional space $\mathbb{C}^{qr}$, are mutually orthogonal with unit norm. Thus, these vectors form an orthonormal basis of $\mathbb{C}^{qr}$. One can easily see from the equation (3.18), that the vector

$$\Big(A_{\sigma}\big(\xi + \mathfrak{p}u(t')\big) : 1 \le \sigma \le r,\, t' \in \mathbb{N}_0\Big) \tag{3.22}$$

is orthogonal to each element of the orthonormal basis defined by (3.21) of the qr-dimensional space $\mathbb{C}^{qr}$. Thus $A_\sigma(\xi) = 0 \;\forall\; 1 \le \sigma \le r$. From this, we deduce that $\psi = 0$. Hence, our claim is true. This completes the proof. $\square$

Acknowledgments. We are deeply indebted to the referee for his/her valuable suggestions which greatly improved the presentation of this paper.

This work is supported by the UGC-BSR Research Start Up Grant (No. F.30-498/2019(BSR)) provided by UGC, Govt. of India.

References

[1] J.J. Benedetto, R.L. Benedetto, A wavelet theory for local fields and related groups, *J. Geom. Anal.* **14** (2004), 423–456.

[2] C.K. Chui, J. Lian, A study on orthonormal multiwavelets, *J. Appl. Numer. Math.* **20** (1996), 273–298.

[3] M. Cotronei, L.B. Montefusco, L. Puccio, Multiwavelet analysis and signal processing, *IEEE Trans. on Circuits and Systems II* **45** (1998), 970–987.

[4] J.S. Geronimo, D.P. Hardin, P.R. Massopust, Fractal functions and wavelet expansions based on several scaling functions, *J. Approx. Theory* **78** (1994), 373–401.

[5] Q. Jiang, Orthogonal multiwavelets with optimum time frequency resolution, *IEEE Trans. Signal Process* **46** (1998), 830–840.

[6] J. Krommweh, Tight frame characterization of multiwavelet vector functions in terms of the polyphase matrix, *Int. J. Wavelets, Multiresol. Informat. Process.* **7** (2009), 9–21.

[7] J. Lebrun, M. Vetterli, Balanced multiwavelets theory and design, *IEEE Trans. Signal Process* **46** (1998), 1119–1125.

[8] S.F. Lukomskii, Step refinable functions and orthogonal MRA on Vilenkin groups, *J. Fourier Anal. Appl.* **20** (2014), 42–65.

[9] D. Ramakrishnan, R.J. Valenza, *Fourier Analysis on Number Fields*, In: Graduate Texts in Mathematics, vol. 186, Springer, New York, 1999.

[10] M.H. Taibleson, *Fourier Analysis on Local Fields*, Princeton University Press, Princeton, 1975.

Mohammad Younus Bhat,
Department of Mathematical Sciences, Islamic University of Science and Technology,
Awantipora-192122, Jammu and Kashmir, India
E-mail address: gyounusg@gmail.com