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New generalized inequalities using arbitrary operator means and their duals

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Abstract. In this article, we present some operator inequalities via arbitrary operator means and unital positive linear maps. For instance, we show that if $A, B \in \mathbb{B}(\mathbb{H})$ are two positive invertible operators such that $0 < m \leq A, B \leq M$ and σ is an arbitrary operator mean, then

$$\Phi^p(A\sigma B) \leq K^p(h)\Phi^p(B\sigma^{\perp} A),$$

where $\sigma^{\perp}$ is dual σ, $p \geq 0$ and $K(h) = \frac{(M+m)^2}{4Mm}$ is the classical Kantorovich constant. We also generalize the above inequality for two arbitrary means σ_1, σ_2 which lie between σ and $\sigma^{\perp}$.

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1. Introduction

In this paper, $\mathbb{B}(\mathbb{H})$ denote the C^*-algebra of all bounded linear operators on a complex Hilbert space $(\mathbb{H}, \langle \cdot, \cdot \rangle)$. I stands for the identity operator. A self-adjoint operator $A \in \mathbb{B}(\mathbb{H})$ is said to be positive if $\langle Ax, x \rangle \geq 0$ for all $x \in \mathbb{H}$, and in this case we write $A \geq 0$. For self-adjoint operators $A, B \in \mathbb{B}(\mathbb{H})$, the order relation $A \leq B$ means that $B - A \geq 0$. A linear map Φ is positive if $\Phi(A) \geq 0$ whenever $A \geq 0$. It is said to be unital provided that it preserves the identity operator, that is, $\Phi(I) = I$.

The axiomatic theory for pairs of positive operators has been developed by Kubo and Ando [10]. If $A, B \in \mathbb{B}(\mathbb{H})$ are two positive invertible operators, then the $\nu-$weighted arithmetic mean, geometric mean and harmonic mean of A and B are denoted by $A\nabla_\nu B$, $A\sharp_\nu B$ $A!_\nu B$, respectively, and are defined as follows

$$A\nabla_\nu B = \nu A + (1-\nu)B, \qquad A\sharp_\nu B = A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\nu}A^{\frac{1}{2}},$$

and

$$A!_\nu B = (\nu A^{-1} + (1-\nu)B^{-1})^{-1}.$$

When $\nu = \frac{1}{2}$, we write $A\nabla B$, $A\sharp B$ and $A!B$ for the arithmetic mean, geometric mean and harmonic mean, respectively. The $\nu-$weighted arithmetic-geometric (AM-GM) operator inequality, which is proved in [16] says that if $A, B \in \mathbb{B}(\mathbb{H})$ are two positive operators and $0 \leq \nu \leq 1$, then $A\sharp_\nu B \leq A\nabla_\nu B$. For a particular case, when $\nu = \frac{1}{2}$, we obtain the AM-GM operator inequality

$$A\sharp B \leq \frac{A+B}{2}. \tag{1}$$

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For two positive operators $A, B \in \mathbb{B}(\mathbb{H})$, the Löwner–Heinz inequality states that, if $A \leq B$, then

$$A^p \leq B^p, \qquad (0 \leq p \leq 1). \tag{2}$$

In general (2) is not true for $p > 1$.
Lin [13, Theorem 2.1] showed a squaring of a reverse of (1), namely that the inequality

$$\Phi^2\left(\frac{A+B}{2}\right) \leq \left(\frac{(M+m)^2}{4Mm}\right)^2 \Phi^2(A\sharp B) \tag{3}$$

as well as

$$\Phi^2\left(\frac{A+B}{2}\right) \leq \left(\frac{(M+m)^2}{4Mm}\right)^2 (\Phi(A)\sharp\Phi(B))^2, \tag{4}$$

where Φ is a positive unital linear map.
The Löwner–Heinz inequality and two inequalities (3) and (4) follow that for $0 < p \leq 2$,

$$\Phi^p\left(\frac{A+B}{2}\right) \leq \left(\frac{(M+m)^2}{4Mm}\right)^p \Phi^p(A\sharp B) \tag{5}$$

and

$$\Phi^p\left(\frac{A+B}{2}\right) \leq \left(\frac{(M+m)^2}{4Mm}\right)^p (\Phi(A)\sharp\Phi(B))^p. \tag{6}$$

In [6], the authors showed that inequalities (5) and (6) for $p \geq 2$ hold.
For more improvements and refinements on the above inequalities see [7, 14, 15] and references therein.
Let σ be an operator mean with the representing function f. The operator mean with the representing function $\frac{t}{f(t)}$ is called the dual of σ and denoted by $\sigma^\perp$. For $A, B \in \mathbb{B}(\mathbb{H})$,

$$A\sigma^\perp B = (B^{-1}\sigma A^{-1})^{-1}.$$

It is trivial that for two invertible operators $A, B \in \mathbb{B}(\mathbb{H})$, $A\nabla^\perp B = A!B$ and $A!B \leq A\sharp B$.
Let $0 < m \leq A, B \leq M$, Φ be a positive unital linear map and σ, τ be two arbitrary means between the harmonic and arithmetic means. In [8], the authors obtained the following inequality:

$$\Phi^2(A\sigma B) \leq K^2(h)\Phi^2(A\tau B), \tag{7}$$

where $K(h) = \frac{(h+1)^2}{4h}$ with $h = \frac{M}{m}$ is the Kantorovich constant.
The authors in [5] generalized inequality (7) for the higher powers as follows:

$$\Phi^p(A\sigma B) \leq K^p(h)\Phi^p(A\tau B), \tag{8}$$

where $p > 0$.
Motivated by the above discussion, in this paper we first obtain the following inequality:

$$\Phi^2(A\sigma B) \leq K^2(h)\Phi^2(B\sigma^\perp A), \tag{9}$$

where $0 < m \le A, B \le M$, σ is an arbitrary mean and $\sigma^{\perp}$ is its dual and $K(h) = \frac{(M+m)^2}{4Mm}$ is the Kantorovich constant. Then, we generalize inequality (9) for two arbitrary means σ_1 and σ_2 between σ and $\sigma^{\perp}$.

2. Main results

To obtain the main results we need to recall the following Lemmas.

Lemma 2.1. [3](*Choi's inequality*) *Let $A \in \mathbb{B}(\mathbb{H})$ be positive and Φ be a positive unital linear map. Then*

$$\Phi(A)^{-1} \le \Phi\left(A^{-1}\right). \tag{10}$$

Lemma 2.2. [16] *Suppose that $0 < m \le A \le M$. Then*

$$A + MmA^{-1} \le M + m.$$

Lemma 2.3. [4, 1, 2] *Let $A, B \in \mathbb{B}(\mathbb{H})$ be positive and $\lambda > 0$. Then*
(i) $||\mathrm{AB}|| \le \frac{1}{4}||\mathrm{A}+\mathrm{B}||^2$.
(ii) *If $\lambda > 1$, then $||A^{\lambda} + B^{\lambda}|| \le ||(A+B)^{\lambda}||$.*
(iii) $\mathrm{A} \le \lambda\mathrm{B}$ *if and only if* $||A^{\frac{1}{2}}B^{-\frac{1}{2}}|| \le \lambda^{\frac{1}{2}}$.

Lemma 2.4. [9] *Let $X \in \mathbb{B}(\mathbb{H})$. Then $\|X\| \le t$ if and only if*

$$\begin{pmatrix} tI & X \\ X^* & tI \end{pmatrix} \ge 0.$$

Theorem 2.5. *Let $0 < m \le A, B \le M$ such that $0 < m < M$ and σ be an arbitrary mean. Then*

$$\Phi^2(A\sigma B) \le K^2(h)\Phi^2(B\sigma^{\perp}A), \tag{11}$$

where $\sigma^{\perp}$ is dual σ and $K(h) = \frac{(M+m)^2}{4Mm}$ is the Kantorovich constant.

Proof. It follows from $0 < m \le A, B \le M$ that $(M-A)(m-A)A^{-1} \le 0$ and $(M-B)(m-B)B^{-1} \le 0$. Therefore

$$A + MmA^{-1} \le M + m \quad \text{and} \quad B + MmB^{-1} \le M + m.$$

Now, the subadditivity and monotonicity properties of the operator mean to conclude that

$$\begin{aligned} A\sigma B + Mm(A^{-1}\sigma B^{-1}) &\le (A + MmA^{-1})\sigma(B + MmB^{-1}) \\ &\le (M+m)\sigma(M+m) \\ &= M + m. \end{aligned}$$

Using the linearity and positivity of Φ and the latter inequality, we get

$$\Phi(A\sigma B) + Mm\Phi(A^{-1}\sigma B^{-1}) \le M + m. \tag{12}$$

Applying two inequalities (10) and (12), respectively, we have

$$\begin{aligned} \Phi(A\sigma B) + Mm\Phi^{-1}(B\sigma^{\perp}A) &\le \Phi(A\sigma B) + Mm\Phi(B\sigma^{\perp}A)^{-1} \\ &\le \Phi(A\sigma B) + Mm\Phi(A^{-1}\sigma B^{-1}) \\ &\le M + m. \end{aligned}$$

By Lemma 2.3(i) and the latter inequality, we get

$$\begin{aligned}\left\|\Phi(A\sigma B)Mm\Phi^{-1}(B\sigma^{\perp}A)\right\| &\leq \frac{1}{4}\left\|\Phi(A\sigma B)+Mm\Phi(B\sigma^{\perp}A)^{-1}\right\|^{2}\\ &\leq \Phi(A\sigma B)+Mm\Phi(A^{-1}\sigma B^{-1})\\ &\leq M+m.\end{aligned}$$

This proves the assertion as desired. □

Remark 2.1. In special case, when $\sigma=\nabla$, since $\sigma^{\perp}=!$ and $!\leq\sharp$, inequality (11) becomes inequality (3).

Corollary 2.6. *Let $0<m\leq A,B\leq M$ such that $0<m<M$, σ be an arbitrary mean and let $p\geq 0$. Then*

$$\Phi^{p}(A\sigma B)\leq K^{p}(h)\Phi^{p}(B\sigma^{\perp}A), \tag{13}$$

where $\sigma^{\perp}$ is dual σ and $K(h)=\frac{(M+m)^{2}}{4Mm}$ is the Kantorovich constant.

Proof. If $0\leq p\leq 2$, then $0\leq\frac{p}{2}\leq 1$. Applying inequality (11) we obtain the desired result. If $p>2$, then

$$\begin{aligned}&\left\|\Phi^{\frac{p}{2}}(A\sigma B)M^{\frac{p}{2}}m^{\frac{p}{2}}\Phi^{-\frac{p}{2}}(B\sigma^{\perp}A)\right\|\\ &\leq\frac{1}{4}\left\|\Phi^{\frac{p}{2}}(A\sigma B)+M^{\frac{p}{2}}m^{\frac{p}{2}}\Phi^{-\frac{p}{2}}(B\sigma^{\perp}A)\right\|^{2}\ \text{(by Lemma 2.3 (i))}\\ &\leq\frac{1}{4}\left\|\Phi(A\sigma B)+Mm\Phi^{-1}(B\sigma^{\perp}A)\right\|^{p}\ \text{(by Lemma 2.3 (ii))}\\ &\leq\frac{1}{4}\left\|\Phi(A\sigma B)+Mm\Phi((B\sigma^{\perp}A))^{-1}\right\|^{p}\ \text{(by (10))}\\ &=\frac{1}{4}\left\|\Phi(A\sigma B)+Mm\Phi(A^{-1}\sigma^{\perp}B^{-1})\right\|^{p}\\ &\leq\frac{1}{4}(M+m)^{p}\ \text{(by inequality (12)).}\end{aligned}$$

Therefore, by Lemma 2.3(iii) we have

$$\Phi^{p}(A\sigma B)\leq K^{p}(h)\Phi^{p}(B\sigma^{\perp}A).$$

□

Remark 2.2. Using the same reason as in Remark 2.1 says that inequality (13) is a generalization of inequality (5) which is presented in [6].

In the following theorem, we generalize inequality (7).

Theorem 2.7. *Let $0<m\leq A,B\leq M$, σ_1 and σ_2 be two arbitrary means which lie between σ and $\sigma^{\perp}$ and let $p\geq 0$. Then for every positive unital linear map Φ,*

$$\Phi^{p}(A\sigma_2 B)\leq K^{p}(h)\Phi^{p}(B\sigma_1 A), \tag{14}$$

where $\sigma^{\perp}$ is dual σ and $K(h)=\frac{(M+m)^{2}}{4Mm}$ is the Kantorovich constant.

Proof. To prove (14), let $\sigma_1 \geq \sigma^{\perp}$ and $\sigma_2 \leq \sigma$. Therefore,

$$\begin{aligned}\Phi(A\sigma_2 B) + Mm\Phi^{-1}(B\sigma_1 A) &\leq \Phi(A\sigma_2 B) + Mm\Phi(B\sigma_1 A)^{-1} \text{ (by (10))} \\ &\leq \Phi(A\sigma B) + Mm\Phi(B\sigma^{\perp} A)^{-1} \\ &= \Phi(A\sigma B) + Mm\Phi(A^{-1}\sigma B^{-1}) \\ &\leq M + m \text{ (by (12))}.\end{aligned}$$

Using the same ideas as used in the proof of Theorem 2.5 and Corollary 2.6, one can obtain the desired result. □

To find a better bound than the obtained bound in inequality (13), we need to state the following Lemma.

Lemma 2.8. [13] *Let* $0 < m \leq A, B \leq M$ *and* σ *be an arbitrary mean. Then for every positive unital linear map* Φ

$$\|\Phi^2(A\sigma B) + M^2m^2\Phi^n((A\sigma B)^{-1})\| \leq M^2 + m^2.$$

Theorem 2.9. *Let* $0 < m \leq A, B \leq M$, σ *be an arbitrary mean and* $p \geq 4$. *Then*

$$\Phi^p(A\sigma B) \leq \left(\frac{K(h)(M^2+m^2)}{2^{\frac{4}{p}}Mm}\right)^p \Phi^p(B\sigma^{\perp}A), \tag{15}$$

where $\sigma^{\perp}$ *is dual* σ *and* $K(h) = \frac{(M+m)^2}{4Mm}$ *is the Kantorovich constant.*

Proof. By Theorem 2.5 we have

$$\Phi^{-2}(B\sigma^{\perp}A) \leq K^2(h)\Phi^{-2}(A\sigma B). \tag{16}$$

A simple computation shows that

$$\begin{aligned}&\left\|\Phi^{\frac{p}{2}}(A\sigma B)M^{\frac{p}{2}}m^{\frac{p}{2}}\Phi^{-\frac{p}{2}}(B\sigma^{\perp}A)\right\| \\ &\leq \frac{1}{4}\left\|K^{\frac{p}{4}}(h)\Phi^{\frac{p}{2}}(A\sigma B) + \left(\frac{M^2m^2}{K(h)}\right)^{\frac{p}{4}}\Phi^{-\frac{p}{2}}(B\sigma^{\perp}A)\right\|^2 \text{ (by Lemmas 2.3(i))} \\ &\leq \frac{1}{4}\left\|K\Phi^2(A\sigma B) + \frac{M^2m^2}{K(h)}\Phi^{-2}(B\sigma^{\perp}A)\right\|^{\frac{p}{2}} \text{ (by Lemmas 2.3(ii))} \\ &\leq \frac{1}{4}\left\|K(h)\Phi^2(A\sigma B) + M^2m^2K(h)\Phi^{-2}(A\sigma B)\right\|^{\frac{p}{2}} \text{ (by (16))} \\ &\leq \frac{1}{4}K^{\frac{p}{2}}(h)\left\|\Phi^2(A\sigma B) + M^2m^2\Phi^2(A\sigma B)^{-1}\right\|^{\frac{p}{2}} \text{ (by (10))} \\ &\leq \frac{1}{4}\left(K(h)\left(M^2+m^2\right)\right)^{\frac{p}{2}} \text{ (by Lemma 2.8)}.\end{aligned}$$

Therefore

$$\left\|\Phi^{\frac{p}{2}}(A\sigma B)\Phi^{-\frac{p}{2}}(B\sigma^{\perp}A)\right\| \leq \frac{1}{4}\left(\frac{K(h)\left(M^2+m^2\right)}{Mm}\right)^{\frac{p}{2}}.$$

The latter relation is equivalent to

$$\Phi^p(A\sigma B) \leq \left(\frac{K(h)\left(M^2+m^2\right)}{2^{\frac{4}{p}}Mm}\right)^p \Phi^p(B\sigma^{\perp}A).$$

This proves the desired result. □

Remark 2.3. When $p \geq 4$, the derived result in Theorem 2.9 is tighter than inequality (13).

Moreover, we show that Theorem 2.9 holds for $0 \leq p \leq 4$.

Corollary 2.10. *Let $0 < m \leq A, B \leq M$, σ be an arbitrary mean and let $0 \leq p \leq 4$. Then*

$$\Phi^p(A\sigma B) \leq \left(\frac{K(h)\left(M^2+m^2\right)}{2Mm}\right)^p \Phi^p(B\sigma^{\perp}A),$$

where $\sigma^{\perp}$ is dual σ and $K(h) = \frac{(M+m)^2}{4MM}$.

Proof. By Theorem 2.5 we have

$$\Phi^4(A\sigma B) \leq \left(\frac{K(h)\left(M^2+m^2\right)}{2Mm}\right)^4 \Phi^4(B\sigma^{\perp}A).$$

If $0 \leq p \leq 4$, then $0 \leq \frac{p}{4} \leq 1$. With the aid of the latter inequality and inequality (2), we conclude the desired inequality. □

Theorem 2.11. *Let $0 < m \leq A, B \leq M$, σ_1 and σ_2 be two arbitrary means between σ and $\sigma^{\perp}$, $1 < \alpha \leq 2$ and $p \geq 2\alpha$. Then for every positive unital linear map Φ*

$$\Phi^p(A\sigma_2 B) \leq \frac{(K^{\frac{\alpha}{2}}(h)(M^\alpha+m^\alpha))^{\frac{2p}{\alpha}}}{16M^pm^p}\Phi^p(B\sigma_1 A) \tag{17}$$

where $\sigma^{\perp}$ is dual σ and $K(h) = \frac{(M+m)^2}{4Mm}$ is the Kantorovich constant.

Proof. At once from inequality (14) follows that for $1 < \alpha \leq 2$

$$\Phi^{-\alpha}(B\sigma_1 A) \leq K^\alpha(h)\Phi^{-\alpha}(A\sigma_2 B). \tag{18}$$

Using the fact that $0 < m \leq A, B \leq M$, it deduces that $0 < m \leq A\sigma_2 B \leq M$. Now, the linearity property Φ results that $0 < m \leq \Phi(A\sigma_2 B) \leq M$. Since $1 < \alpha \leq 2$, one can easily prove that

$$\Phi^\alpha(A\sigma_2 B) + M^\alpha m^\alpha \Phi^{-\alpha}(A\sigma_2 B) \leq M^\alpha + m^\alpha. \tag{19}$$

Therefore

$$\begin{aligned}
&\left\|M^{\frac{p}{2}}m^{\frac{p}{2}}\Phi^{\frac{p}{2}}(A\sigma_2 B)\Phi^{-\frac{p}{2}}(B\sigma_1 A)\right\| \\
&\leq \frac{1}{4}\left\|K^{-\frac{p}{4}}(h)M^{\frac{p}{2}}m^{\frac{p}{2}}\Phi^{-\frac{p}{2}}(B\sigma_1 A) + K^{\frac{p}{4}}(h)\Phi^{\frac{p}{2}}(A\sigma_2 B)\right\|^2 \text{ (by Lemma 2.3(i))} \\
&\leq \frac{1}{4}\left\|\left(K^{-\frac{\alpha}{2}}(h)M^\alpha m^\alpha\Phi^{-\alpha}(B\sigma_1 A) + K^{\frac{\alpha}{2}}(h)\Phi^\alpha(A\sigma_2 B)\right)^{\frac{p}{2\alpha}}\right\|^2 \text{ (by Lemma 2.3(ii))} \\
&= \frac{1}{4}\left\|K^{-\frac{\alpha}{2}}(h)M^\alpha m^\alpha\Phi^{-\alpha}(B\sigma_1 A) + K^{\frac{\alpha}{2}}(h)\Phi^\alpha(A\sigma_2 B)\right\|^{\frac{p}{\alpha}} \\
&\leq \frac{1}{4}\left\|K^{\frac{\alpha}{2}}(h)M^\alpha m^\alpha\Phi^{-\alpha}(A\sigma_2 B) + K^{\frac{\alpha}{2}}(h)\Phi^\alpha(A\sigma_2 B)\right\|^{\frac{p}{\alpha}} \text{ (by (18))} \\
&\leq \frac{1}{4}K^{\frac{p}{2}}(h)(M^\alpha+m^\alpha)^{\frac{p}{\alpha}} \text{(by (19))},
\end{aligned}$$

that is

$$\left\|\Phi^{\frac{p}{2}}(A\sigma_2 B)\Phi^{-\frac{p}{2}}(B\sigma_1 A)\right\| \leq \frac{K^{\frac{p}{2}}(h)(M^\alpha+m^\alpha)^{\frac{p}{\alpha}}}{4M^{\frac{p}{2}}m^{\frac{p}{2}}},$$

or equivalently

$$\Phi^p(A\sigma_2 B) \leq \frac{\left(K^{\frac{\alpha}{2}}(h)(M^\alpha + m^\alpha)\right)^{\frac{2p}{\alpha}}}{16Mm}\Phi^p(B\sigma_1 A).$$

□

Remark 2.4. In special case, for $\alpha = 2$, inequality (17) becomes inequality (15).

Remark 2.5. By taking $\sigma = \nabla$ in inequality (17), we get inequality (8).

Theorem 2.12. *Let $0 < m \leq A, B \leq M$ such that $0 < m < M$ and σ be an arbitrary mean. Then for every positive unital linear map Φ and two arbitrary means σ_1 and σ_2 which lie between σ and $\sigma^\perp$ and $p \geq 0$, the following inequality holds*

$$\Phi^p(A\sigma_2 B)\Phi^{-p}(B\sigma_1 A) + \Phi^{-p}(B\sigma_1 A)\Phi^p(A\sigma_2 B) \leq 2K^p(h)\Phi^p(B\sigma_1 A) \tag{20}$$

where $\sigma^\perp$ is dual σ and $K(h) = \frac{(M+m)^2}{4Mm}$ is the Kantorovich constant.

Proof. It follows from (14) that

$$\|\Phi^p(A\sigma_2 B)\Phi^{-p}(B\sigma_1 A)\| \leq K^p(h). \tag{21}$$

Applying Lemma 2.4 we have

$$\begin{pmatrix} K(h)^p I & \Phi^{-p}(B\sigma_1 A)\Phi^p(A\sigma_2 B) \\ \Phi^p(A\sigma_2 B)\Phi^{-p}(B\sigma_1 A) & K(h)^p I \end{pmatrix} \geq 0$$

and

$$\begin{pmatrix} K(h)^p I & \Phi^p(A\sigma_2 B)\Phi^{-p}(B\sigma_1 A) \\ \Phi^{-p}(B\sigma_1 A)\Phi^p(A\sigma_2 B) & K(h)^p I \end{pmatrix} \geq 0.$$

Summing up two above inequalities, we obtain the following inequality

$$\begin{pmatrix} 2K(h)^p I & \beta_1 \\ \beta_2 & 2K(h)^p I \end{pmatrix} \geq 0,$$

where

$$\beta_1 = \Phi^{-p}(B\sigma_1 A)\Phi^p(A\sigma_2 B) + \Phi^p(A\sigma_2 B)\Phi^{-p}(B\sigma_1 A)$$

and

$$\beta_2 = \Phi^p(A\sigma_2 B)\Phi^{-p}(B\sigma_1 A) + \Phi^{-p}(B\sigma_1 A)\Phi^p(A\sigma_2 B).$$

Again using Lemma 2.4 we get the desired result. □

Remark 2.6. Put $\sigma = \nabla$, inequality (20) reduces to some results in [2]

3. A refined inequality for the arithmetic-geometric mean

Let $A, B \in \mathbb{B}(\mathbb{H})$ be two invertible positive operators, $0 \leq \nu \leq 1$ and $-1 \leq q \leq 1$. We use from the notation $A\sharp_{q,\nu}B$ to define the power mean

$$A\sharp_{q,\nu}B = A^{\frac{1}{2}}\left((1-\nu)I + \nu\left(A^{\frac{1}{2}}BA^{\frac{1}{2}}\right)^q\right)^{\frac{1}{q}}A^{\frac{1}{2}}.$$

For more information see [11]. The authors in [12] proved that if $0 < m \leq A, B \leq M$ such that $0 < m < M$ and $0 < \nu \leq \mu < 1$, $-1 \leq q \leq 1$. Then for every positive unital linear map Φ and $p \geq 0$, the following inequality holds

$$\Phi^p \left(A\nabla_\nu B + \frac{\nu}{\mu} Mm \left(A^{-1}\nabla_\mu B^{-1} - A^{-1}\sharp_{q,\mu} B^{-1} \right) \right) \leq K^p(h)\Phi^p(A\sharp_{q,\nu} B), \tag{22}$$

where $K(h) = \frac{(M+m)^2}{4Mm}$ is the Kantorovich constant.
Using the following theorem, we obtain a generalization from inequality (22).

Theorem 3.1. *Suppose that $0 < m \leq A, B \leq M$ such that $0 < m < M$ and $0 < \nu \leq \mu < 1$, $-1 \leq q \leq 1$ and $1 < \alpha \leq 2$. Then for every positive unital linear map Φ and $p \geq 0$, the following inequality holds*

$$\Phi^p \left(A\nabla_\nu B + \frac{\nu}{\mu} Mm (A^{-1}\nabla_\nu B^{-1} - A^{-1}\sharp_{q,\mu} B^{-1}) \right) \leq \frac{\left(K^{\frac{\alpha}{4}}(h)(M^\alpha + m^\alpha) \right)^{\frac{2p}{\alpha}}}{16 M^p m^p} \Phi^p(A\sharp_{q,\nu} B), \tag{23}$$

where $K(h) = \frac{(M+m)^2}{4Mm}$ is the Kantorovich constant.

Proof. For $1 < \alpha \leq 2$, by inequality (22), we have

$$\Phi^\alpha \left(A\nabla_\nu B + \frac{\nu}{\mu} Mm \left(A^{-1}\nabla_\mu B^{-1} - A^{-1}\sharp_{q,\mu} B^{-1} \right) \right) \leq K^\alpha(h)\Phi^\alpha(A\sharp_{q,\nu} B) \tag{24}$$

The last inequality deduces using a process similar to inequality (19). This shows that

$$\left\| \Phi^{\frac{p}{2}} \left(A\nabla_\nu B + \frac{\nu}{\mu} Mm \left(A^{-1}\nabla_\mu B^{-1} - A^{-1}\sharp_{q,\mu} B^{-1} \right) \right) \Phi^{-\frac{p}{2}}(A\sharp_{q,\nu} B) \right\| \leq \frac{K^{\frac{p}{2}}(h)(M^\alpha + m^\alpha)^{\frac{p}{\alpha}}}{4M^{\frac{p}{2}} m^{\frac{p}{2}}}.$$

Then

$$\Phi^p \left(A\nabla_\nu B + \frac{\nu}{\mu} Mm \left(A^{-1}\nabla_\mu B^{-1} - A^{-1}\sharp_{q,\mu} B^{-1} \right) \right) \leq \frac{\left(K^{\frac{\alpha}{4}}(h)(M^\alpha + m^\alpha) \right)^{\frac{2p}{\alpha}}}{16 M^p m^p} \Phi^p(A\sharp_{q,\nu} B).$$

□

Remark 3.1. Taking $\alpha = 2$, inequality (23) becomes inequality (22).

Remark 3.2. By putting $\alpha = 2, \mu = \frac{1}{2}$ and taking $q \to 0$, inequality (23) collapse to the derived result in [2].

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