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# Superposition operators on some new type of order modular spaces

E. Herawati and M. Mursaleen

Abstract. In this paper, we introduce $E$-valued sequence spaces, namely $X(E, \rho_f)$, where $E$ is a Riesz space, $\rho_f$ an order modular and $f$ is an order $\varphi$-function. Further, we characterize the classes $X(E, \rho_f)$ into $\ell_1(E, \rho_h)$ by superposition for $X \in \{c_0, \ell_1\}$.

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## 1. Introduction and preliminaries

Let $E$ be a Riesz space with a cone $E^+$. In this work, we introduce and study $E$-valued sequence spaces defined by an order modular $\rho_f$, where $f$ is an order $\varphi$-function.

For any $E$-sequence space $X(E)$ and a real number $\alpha > 0$ we define the set

$$X(E, \rho_f)) = \Big\{x = (x_k) \in \Omega(E) : \Big(\rho_f\big(\frac{x_k}{\alpha}\big)\Big) \in X(E)\Big\},$$

where $\rho_f\big(\frac{x_k}{\alpha}\big) = f\big(\frac{x_k}{\alpha}\big)$ for every $k$.

We show that $X(E, \rho_f)$ is an ideal Riesz space under coordinate wise ordering. We obtain the sufficient condition for the superposition operator acting from $X(E, \rho_f)$ into $\ell_1(E, \rho_h)$; and the necessary condition of the superposition operator acting from $X(E, \rho_f)$ into $\ell_1(E, \rho_h)$ where $f$ and $h$ are order $\varphi$-function. Finally, we obtain the sufficient and necessary condition of the superposition operator $P_g$ from $X(E, \rho_f)$ into $\ell_1(E, \rho_h)$ where $X \in \{c_0, \ell_1\}$.

An even function $f : E \to E^+$ is called *order $\varphi$-function* if $f$ is vanishing at zero, non-decreasing in $E^+$ and if $x_k \xrightarrow{o} t$ in $E$ implies $f(x_k) \xrightarrow{o} f(t)$. Furthermore, a function $\rho : E \to E^+$ is called an *order modular* if

(i) $\rho(t) = \theta$ if and only if $t = \theta$
(ii) $\rho(t) = \rho(|t|)$
(iii) $\rho(t_1 \vee t_2) \le \rho(t_1) + \rho(t_2)$.

Let $\mathbb{N}$ be the set of all natural numbers and let $g : \mathbb{N} \times E \to E$ be such that $g(k, 0) = 0$ for every $k \in \mathbb{N}$. If $X$ and $Y$ are two $E$-sequence spaces then the function $P_g : X \to Y$ is called a *superposition operator* if

$$P_g(x) = \big(g(k, x_k)\big) \text{ for every } x \in X.$$

In an implicit form, the superposition operator can be found in the terminology as "composite operator". For case $E = \mathbb{R}$, the characterization of $P_g$ was given by several authors. For example, Šragin [1] defined the superposition operator on Orlicz

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sequence space. Appel and Dedegich [2] defined for the classical sequence spaces $\ell_\infty, c_0$, and $\ell_p$ $(1 \leq p < \infty)$. Other researchers such as [3, 4, 5, 6, 7] have shown some properties of superposition operator $P_g$ on some real sequence spaces.

The notation $x_n \downarrow$ (resp. $x_n \uparrow$) is used for decreasing (resp. increasing) sequence in $E$ and $x_n \downarrow x$ (resp.$x_n \uparrow x$) provided that $x_n \downarrow$ and $\inf x_n = x$ exists in $E$ (resp. $x_n \uparrow$ and $\sup x_n = x$ exists in $E$). If every nonempty countable subset of $E$ that is bounded from above has supremum, then $E$ is called *Dedekind $\sigma$-complete* (or, equivalently, if $0 \leq x_n \uparrow \leq x$ implies the existence of sup $\{x_n : n \in \mathbb{N}\}$. A Riesz space $E$ is *Archimedean* if $\frac{1}{n}x \downarrow 0$ for each $x \in E^+$.

A sequence $(x_n) \subset E$ is said to be *$\sigma$-order convergent* to $x \in E$, denoted by $x_n \xrightarrow{o} x$ or $x = o - \lim\limits_{n\to\infty} x_n$, if there exists a sequence $p_n \downarrow 0$ in $E$ such that $|x_n - x| \leq p_n$ holds for all $n$. In this case $x \in E$ is called *order limit* of the sequence $(x_n)$ where $|x| = x \vee (-x)$ for any $x \in E$. A sequence can have at most one order limit. Indeed, if $x_n \xrightarrow{o} x$ and $x_n \xrightarrow{o} y$, then pick two sequences $(p_n)$ and $(q_n)$ with $p_n \downarrow 0$ and $q_n \downarrow 0$ such that $|x_n - x| \leq p_n$ and $|x_n - y| \leq q_n$ for all $n$ and note that

$$0 \leq |x - y| \leq |x_n - x| + |x_n - y| \leq (p_n + q_n) \downarrow 0 \text{ for all } n$$

implies $x = y$. The norm $\|\cdot\|$ in $E$ is called a *Riesz norm* if $|x| \leq |y|$ in $E$ implies $\|x\| \leq \|y\|$. Any Riesz space equipped with Riesz norm is called a *normed Riesz space.* However, norm convergence and $\sigma$-order convergence do not coincide. A complete normed Riesz space is called a *Banach lattice.* Furthermore, we say a Riesz norm $\|\cdot\|$ in $E$ has

(i) *the Riesz-Fischer property*, if for any sequence $(x_k) \subset E^+$ for which $\sum\limits_{k=1}^{\infty} \|x_k\| < \infty$, then the order limit of $\sum\limits_{k=1}^{\infty} x_k$ exists and $\left\|\sum\limits_{k=1}^{\infty} x_k\right\| \leq \sum\limits_{k=1}^{\infty} \|x_k\|$.

(ii) *the $\sigma$-Fatou property*, if $\|x_k\| \uparrow \|x\|$ whenever $\theta \leq x_k \uparrow x$

It is easy to see that $\sigma$-Fatou property property implies the Riesz-Fischer property, but not conversely.

For notation and the facts regarding Riesz spaces we refer to [8, 9]. We have

**Theorem 1.1.** *The normed Riesz space $E$ is a Banacah lattice if and only if $E$ has the Riesz-Fischer property.*

The space of all $E$-valued sequences is denoted by $\Omega(E)$. Any linear subspace of $\Omega(E)$ is called $E$-valued sequence space. We denote the $k$th term of a sequence $x$ in an $E$-sequence space by $x_k$ in $E$, and write $x = (x_k)$.

As examples of $E$-valued sequence spaces, we recall the following spaces [10, 11] which are needed in this paper.

- $\ell_1(E) = \left\{x = (x_k) \in \Omega(E) : \ (\exists a \in E),\ \sum\limits_{k=1}^{n} |x_k| \xrightarrow{o} a\right\}$
- $c(E) = \left\{x = (x_k) \in \Omega(E) : \ (\exists a \in E),\ x_k \xrightarrow{o} a\right\}$
- $c_0(E) = \left\{x = (x_k) \in \Omega(E) : \ x_k \xrightarrow{o} 0\right\}$.

Furthermore, let $E$ be a real Köthe sequence space and $\varphi$ be an Orlicz function. The space $E_\varphi$ introduced by Calederon-Lozanovskìı [12] is defined as follows

$$E_\varphi = \left\{x = (x_k) \in \Omega(\mathbb{R}) \ : \ I_\varphi(cx) < \infty \text{ for some } c > 0\right\}$$

where $I_\varphi$ is a convex semi-modular on $\Omega(\mathbb{R})$ defined by

$$I_\varphi = \begin{cases} \|\varphi(x)\|_E & \text{if } \varphi(x) \in E \\ \infty & \text{otherwise} \end{cases}$$

and equipped with Luxemberg-Nakano norm

$$\|x\|_\varphi = \inf\{\lambda > 0 \ : \ I_\varphi(x/\lambda) \leq 1\}$$

(se [13, 14, 15] and [16]).

**Lemma 1.2.** *Let $\rho$ be an order modular on a Riesz space $E$. Then*

(i) *if $0 \leq t_1 \leq t_2$ in $E$ implies $\rho(t_1) \leq \rho(t_2)$.*

(ii) *$\rho(\alpha t) = |\alpha|\ \rho(t)$ for $|\alpha| \leq 1$.*

(iii) *if $\alpha_i \in \mathbb{R}^+$ and $t_i \in E$ for every $i = 1, \cdots n$ such that $\sum_{k=1}^{n} \alpha_i = 1$, then $\rho\big(\sum_{k=1}^{n} \alpha_i t_i\big) \leq \sum_{k=1}^{n} \rho\big(t_i\big)$.*

For any $E$-sequence space $X(E)$ and a real number $\alpha > 0$ we define the set

$$X(E, \rho_f)) = \Big\{x = (x_k) \in \Omega(E) : \Big(\rho_f\big(\frac{x_k}{\alpha}\big)\Big) \in X(E)\Big\}$$

where $\rho_f\big(\frac{x_k}{\alpha}\big) = f\big(\frac{x_k}{\alpha}\big)$ for every $k$.

The following example shows that the set $X(E, \rho_f)$ may not be linear.

**Example 1.1.** Let $X(E) = \ell_\infty(E)$ and $u > 0$ be a unit in $E$. Then there exists a real number $\lambda > 0$ such that $|t| \leq \lambda u$ for every $t \in E$. Therefore, there exists a positive real number sequence $(\lambda_k)$ where $\lambda \leq \lambda_k \uparrow$. If we define order $\varphi$-function by

$$f(t) = \begin{cases} \dfrac{|t|}{2\lambda} & \text{if } t \leq \dfrac{\lambda u}{\alpha} \\[2ex] \dfrac{t}{2\lambda} & \text{if } t > \dfrac{\lambda u}{\alpha} \end{cases}$$

and take a sequence $x = (\lambda u_k)$ where $u_k = u$ for every $k$, then $\sup_{k\geq 1} f\big(\frac{\lambda u_k}{\alpha}\big) = \frac{u}{2\alpha} \in E$. Hence $x \in \ell_\infty(E, \rho_f)$. But $2x = (2\lambda u_k) \notin \ell_\infty(E, \rho_f)$ for every $k$, because $\sup_{k\geq 1} f\big(\frac{2\lambda u_k}{\alpha}\big) \leq \sup_{k\geq 1} f\big(\frac{2\lambda_k u}{\alpha}\big) = \frac{u}{\alpha} \sup_{k\geq 1} \lambda_k$ undefined in $E$. Therefore $\ell_\infty(E, \rho_f)$ is not a sequence space.

It is easy to check that $X(E, \rho_f)$ is $E$-sequence space under restriction on $X(E)$.

## 2. Topological properties of $X(E, \rho_f)$

We have the following basic result (see [4]).

**Theorem 2.1.** *If $X(E)$ be an ideal in $\Omega(E)$, then $X(E, \rho_f)$ is an ideal Riesz space under coordinate-wise ordering.*

Now we prove the following:

**Theorem 2.2.** *Let $X(E) = \big(X(E), \|\cdot\|_{X(E)}\big)$ be an ideal normed Riesz space in $\Omega(E)$. If $X(E)$ has the $\sigma$-Fatou property, then $X(E, \rho_f)$ is a Banach lattice equipped* *with the following norm*

$$\|x\| = \inf\left\{\varepsilon > 0 \ : \ \|\rho_f\big(\frac{x}{\alpha}\big)\|_{X(E)} \le 1\right\}$$

*for every* $x \in X(E, \rho_f)$, *where* $\rho_f\big(\frac{x}{\alpha}\big) = \Big(\rho_f\big(\frac{x_k}{\alpha}\big)\Big)$.

*Proof.* Theorem 2.1 gives that $X(E, \rho_f)$ is an ideal Riesz space. It is easy to show that $X(E, \rho_f)$ is a normed Riesz space. Furthermore, we will show the norm $\|\cdot\|$ has the $\sigma$-Fatou property. If it fails, then there exists a sequence $(x^n) \subset X(E, \rho_f)$

$$x^n = (x^n_k) = (x^n_1,\ x^n_2,\ \cdots)$$

for every $n$ and $x \in \Omega(E)$; where $0 \le x^n \uparrow_n \le x$ and $\|x^n\| \nrightarrow \|x\|$ as $n \to \infty$. Since $\|x^n\|\uparrow_n < \infty$, it follows that there exists $r \in \mathbb{R}$ such that $r = \sup\{\|x^n\| \ : \ n \in \mathbb{N}\}$. Since $f$ is a nondecreasing function on $E^+$, we have $\rho_f\big(x^n/(r+1)\big) \uparrow_n \rho_f(x/r)$. By the definition of $\|\cdot\|_{X(E)}$, we have

$$\Big\|\rho_f\Big(\frac{x^n}{r+1}\Big)\Big\|_{X(E)} \le \ \Big\|\rho_f\Big(\frac{x^n}{\|x^n\|_\phi}\Big)\Big\|_{X(E)} \le 1.$$

for every $n \in \mathbb{N}$. Next, since $X(E)$ has the $\sigma$-Fatou property, it follows that

$$\Big\|\rho_f\Big(\frac{x}{r+1}\Big\|_{X(E)} = \lim_{n\to\infty} \Big\|\rho_f\Big(\frac{x^n}{r+1}\Big)\Big\|_{X(E)} \le 1.$$

Hence, $\|x^n\| \le r+1$. Since $\|x^n\| \nrightarrow_n \|x\|$, there exists a real number $\varepsilon_0 > 0$ and subsequence $\big(\|x^{n_j}\|\big) \subset (\| x^n\|)$ such that $\|x^{n_j}\| \le \|x\| - \varepsilon_0$ for every $j \in \mathbb{N}$. Therefore

$$\Big\|\rho_\phi\Big(\frac{x^{n_j}}{\|x\| - \varepsilon_0}\Big)\Big\|_{X(E)} = \ \Big\|\rho_\phi\Big(\frac{x^{n_j}}{\|x^{n_j}\|}\Big)\Big\|_{X(E)} \le 1.$$

Since $\big(\|x\| - \varepsilon_0\big)^{-1} x^{n_j} \uparrow_j \big(\|x\| - \varepsilon_0\big)^{-1} x$ and $X(E)$ has $\sigma$-Fatou, we get

$$\Big\|\rho_\phi\Big(\frac{x}{\|x\| - \varepsilon_0}\Big)\Big\|_{X(E)} = \lim_{j\to\infty}\Big\|\rho_\phi\Big(\frac{x^{n_j}}{\|x^{n_j}\|}\Big)\Big\|_{X(E)} \le 1.$$

Hence, $\|x\| \le \|x\| - \varepsilon_0$ which is impossible. Therefore, $X(E, \rho_f)$ has the $\sigma$-Fatou property. Consequently, $X((E, \rho_f)$ is a Banach lattice. □

## 3. Superposition operators on $X(E, \rho_f)$

Let $E$ be a Dedekind $\sigma$-complete Riesz space equipped with the Riesz norm $\|\cdot\|_E$ and let $g : \mathbb{N} \times E \to E$ be such that $g(k, 0) = 0$ for each $k \in \mathbb{N}$.

We begin with the following theorem which gives the sufficient condition for the superposition operator acting from $X(E, \rho_f)$ into $\ell_1(E, \rho_h)$ where $f$ and $h$ are order $\varphi$-function.

**Theorem 3.1.** *Let $X(E, \rho_f)$ be an $E$-valued sequence space, where $X \in \{c_0, \ell_1\}$. If there exist an $E$-sequence $d = (d_k) \in \ell_1(E)^+, c \in E^+$ and a non negative real number $\gamma$ such that for every $t \in E$, $f(t) \le c$ implies $h\big(g(k,t)\big) \le d_k + \gamma\ f(t)$. Then the superposition operator $P_g$ maps the space $X(E, \rho_f)$ into the space $\ell_1(E, \rho_h)$.*

*Proof.* Firstly, consider the case $X = \ell_1$. For any $x = (x_k) \in \ell_1(E, \rho_f)$, we will show that there exists $x_0 \in E$ such that

$$\sum_{k=1}^{n} h\big(g(k, x_k)\big) \xrightarrow{o}_n x_0.$$

Since $x = (x_k) \in \ell_1(E, \rho_f)$, it follows that there exists $c_1 \in E$ such that $\sum_{k=1}^{n} (x_k) \uparrow_n c_1$. Therefore, under the hypotheses

$$h\big(g(k, x_k)\big) \le d_k + \gamma f(x_k) \text{ for every } k,$$

we have

$$\sum_{k=1}^{n} h\big(g(k, x_k)\big) \le \sum_{k=1}^{n} d_k + \gamma \sum_{k=1}^{n} f(x_k) \le \sum_{k=1}^{n} d_k + \gamma c_1.$$

Since $(d_k) \in \ell_1(E)^+$, we see that there exists $c_2 \in E^+$ such that $\sum_{k=1}^{n} d_k \uparrow_n c_2$. Then

$$\sum_{k=1}^{n} h\big(g(k, x_k)\big) \uparrow_n \le c \text{ where } c = c_2 + \gamma c_1.$$

Since $E$ is Dedekind $\sigma$-complete, there exists $x_0 \in E$ such that $\sum_{k=1}^{n} h\big(g(k, x_k)\big) \uparrow x_0$, which shows that $P_g(x) = \big(g(k, x_k)\big) \in \ell_1(E, \rho_h)$ for every $x \in \ell_1(E, \rho_f)$.

Secondly, suppose $X = c_0$ and consider $x = (x_k) \in c_0(E, \rho_f)$, then there exists a sequence $p_k \downarrow 0$ in $E$ such that $f(x_k) \le p_k$ for every $k$. It follows that $f(x_k) \le p_1$ for every $k \ge k_0$. Furthermore, under the hypotheses and the same argument of the first case, we have $P_g(f) = \big(g(k, x_k)\big) \in \ell_1(E, \rho_h)$. This shows that the superposition operator $P_g$ maps the space $X(E, \rho_f)$ into the space $\ell_1(E, \rho_h)$.

This completes the proof. □

The next theorem gives the necessary condition of the superposition operator acting from $X(E, \rho_f)$ into $\ell_1(E, \rho_h)$ where $f$ and $h$ are order $\varphi$-function.

**Theorem 3.2.** *Let $X(E)$ be $E$-valued sequence space, where $X \in \{c_0, \ell_1\}$ and let $g : \mathbb{N} \times E \to E$ be a function such that $g(k, 0) = 0$ for every $k \in \mathbb{N}$. If $P_g : X(E, \rho_f) \to \ell_1(E, \rho_h)$ is superposition operator, then there exist a real number $\delta > 0$ and $c \in E^+$ and sequence $a = (a_k) \in \ell_1(E)^+$ such that for every sequence $(x^n) \in X(E, \rho_f)$, $\sum_{k=1}^{n} f\big(\delta^{-1} x_k^n\big) \le c$ implies $\sum_{k=1}^{n} h\big(g(k, x_k)\big) \le \sum_{k=1}^{n} a_k$, for every $n \in \mathbb{N}$.*

*Proof.* We take the case $X = \ell_1$. That is $P_g : \ell_1(E, \rho_f) \to \ell_1(E, \rho_h)$ is a superposition operator. Let $x = (x_k) \in \ell_1(E, \rho_f)$ where $\ell_1(E, \rho_f)$ has $\sigma$-Fatou property. Then there exists $x_0 \in E$ such that $\|s_n\| \uparrow_n \|x_0\|$, where $s_n = \sum_{k=1}^{n} f(x_k)$. Therefore, there exists real number $\delta > 0$ such that $0 < \delta < \|x_0\|$. Furthermore, for any $n \in \mathbb{N}$, we define the function $G_n : \Omega(E) \to E$ by

$$G_n(x = \sum_{j=1}^{n} \varphi\big(g(j, x_j)\big).$$

For the real number $\delta > 0$, we define the operator

$$F : \Omega(E) \to E^+ \text{ by } F(x) = \sum_{j=1}^{n} f\Big(\frac{x_j}{\delta}\Big).$$

Since $\Phi\Big(\Omega(E)\Big) \subset E$ is an order bounded subset, it follows that there exists $v \in E^+$ such that $F\big(\Omega(E)\big) \leq c$. If for any $k, n \in \mathbb{N}$, we take $z_n^k = \sum_{j=1}^{n} f\big(\delta^{-1}x_j^k\big)$, then for every $k \in \mathbb{N}$ we get $z_n^k \uparrow_n \leq c$. Since $E$ is a Dedekind $\sigma$-complete, we see that for every $k \in \mathbb{N}$ there exists $y = (y_k)$ such that $z_n^k \uparrow_n y_k$. It means $z^k = (z_n^k)_{n\geq 1} \in \ell_1(E, \rho_f)$ and $\Phi(z^k) \uparrow_k F(y)$. Then

$$F(y) = \sup\{F(z^k) \mid F(z^k) \leq h\}.$$

Therefore, for every $k \in \mathbb{N}$ there exists $y_k \in E$ such that $\sum_{k=1}^{p} f\big(\frac{x_k}{\delta}\big) \leq c$ and

$$\sum_{k=1}^{p} h\big(g(k, y_k)\big) = \sup \Big\{ \sum_{k=1}^{p} h\big(g(k, z_n^k)\big) \mid \sum_{k=1}^{p} f\big(\delta^{-1}z_n^k\big) \leq h \Big\}.$$

Next, we shall show that there exists $x_0 \in E$ such that $\sum_{k=1}^{p} h\big(g(k, y_k)\big) \xrightarrow{o}_p x_0$.

Since $F(y) = \sum_{k=1}^{n} f\big(\frac{y_k}{\delta}\big) \leq c$ in the Dedekind $\sigma$-complete Riesz space $E$, there exists $u \in E^+$ such that $\sum_{k=1}^{n} f\big(\frac{y_k}{\delta}\big) \uparrow_p u$. Hence, a sequence $y = (y_k) \in \ell_1(E, \rho_f)$. Since $P_g$ is a superposition operator, there exists $x_0 \in E$ such that $\sum_{k=1}^{n} f\big(\frac{y_k}{\delta}\big) \xrightarrow{o}_p f_0$.

Let $a_k = h\big(g(k, y_k)\big)$ be any sequence, then there exist a sequence $(a_k) \in \ell_1(E, \rho_h)^+$ and a real number $\delta > 0$ such that

$$\sum_{k=1}^{n} h\big(g((k, x_k) \leq \sum_{k=1}^{n} a_k \text{ for every } n \in \mathbb{N}.$$

This proves the theorem. ☐

By using Theorem 3.2, we get the following theorem:

**Theorem 3.3.** *Let $X(E)$ be $E$-valued sequence space, where $X \in \{\ell_1, c_0\}$ and let $g : \mathbb{N} \times E \to E$ be such that $g(k, 0) = 0$ for every $k \in \mathbb{N}$. If $P_g : X(E, \rho_f) \to \ell_1(E, \rho_h)$ is a superposition operator, then there exist an $E$- sequence $(d_k) \in \ell_1(E)^+$, a real number $\gamma \geq 0$ and $c \in E^+$ such that $f(t) \leq c$ for every $t \in E$ which implies that $h\big(g(k, t)\big) \leq d_k + \gamma f(t)$.*

*In this case $\gamma = 0$, whenever $X = c_0$.*

*Proof.* Let $P_g : X(E, \rho_f) \to \ell_1(E, \rho_h)$ be a superposition operator where $X = \ell_1$ and let $x = (x_k) \in \ell_1(E, \rho_f)$ be a any sequence. Then under the same arguments as in Theorem 3.2, there exist $\delta \in (0, 1)$ and $c \in E^+$ such that $f(x_k) \leq \delta c$. We define

$$\psi(k, t) = \begin{cases} h\big(g(k, t)\big) - \delta^{-1} f(t/\delta) & \text{if } h\big(g(k, t)\big) \geq \delta^{-1} f(t/\delta) \\ 0 & \text{otherwise} \end{cases}$$

Therefore $\psi(k, t) \geq 0$ for every $k \in \mathbb{N}$. Since the set $\big\{t \in E \ : \ f(t/\delta) \leq c\big\} = \big\{t \in E \ : \ t/\delta \leq f^{-1}(c)\big\}$ is bounded from above in the Dedekind $\sigma$-complete Riesz space $E$, there exists $v = \sup\big\{t \in E \ : \ t/\delta \leq f^{-1}(h)\big\} \in E$. Next, let $t = z_k$. Then for every $N \in \mathbb{N}$ we should show the following summation $\sum_{k=1}^{N} f\Big(\frac{z_k}{\delta}\Big)$ could be decomposed as

$$\sum_{k=1}^{N} f\left(\frac{z_k}{\delta}\right) = \sum_{k=1}^{N_1} f\left(\frac{z_k}{\delta}\right) + \sum_{k=N_1+1}^{N-2} f\left(\frac{z_k}{\delta}\right) + \cdots + \sum_{k=N_{\ell-1}+1}^{N-\ell} f\left(\frac{z_k}{\delta}\right)$$
$$= \sum_{(1)} f\left(\frac{z_k}{\delta}\right) + \sum_{(2)} f\left(\frac{z_k}{\delta}\right) + \cdots + \sum_{(\ell)} f\left(\frac{z_k}{\delta}\right)$$

where

$$\sum_{(i)} f\left(\frac{z_k}{\delta}\right) = \sum_{k=N_{i-1}+1}^{N_i} f\left(\frac{z_k}{\delta}\right) < c \text{ for every } i = 1, 2, \cdots, \ell - 1 \text{ with}$$
$$N_0 = 0 \text{ and } \frac{h}{2} \le \sum_{(\ell)} f\left(\frac{z_k}{\delta}\right) \le c.$$

If $\sum_{k=1}^{N} f\left(\frac{z_k}{\delta}\right) \le h$, then $\ell = 1$ and $\sum_{(1)} f\left(\frac{z_k}{\delta}\right) = \sum_{k=1}^{N} f\left(\frac{z_k}{\delta}\right)$.

If it is not so, take the least natural number $N_1$ such that

$$\sum_{k=1}^{N_1} f\left(\frac{z_k}{\delta}\right) \ge c \text{ and } \sum_{k=1}^{N_1-1} f\left(\frac{z_k}{\delta}\right) < c.$$

Since $f\left(z_{N_1}\right) \le c$, we have

$$\sum_{k=1}^{N_1} f\left(\frac{z_k}{\delta}\right) = \sum_{k=1}^{N_1-1} f\left(\frac{z_{N_1}}{\delta}\right) + f\left(z_{N_1}\right) < 2c.$$

Next, if $\sum_{(1)} f\left(\frac{z_k}{\delta}\right) = \sum_{k=1}^{N_1} f\left(\frac{z_k}{\delta}\right)$ and $\sum_{k=N_1+1}^{N} f\left(\frac{z_k}{\delta}\right) \le h$, then $\ell = 2$.

If it is not so, take the least natural number $N_2$ such that

$$\sum_{N_1+1}^{N_2} f\left(\frac{z_k}{\delta}\right) \ge c \text{ and } \sum_{N_1+1}^{N_2-1} f\left(\frac{z_k}{\delta}\right) < c.$$

Since $f\left(z_{N_2}\right) < c$, we see that

$$c \le \sum_{k=N_1+1}^{N_2} f\left(\frac{z_k}{\delta}\right) = \sum_{k=N_1+1}^{N_2-1} f\left(\frac{z_2}{\delta}\right) + f\left(z_{N_2}\right) < 2c.$$

If $\sum_{(2)} f\left(\frac{z_k}{\delta}\right) = \sum_{k=N_1+1}^{N_2} f\left(\frac{z_k}{\delta}\right)$ and $\sum_{k=N_2+1}^{N} f\left(\frac{z_k}{\delta}\right) < c$, then $\ell = 3$,

If we continue this process then the decomposition like above will be obtained.

As in the proof of Theorem 3.3, there exists a sequence $(a_k) \in \ell_1(E, \rho_h)^+$. Further, if we put $c_k = \psi(k, z_k)$, we shall show that for every $N \in \mathbb{N}$

$$\sum_{k=1}^{N} c_k \le \sum_{k=1}^{N} a_k.$$

Since

$$h\big(g(k, z_k)\big) \ge f\left(\frac{z_k}{\delta}\right) \text{ for every } k \in \mathbb{N},$$

we have

$$\sum_{k=1}^{N} c_k = \sum_{k=1}^{N} h\big(g(k, z_k)\big) - \sum_{k=1}^{N} f\Big(\frac{z_k}{\delta}\Big) = \sum_{j=1}^{\ell_1} h\big(g(k, z_k)\big) - \sum_{j=1}^{\ell-1} f\Big(\frac{z_k}{\delta}\Big)$$

for every $j$, where $1 \leq j \leq \ell$. Putting $z_j = (z_k^{(j)})$ by

$$z_k^{(j)} = \begin{cases} z_k & \text{for some index } k \text{ in } \sum_j \\ 1 & \text{otherwise} \end{cases}$$

Then for every $j$, $1 \leq j \leq \ell$, we obtain

$$\sum_{k=1}^{N} f\Big(\frac{z_k^{(j)}}{\delta}\Big) = \sum_j f\Big(\frac{z_k}{\delta}\Big) \leq c \text{ and } \sum_{k=1}^{N} h\big(g(k, z_k^{(j)})\big) = \sum_j h\big(g(k, z_k^{(j)})\big),$$

which means $\sum_{k=1}^{N} h\big(g(k, z_k^{(j)})\big) \leq \sum_{k=1}^{N} a_k$. Furthermore, for $j$, $1 \leq j \leq \ell$, we have

$$\sum_j h\big(g(k, z_k^{(j)})\big) \leq \sum_{k=1}^{N} a_k.$$

Therefore, for every $N \in \mathbb{N}$

$$\begin{aligned} \sum_{k=1}^{N} c_k &\leq \ell \sum_j h\big(g(k, z_k^{(j)})\big) - (\ell - 1) \sum_j f\Big(\frac{z_k}{\delta}\Big) \\ &\leq \ell \sum_{k=1}^{N} a_k - (\ell - 1)c. \end{aligned}$$

Since $\sum_{k=1}^{N} a_k \uparrow c$ for $N \to \infty$ and $E$ is a Dedekind $\sigma$-complete, there exists $v \in E$ such that $\sum_{k=1}^{N} c_k \xrightarrow{o} v$ in $E$ for $N \to \infty$. Hence, there exists $(c_k) \in \ell(E)^+$.

Furthermore, since for every $k \in \mathbb{N}$, $h\big(g(k, t)\big) \geq f(t)$, we have

$$h\big(g(k, t)\big) \leq c_k + \delta^{-1} f(t) \text{ for } f(t) \leq f\Big(\frac{t}{\delta}\Big) \leq \delta c.$$

On the other hand, if $h\big(g(k, t)\big) < \delta^{-1} f(t)$, then $h\big(g(k, t)\big) < c_k + 2\delta^{-1} f(t)$. Take $\gamma = 2\delta^{-1}$, then there exists $\gamma > 0$ such that

$$h\big(g(k, t)\big) < c_k + \gamma f(t) \text{ for } f(t) \leq c.$$

This completes the proof. □

Finally, we shall apply the Theorem 3.1 and 3.3 to obtain the sufficient and necessary conditions of the superposition operator $P_g$ from $X(E, \rho_f)$ into $\ell_1(E, \rho_h)$, where $X \in \{c_0, \ell_1\}$.

**Theorem 3.4.** *Let $X \in \{c_0, \ell_1\}$. Then $P_g : X(E, \rho_f) \to \ell_1(E, \rho_h)$ is a superposition operator if and only if there exist a sequence $(d_k) \in \ell_1(E)^+$, $c \in E^+$ and a real number $\gamma \geq 0$ such that*

*$h\big(g(k, t)\big) \leq d_k + \gamma\ f(t)$ whenever $f(t) \leq c$.*

*In this case $\gamma = 0$, whenever $X = c_0$.*

## 4. Conclusion

In this paper, we introduced $E$-valued sequence spaces, namely $X(E,\rho_f)$, where $E$ is a Riesz space, $\rho_f$ an order modular and $f$ is an order $\varphi$-function. We proved that If an ideal normed Riesz space $X(E)$ has the $\sigma$-Fatou property, then $X(E,\rho_f)$ is a Banach lattice. Further, we have characterized here the classes $X(E,\rho_f)$ into $\ell_1(E,\rho_h)$ by superposition for $X \in \{c_0, \ell_1\}$.

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(E. Herawati) Department of Mathematics, Universitas Sumatera Utara, Medan 20155, Indonesia
*E-mail address*: herawaty.elv@gmail.com or elvina@usu.ac.id

(M. Mursaleen) Department of Medical Research, China Medical University Hospital, China Medical University (Taiwan), Taichung, Taiwan
Department of Mathematics, Aligarh Muslim University, Aligarh 202002, India
*E-mail address*: mursaleenm@gmail.com