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Applications of a global inversion theorem to unique solvability of second order Dirichlet problems

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Abstract. Banach-Mazur-Caccioppoli global inversion theorem is applied to obtain a generalization of a previous result of the authors and a result due to Ambrosetti and Prodi concerning unique solvability of a Dirichlet problem for a second order differential equation.

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In [1] and [2] Ambrosetti and Prodi had proved the following result.

Theorem 1. *Let $\psi : R \to R$ be a C^2 map such that $\psi(0) = 0$, $\psi'(0) = \pi^2$, $t\psi''(t) < 0$ if $t \neq 0$. Then for every continuous map $f : [0,1] \to R$ the Dirichlet problem*

$$\ddot{x}(t) + \psi(x(t)) = f(t), \qquad x(0) = x(1) = 0 \tag{1}$$

has a unique solution.

The above result was generalised in [15], in a Hilbert space framework by the authors of this paper who proved the following two theorems:

Theorem 2.[*15, Thm.4*] *Let H be a real finite dimensional Hilbert space and $f : [0,1] \times H \to H$ be a continuous map such that f_x', the derivative of f with respect to the second argument exists and is continuous. Consider $b \geq 0$, $a \in [0, \pi^2)$ and suppose that:*

$$\langle x, f(t,x)\rangle \leq a\|x\|^2 + b\|x\|, \qquad x \in H \tag{2}$$

$$\mu(f_x'(t,x)) \leq \pi^2 \qquad t \in [0,1],\ x \in H \tag{3}$$

Denote $W = \{x \in H : \quad \mu(f_x'(t,x))) = \pi^2 \quad \forall t \in [0,1]\}$.

If W is at most countable then for every continuous map $p : [0,1] \to H$ the problem:

$$\ddot{x}(t) + f(t, x(t)) = p(t), \qquad x(0) = x(1) = 0 \tag{4}$$

has exactly one solution.

Theorem 3.[*15, Thm.6*] *Let H be a real finite dimensional Hilbert space and $\psi : H \to H$ be a C^1 map with the properties*

$$\psi(0) = 0,\ \mu\left(\psi'(0)\right) = \pi^2 \text{ and } \mu\left(\psi'(x)\right) \leq \pi^2, \text{ for every } x \in H \tag{5}$$

For every $r \geq 0$ denote

$$m\left(r\right) = \sup\left\{\left\|\psi\left(x\right)\right\| : \left\|x\right\| \leq 2r\right\},$$

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$$M(r) = \sup\{\mu(\psi'(x)) : \|x\| \geq r\}.$$

Suppose that there exist $r > 0$ and $b > 0$ such that

$$m(r) \leq b \text{ and } M(r) < \pi^2. \tag{6}$$

Denote $W = \{x \in H : \quad \mu(\psi'(x)) = \pi^2\}$.

If W is at most countable then for every continuous map $f : [0,1] \to H$ the problem:

$$\ddot{x}(t) + \psi(x(t)) = f(t),\ x(0) = x(1) = 0 \tag{7}$$

has exactly one solution.

In the above two theorems we denoted by μ the logarithmic norm. We recall that if E be a linear normed space, then the logarithmic norm is a functional defined on $L(E,E)$ by

$$\mu(A) = \lim_{t\downarrow 0}\tfrac{\|I+tA\|-1}{t} \qquad A \in L(E,E)$$

One can easily see that:

$$\begin{aligned} \mu(A) &\leq \|A\| \\ \mu(\alpha A) &= \alpha\mu(A), \text{ for every nonnegative number } \alpha \\ |\mu(A) - \mu(B)| &\leq \|A - B\|. \end{aligned}$$

The functional μ is not a norm since it may take negative values.

In case E is a Hilbert space, we have:

$\mu(A) = \sup_{\|h\|=1} \Re e\langle Ah, h\rangle =$ the greatest eigenvalue of $\frac{(A+A^*)}{2}$.

More properties of the logarithmic norm can be found in [7], [11].

The proofs of Theorem 2 and Theorem 3 were based on the application of the Banach-Mazur-Caccioppoli global inversion theorem.

Caccioppoli [5] and Banach and Mazur [3] gave an interesting and simple condition for global invertibility of a mapping.

Theorem 4. *Let E,F be two Banach spaces. Then $f : E \to F$ is a global homeomorphism if and only if f is a local homeomorphism and a proper map.*

Here by a global homeomorphism (diffeomorphism of class C^p for some $p \geq 1$) we understand a homeomorphism (diffeomorphism of class C^p) which is onto. More information on global inversion theorems can be found in [9]-[15]. Several authors applied the Banach-Mazur-Caccioppoli theorem to existence and uniqueness results for boundary value problems. See for example [1], [2], [4], [6],[8],[15].

In the following we shall give a generalization of a result due to Ambrosetti and Prodi [1] based on the Banach-Mazur-Caccioppoli theorem. Our approach uses a different argument than that given in [1] to prove the invertibility of a nonlinear operator which is a local homeomorphism except a small set. Our result represents a slight generalization of our previous result (Theorem 2 in this paper).

Let H be a Hilbert space. Consider the following norms:

$$\|x\|_0 = \sup_{t\in[0,1]} \|x(t)\|, \qquad x \in C([0,1], H), \tag{8}$$

$$\|x\|_2 = \left(\int_0^1 \|x(t)\|^2 \, dt\right)^{\frac{1}{2}}, \quad x \in L^2([0,1], H). \tag{9}$$

To prove our result we need the following two lemmas:

Lemma 1. *Let H be a Hilbert space and $x : [0,1] \to H$ be a C^1 map such that $x(0) = x(1) = 0$.*

Then the following inequalities hold:

$$\left\|\dot{x}\right\|_2 \geq \pi \|x\|_2, \tag{10}$$

$$\left\|\dot{x}\right\|_2 \geq 2 \|x\|_0. \tag{11}$$

Inequality (10) is known as the Wirtinger's inequality and inequality (11) is known as the Lees' inequality. The following lemma is due to Plastock [8].

Lemma 2. *Let E, F be two Banach spaces whose dimensions are 3 or greater and $f : E \to F$ be a C^1 Fredholm proper map of zero index. Denote $B = \{x \in E : f'(x)$ is not surjective$\}$. If $b \in B$ is isolated in B then f is a local homeomorphism about b.*

Theorem 5. *Let H be a real finite dimensional Hilbert space and $f : [0,1] \times H \to H$ be a continuous map such that f_x', the derivative of f with respect to the second argument exists and is continuous. Consider $b \geq 0$, $a \in [0, \pi^2)$ and suppose that:*

$$\langle x, f(t,x)\rangle \leq a\|x\|^2 + b\|x\|, \qquad x \in H \tag{12}$$

$$\mu(f_x'(t,x)) \leq \pi^2 \qquad t \in [0,1],\ x \in H \tag{13}$$

Denote $W = \{x \in H : \quad \mu(f_x'(t,x))) = \pi^2 \quad \forall t \in [0,1]\}$.

If W is at most countable then for every continuous map $p : [0,1] \to H$ and every real constant c, the problem:

$$\ddot{x}(t) + c\dot{x}(t) + f(t, x(t)) = p(t),\ x(0) = x(1) = 0 \tag{14}$$

has exactly one solution.

Proof. Let $X = \{x \in C^2([0,1], H) : x(0) = x(1) = 0\}$, $Y = C([0,1], H)$. Endow Y with the the sup norm, that is with the norm $\|x\|_0$ from (1). Endow X with the norm:

$$\|x\|_1 = \|x\|_0 + \|\ddot{x}\|_0, \ x \in X.$$

Let $\|x\|_2$ be the L^2 norm as it was defined in (9). Consider the operator $S : X \to Y$,

$$(Sx)(t) = \ddot{x}(t) + c\dot{x}(t) + f(t, x(t)), \qquad t \in [0,1],\ x \in X.$$

Let $x \in X$. Then applying Wirtinger's inequality we obtain:

$$\begin{aligned}
\|Sx\|_2 \|x\|_2 &\geq -\int_0^1 \langle x(t), (Sx)(t) \rangle \, dt = \\
&= \int_0^1 \left\| \dot{x}(t) \right\|^2 dt - c \int_0^1 \left\langle x(t), \dot{x}(t) \right\rangle dt - \\
&\quad - \int_0^1 \langle x(t), f(t, x(t)) \rangle \, dt = \\
&= \int_0^1 \left\| \dot{x}(t) \right\|^2 dt - \frac{1}{2} c \int_0^1 \frac{d}{dt} \|x(t)\|^2 dt - \\
&\quad - \int_0^1 \langle x(t), f(t, x(t)) \rangle \, dt = \\
&= \int_0^1 \left\| \dot{x}(t) \right\|^2 dt - \int_0^1 \langle x(t), f(t, x(t)) \rangle \, dt \geq \\
&\geq \left\| \dot{x} \right\|_2^2 - a \int_0^1 \|x(t)\|^2 dt - b \int_0^1 \|x(t)\| \, dt \geq \\
&\quad \geq \left\| \dot{x} \right\|_2 \pi \|x\|_2 - a \|x\|_2^2 - b \|x\|_2 = \\
&= \|x\|_2 \left(\pi \left\| \dot{x} \right\|_2 - a \|x\|_2 - b \right) \geq \|x\|_2 \left(\pi \left\| \dot{x} \right\|_2 - \frac{a}{\pi} \left\| \dot{x} \right\|_2 - b \right).
\end{aligned}$$

If we divide by $\|x\|_2$ and apply Lees inequality we obtain:

$$\|Sx\|_0 \geq \|Sx\|_2 \geq \frac{\pi^2 - a}{\pi} \left\| \dot{x} \right\|_2 - b \geq 2 \frac{\pi^2 - a}{\pi} \|x\|_0 - b. \tag{15}$$

We shall prove that (15) implies that $S : (X, \|\cdot\|_1) \to (Y, \|\cdot\|_0)$ is a proper map.

To prove this let $(x_n)_{n \geq 1}$ be a sequence of X and $y \in Y$ such that $Sx_n \to y$ as $n \to \infty$.

By (15) it follows that there exists $M > 0$ such that $\|x_n\|_0 \leq M$ for every $n \geq 1$.

Consider the operators $L, N : X \to Y$, $Lx = \ddot{x} + c\dot{x}$, $(Nx)(t) = f(t, x(t))$, $x \in X$, $t \in [0, 1]$.

Note that L is one-to-one and onto and $L^{-1} : (Y, \|\cdot\|_0) \to (X, \|\cdot\|_0)$ is continuous.

$\left(L^{-1} y\right)(t) = \frac{1 - e^{-ct}}{e^{-c} - 1} \int_0^1 e^{-cs} \left(\int_0^s e^{c\tau} y(\tau) \, d\tau \right) ds + \int_0^t e^{-cs} \left(\int_0^s e^{c\tau} y(\tau) \, d\tau \right) ds, \quad t \in [0, 1], \ y \in Y.$

Since $N : (X, \|\cdot\|_0) \to (Y, \|\cdot\|_0)$ is a compact operator it follows that the sequence $(Nx_n)_{n \geq 1}$ contains a convergent subsequence. Without loss of generality we may suppose that $(Nx_n)_{n \geq 1}$ is convergent to $z \in Y$. Letting $n \to \infty$ in the equality $x_n = L^{-1} S(x_n) - L^{-1} N(x_n)$ we obtain

$$\lim_{n \to \infty} \left\| x_n - L^{-1} y - L^{-1} z \right\|_0 = 0. \tag{16}$$

By (16) and

$$\lim_{n \to \infty} \left\| Lx_n - L(L^{-1} y - L^{-1} z) \right\|_0 = \lim_{n \to \infty} \| (Sx_n - y) - (Nx_n - z) \|_0 = 0,$$

it follows that

$$\lim_{n \to \infty} \left\| x_n - L^{-1}(y - z) \right\|_1 = 0.$$

Consequently $S : (X, \|\cdot\|_1) \to (Y, \|\cdot\|_0)$ is a proper map.

For every $w \in W$ consider the map $\widetilde{w} \in X$, $\widetilde{w}(t) = w$, $t \in [0,1]$. Denote $\widetilde{W} = \{\widetilde{w} : w \in W\}$.

Note that $\left(S^{'}x\right)(h)(t) = \ddot{h}(t) + c\dot{h}(t) + f\,'_x(t, x(t))\,h(t)$, $t \in [0,1]$, $x, h \in X$.

We shall prove that $S^{'}x : X \to Y$ is one-to-one for every $x \in X - \widetilde{W}$.

Suppose that $x \in X - \widetilde{W}$. Denote $\varphi(t) = \mu(f\,'_x(t, x(t)))$, $t \in [0,1]$.

Let $J = t \in [0,1] : x(t) \in H - W\}$. One can easily see that J is infinite and $\varphi(t) < \pi^2$ for every $t \in J$.

If $h \in X - \{0\}$ and $\left(S^{'}x\right)(h) = 0$, then

$$\begin{aligned}
0 &= -\left\langle \left(S^{'}x\right)(h), h \right\rangle = \\
&= \int_0^1 \left\|\dot{h}(t)\right\|^2 dt - c\int_0^1 \left\langle \dot{h}(t), h(t) \right\rangle dt - \\
&\quad - \int_0^1 \langle f\,'_x(t, x(t))\,h(t), h(t)\rangle\, dt = \\
&= \int_0^1 \left\|\dot{h}(t)\right\|^2 dt - \frac{1}{2}c\int_0^1 \frac{d}{dt}\|h(t)\|^2 dt - \\
&\quad - \int_0^1 \langle f\,'_x(t, x(t))\,h(t), h(t)\rangle\, dt = \\
&= \int_0^1 \left\|\dot{h}(t)\right\|^2 dt - \int_0^1 \langle f\,'_x(t, x(t))\,h(t), h(t)\rangle\, dt \geq \\
&\geq \int_0^1 \left[\pi^2 - \varphi(t)\right] \|h(t)\|^2 dt,
\end{aligned}$$

hence $h = 0$ on J. If t_0 is a limit point of J then $h(t_0) = h'(t_0) = 0$. Since h is the unique solution of an initial value problem we obtain that $h = 0$ on $[0,1]$. Hence we obtain a contradiction. Consequently $\left(S^{'}x\right)(h) = 0$ implies that $h = 0$.

Since $L^{-1}\left(S^{'}x\right)$ is a compact perturbation of the identity for every $x \in X - \widetilde{W}$ the Fredholm alternative implies that $S^{'}x$ is bijective for every $x \in X - \widetilde{W}$.The local inversion theorem implies that S is a local diffeomorphism on $X - \widetilde{W}$.

Lemma 2 implies that S is a local homeomorphism on X. Now the Banach-Mazur-Caccioppoli theorem implies that $S : (X, \|\|_1) \to (Y, \|\|_0)$ is a global homeomorphism.

We recall the following lemma from [15].

Lemma 3. *Let H be a real Hilbert space and $\psi : H \to H$ be a C^1 map with the properties* (5) *and (6). Then there exists $a \in \left(0, \pi^2\right)$ such that*

$$\langle x, \psi(x)\rangle \leq a\|x\|^2 + b\|x\|, \qquad x \in H. \tag{17}$$

Theorem 6. *Let H be a real finite dimensional Hilbert space and $\psi : H \to H$ be a C^1 map with the properties* (5) *and* (6). *Denote $W = \{x \in H : \ \mu(\psi'(x)) = \pi^2\}$.*

If W is at most countable then for every continuous map $f : [0,1] \to H$ the problem:

$$\ddot{x}(t) + c\dot{x}(t) + \psi(x(t)) = f(t), \qquad x(0) = x(1) = 0 \tag{18}$$

has exactly one solution.

Proof. The result from the statement follows at once from Theorem 5 and Lemma 3.

References

[1] A. Ambrosetti, G. Prodi, *Analisi Non Lineare*, Pisa, Scuola Normale Superiore Pisa, 1973.

[2] A. Ambrosetti, G. Prodi, *A Primer of Nonlinear Analysis*, Cambridge, Cambridge University Press, 1993.

[3] S. Banach and S. Mazur, Über mehrdeutige stetige Abbildungen, *Studia Math*, **5**, 174-178 (1934).

[4] M.S. Berger, *Nonlinearity and Functional Analysis*, New York, Academic Press, 1977.

[5] R. Caccioppoli, Sugli elementi uniti delle transformazioni funzionali: un teorema di esistenza e di unicita ed alcune sue applicazioni, *Rend. Seminar Mat. Padova***3**, 1-15 (1932).

[6] S.N. Chow, J.K.Hale, *Methods of Bifurcation Theory*, New York, Springer Verlag, 1982.

[7] C.A. Desoer, H. Haneda, The measure of a matrix as a tool to analyse computer algorithms for circuit analysis, *IEEE Trans. Circuit Theory*, **CT-19**, 480-486 (1972).

[8] R. Plastock, Nonlinear Fredholm maps of index zero and their singularities, *Proc. Amer. Math. Soc.*, **68**, 317-322 (1973).

[9] M. Rădulescu, S. Rădulescu, *Theorems and Problems in Mathematical Analysis*, Bucharest, Ed. Didactică şi Pedagogică, 1982. (Romanian).

[10] M. Rădulescu, S. Rădulescu, Global inversion theorems and applications to differential equations, *Nonlinear Analysis*, **4**, 951-965 (1980).

[11] M. Rădulescu, S. Rădulescu, Global univalence and global inversion theorems in Banach spaces, *Nonlinear Analysis*, **13**, 539-553 (1989).

[12] M. Rădulescu, S. Rădulescu, Global inversion theorems, global univalence theorems and the Jacobian conjecture, Proceedings of the workshop *Recent results on the Global Asymptotic Stability Jacobian Conjecture*, Universita degli studi di Trento, Trento, 1993, 15p, (Editor Marco Sabatini) UTM **429**, March 1994.

[13] M. Rădulescu, S. Rădulescu, Applications of global inversion theorems to differential equations, numerical analysis and nonlinear circuits, Proceedings of the workshop *Recent results on the Global Asymptotic Stability Jacobian Conjecture'*, Universita degli studi di Trento, Trento, 1993, 26p, (Editor Marco Sabatini) UTM **429**, March 1994.

[14] M. Rădulescu, S. Rădulescu, Global inversion theorems and applications to boundary value problems for differential equations, *International J. of diff. eqs appl.*, **1**(2), 159-166 (2000).

[15] M. Rădulescu, S. Rădulescu, Global inversion theorem to unique solvability of Dirichlet problems, *J. Math. Anal. Appl.*, **272**(1), 362-367 (2002).

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