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Ewens distribution on $\mathbb{S}_n$ is a wavy probability distribution with respect to n partitions

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Abstract. We show that the Ewens distribution on $\mathbb{S}_n$, the set of permutations of order n, is a wavy probability distribution with respect to an order relation and n partitions which will be specified — the fact that the number of partitions is n is important. We then construct a Gibbs sampler in a generalized sense for the Ewens distribution. This chain leads

1) to a fast exact (not approximate) Markovian method for sampling from $\mathbb{S}_n$ according to the Ewens distribution and, as a result, to a fast exact method for sampling from $\mathbb{A}_n$, a set which will be specified, according to the Ewens sampling formula;

2) to the computation of normalization constant of Ewens distribution;

3) to the computation, by Uniqueness Theorem, of certain important probabilities for the Ewens distribution and, as a result, to upper bounds for the cumulative distribution function of number of cycles of permutation chosen from $\mathbb{S}_n$ according to the Ewens distribution.

Our sampling Markovian method has something in common with the swapping method. The number of steps of our sampling Markovian method is equal to the number of steps of swapping method, *i.e.*, $n-1$; moreover, both methods use the best probability distributions on sampling, the swapping method uses uniform probability distributions while our method uses almost uniform probability distributions (all the components of an almost uniform probability distribution are, here, identical, excepting at most one of them).

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1. Basic things, I

In this section, we present some basic things on nonnegative matrices, products of stochastic matrices, our hybrid Metropolis-Hastings chain, our Gibbs sampler in a generalized sense, and our wavy probability distributions.

Set

$$\operatorname{Par}(E) = \{\Delta \mid \Delta \text{ is a partition of } E\},$$

where E is a nonempty set. We shall agree that the partitions do not contain the empty set.

Definition 1.1. Let $\Delta_1, \Delta_2 \in \operatorname{Par}(E)$. We say that Δ_1 *is finer than* Δ_2 if $\forall V \in \Delta_1$, $\exists W \in \Delta_2$ such that $V \subseteq W$.

Write $\Delta_1 \preceq \Delta_2$ when Δ_1 is finer than Δ_2.

In this article, a vector is a row vector and a stochastic matrix is a row stochastic matrix.

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The entry (i,j) of a matrix Z will be denoted Z_{ij} or, if confusion can arise, $Z_{i\to j}$. Set

$$\langle m\rangle = \{1,2,...,m\} \ \ (m\geq 1),$$
$$N_{m,n} = \{P \mid P \text{ is a nonnegative } m\times n \text{ matrix}\},$$
$$S_{m,n} = \{P \mid P \text{ is a stochastic } m\times n \text{ matrix}\},$$
$$N_n = N_{n,n},$$
$$S_n = S_{n,n}.$$

Let $P=(P_{ij})\in N_{m,n}$. Let $\emptyset\neq U\subseteq\langle m\rangle$ and $\emptyset\neq V\subseteq\langle n\rangle$. Set the matrices

$$P_U = (P_{ij})_{i\in U, j\in\langle n\rangle}, \ P^V = (P_{ij})_{i\in\langle m\rangle, j\in V}, \text{ and } P_U^V = (P_{ij})_{i\in U, j\in V}.$$

Set

$$(\{i\})_{i\in\{s_1,s_2,...,s_t\}} = (\{s_1\},\{s_2\},...,\{s_t\});$$
$$(\{i\})_{i\in\{s_1,s_2,...,s_t\}} \in \operatorname{Par}(\{s_1,s_2,...,s_t\}) \ \ (t\geq 1).$$

E.g.,

$$(\{i\})_{i\in\langle n\rangle} = (\{1\},\{2\},...,\{n\}).$$

Definition 1.2. Let $P\in N_{m,n}$. We say that P is a *generalized stochastic matrix* if $\exists a\geq 0$, $\exists Q\in S_{m,n}$ such that $P=aQ$.

Definition 1.3. ([8].) Let $P\in N_{m,n}$. Let $\Delta\in\operatorname{Par}(\langle m\rangle)$ and $\Sigma\in\operatorname{Par}(\langle n\rangle)$. We say that P is a $[\Delta]$-*stable matrix on* Σ if P_K^L is a generalized stochastic matrix, $\forall K\in\Delta, \forall L\in\Sigma$. In particular, a $[\Delta]$-stable matrix on $(\{i\})_{i\in\langle n\rangle}$ is called $[\Delta]$-*stable* for short.

Definition 1.4. ([8].) Let $P\in N_{m,n}$. Let $\Delta\in\operatorname{Par}(\langle m\rangle)$ and $\Sigma\in\operatorname{Par}(\langle n\rangle)$. We say that P is a Δ-*stable matrix on* Σ if Δ is the least fine partition for which P is a $[\Delta]$-stable matrix on Σ. In particular, a Δ-stable matrix on $(\{i\})_{i\in\langle n\rangle}$ is called Δ-*stable* while a $(\langle m\rangle)$-stable matrix on Σ is called *stable on* Σ for short. A stable matrix on $(\{i\})_{i\in\langle n\rangle}$ is called *stable* for short.

Let $\Delta_1\in\operatorname{Par}(\langle m\rangle)$ and $\Delta_2\in\operatorname{Par}(\langle n\rangle)$. Set (see [8] for G_{Δ_1,Δ_2} and [9] for $\overline{G}_{\Delta_1,\Delta_2}$)

$$G_{\Delta_1,\Delta_2} = \{P \mid P\in S_{m,n} \text{ and } P \text{ is a } [\Delta_1]\text{-stable matrix on } \Delta_2\}$$

and

$$\overline{G}_{\Delta_1,\Delta_2} = \{P \mid P\in N_{m,n} \text{ and } P \text{ is a } [\Delta_1]\text{-stable matrix on } \Delta_2\}.$$

When we study or even when we construct products of nonnegative matrices (in particular, products of stochastic matrices) using G_{Δ_1,Δ_2} or $\overline{G}_{\Delta_1,\Delta_2}$, we shall refer this as the G *method*. G comes from the verb *to group* and its derivatives.

Below we give an important beautiful result on products of stochastic matrices.

Theorem 1.1. ([8].) *Let* $P_1\in G_{(\langle m_1\rangle),\Delta_2}\subseteq S_{m_1,m_2}$, $P_2\in G_{\Delta_2,\Delta_3}\subseteq S_{m_2,m_3}$, ..., $P_{n-1}\in G_{\Delta_{n-1},\Delta_n}\subseteq S_{m_{n-1},m_n}$, $P_n\in G_{\Delta_n,(\{i\})_{i\in\langle m_{n+1}\rangle}}\subseteq S_{m_n,m_{n+1}}$. *Then*

$$P_1P_2...P_n$$

is a stable matrix (*i.e., a matrix with identical rows, see Definition* 1.4).

Proof. See [8]. □

Definition 1.5. (See, *e.g.*, [16, p. 80].) Let $P \in N_{m,n}$. We say that P is a *row-allowable matrix* if it has at least one positive entry in each row.

Let $P \in N_{m,n}$. Set

$$\overline{P} = \left(\overline{P}_{ij}\right) \in N_{m,n},\ \overline{P}_{ij} = \begin{cases} 1 \text{ if } P_{ij} > 0, \\ 0 \text{ if } P_{ij} = 0, \end{cases}$$

$\forall i \in \langle m \rangle, \forall j \in \langle n \rangle$. We call $\overline{P}$ the *incidence matrix of* P (see, *e.g.*, [7, p. 222]).

In this article, the transpose of a vector x is denoted x'. Set $e = e(n) = (1, 1, ..., 1) \in \mathbb{R}^n$, $\forall n \geq 1$.

In this article, some statements on the matrices hold eventually by permutation of rows and columns. For simplification, further, we omit to specify this fact.

Warning! In this article, if a Markov chain has the transition matrix $P = P_1 P_2 ... P_s$, where $s \geq 1$ and P_1, P_2, ..., P_s are stochastic matrices, then any 1-step transition of this chain is performed via P_1, P_2, ..., P_s, *i.e.*, doing s transitions: one using P_1, one using P_2, ..., one using P_s.

Let S be a finite set with $|S| = r$, where $r \geq 2$ ($|\cdot|$ is the cardinal; for "$r \geq 2$", see below). Let $\pi = (\pi_i)_{i \in S}$ be a positive probability distribution on S. One way to sample approximately or, at best, exactly from S is by means of our hybrid Metropolis-Hastings chain from [9]. Below we define this chain.

Let E be a nonempty set. Set $\Delta \succ \Delta'$ if $\Delta' \preceq \Delta$ and $\Delta' \neq \Delta$, where Δ, $\Delta' \in \mathrm{Par}(E)$.

Let Δ_1, Δ_2, ..., $\Delta_{t+1} \in \mathrm{Par}(S)$ with $\Delta_1 = (S) \succ \Delta_2 \succ ... \succ \Delta_{t+1} = (\{i\})_{i \in S}$, where $t \geq 1$. ($\Delta_1 \succ \Delta_2$ implies $r \geq 2$.) Let Q_1, Q_2, ..., $Q_t \in S_r$, $Q_1 = \left((Q_1)_{ij}\right)_{i,j \in S}$, $Q_2 = \left((Q_2)_{ij}\right)_{i,j \in S}$, ..., $Q_t = \left((Q_t)_{ij}\right)_{i,j \in S}$, such that

(C1) $\overline{Q}_1$, $\overline{Q}_2$, ..., $\overline{Q}_t$ are symmetric matrices;

(C2) $(Q_l)_K^L = 0, \forall l \in \langle t \rangle - \{1\}, \forall K, L \in \Delta_l, K \neq L$ (this assumption implies that Q_l is a block diagonal matrix and Δ_l-stable matrix on $\Delta_l, \forall l \in \langle t \rangle - \{1\}$);

(C3) $(Q_l)_K^U$ is a row-allowable matrix, $\forall l \in \langle t \rangle$, $\forall K \in \Delta_l$, $\forall U \in \Delta_{l+1}$, $U \subseteq K$.

Define the matrices

$$P_l = \left((P_l)_{ij}\right)_{i,j \in S} \ (P_l \in S_r),$$

$$(P_l)_{ij} = \begin{cases} 0 & \text{if } j \neq i \text{ and } (Q_l)_{ij} = 0, \\ (Q_l)_{ij} \min\left(1, \frac{\pi_j (Q_l)_{ji}}{\pi_i (Q_l)_{ij}}\right) & \text{if } j \neq i \text{ and } (Q_l)_{ij} > 0, \\ 1 - \sum\limits_{k \neq i} (P_l)_{ik} & \text{if } j = i, \end{cases}$$

$\forall l \in \langle t \rangle$. Set $P = P_1 P_2 ... P_t$.

Theorem 1.2. ([9].) *Concerning P above we have $\pi P = \pi$ and $P > 0$.*

Proof. See [9]. □

By Theorem 1.2, $P^n \to e'\pi$ as $n \to \infty$. We call the Markov chain with transition matrix P the *hybrid Metropolis-Hastings chain.* In particular, we call this chain the *hybrid Metropolis chain* when Q_1, Q_2, ..., Q_t are symmetric matrices.

The next result is a corrected version of Theorem 2.1 from [14].

Theorem 1.3. ([15].) *Consider a hybrid Metropolis-Hastings chain with state space S above ($|S| = r \geq 2$) and transition matrix $P = P_1 P_2 ... P_t$, P_1, P_2, ..., P_t corresponding to Q_1, Q_2, ..., Q_t, respectively. Suppose that $\forall l \in \langle t \rangle$, $\forall i, j \in S$,*

$$(Q_l)_{ij} = \frac{\pi_j}{\sum\limits_{k \in S, (Q_l)_{ik} > 0} \pi_k} \text{ if } (Q_l)_{ij} > 0$$

(see above for Q_l, $l \in \langle t \rangle$, $\pi = (\pi_i)_{i \in S}$, ...). Then

$$(P_l)_{ij} = \begin{cases} 0 & \text{if } j \neq i \text{ and } (Q_l)_{ij} = 0, \\ (Q_l)_{ij} & \text{if } j \neq i \text{ and } \pi_j (Q_l)_{ji} \geq \pi_i (Q_l)_{ij} > 0, \\ \frac{\pi_j}{\sum\limits_{k \in S,\ (Q_l)_{jk} > 0} \pi_k} & \text{if } j \neq i \text{ and } \pi_j (Q_l)_{ji} < \pi_i (Q_l)_{ij}, \\ 1 - \sum\limits_{k \neq i} (P_l)_{ik} & \text{if } j = i, \end{cases}$$

$\forall l \in \langle t \rangle$, $\forall i, j \in S$. If, moreover,

$$\pi_i (Q_l)_{ij} = \pi_j (Q_l)_{ji}, \ \forall l \in \langle t \rangle, \forall i, j \in S,$$

then

$$P_l = Q_l, \ \forall l \in \langle t \rangle.$$

Proof. See [15]. □

We call the hybrid Metropolis-Hastings chain from Theorem 1.3 the *cyclic Gibbs sampler in a generalized sense* — the *Gibbs sampler in a generalized sense* for short.

Further, we consider that $S = \{s_1, s_2, ..., s_r\}$, where $r \geq 2$ ($|S| = r$). Equip S with an order relation, $\leqq$. Suppose that $s_1 \leqq s_2 \leqq ... \leqq s_r$. Let $\pi = (\pi_{s_i})_{i \in \langle r \rangle}$ be a positive probability distribution (on S). Let Δ_1, Δ_2, ..., $\Delta_{t+1} \in \text{Par}(S)$ with $\Delta_1 = (S) \succ \Delta_2 \succ ... \succ \Delta_{t+1} = (\{s_i\})_{i \in \langle r \rangle}$, where $t \geq 1$ and $(\{s_i\})_{i \in \langle r \rangle} = (\{s_1\}, \{s_2\}, ..., \{s_r\})$. ($\Delta_1 \succ \Delta_2$ implies $r \geq 2$.) Consider that $\Delta_l = \left(K_1^{(l)}, K_2^{(l)}, ..., K_{u_l}^{(l)}\right)$, $K_1^{(l)}$ having the first $\left|K_1^{(l)}\right|$ elements of S, $K_2^{(l)}$ having the next $\left|K_2^{(l)}\right|$ elements of S (this condition and the next ones vanish when $l = 1$), ..., $K_{u_l}^{(l)}$ having the last $\left|K_{u_l}^{(l)}\right|$ elements of S, $\forall l \in \langle t+1 \rangle$. Consider that

(c1) $\left|K_1^{(l)}\right| = \left|K_2^{(l)}\right| = ... = \left|K_{u_l}^{(l)}\right|$, $\forall l \in \langle t+1 \rangle$ with $u_l \geq 2$;

(c2) $r = r_1 r_2 ... r_t$ with $r_1 r_2 ... r_l = |\Delta_{l+1}|$, $\forall l \in \langle t-1 \rangle$, and $r_t = \left|K_1^{(t)}\right|$.

We have

$$K_v^{(l)} = \bigcup_{w \in D_{v,b_l} \cup \{vb_l\}} K_w^{(l+1)}, \ \forall l \in \langle t \rangle, \ \forall v \in \langle u_l \rangle,$$

where

$$b_l = \frac{|\Delta_{l+1}|}{|\Delta_l|}, \ \forall l \in \langle t \rangle,$$

and

$$D_{v,b_l} = \{(v-1) b_l + 1, \ (v-1) b_l + 2, \ ..., \ vb_l - 1\}, \ \forall l \in \langle t \rangle, \ \forall v \in \langle u_l \rangle.$$

Suppose that $\forall l \in \langle t \rangle$, $\forall v \in \langle u_l \rangle$, $\forall w \in D_{v,b_l}$, $\exists \alpha_w^{(l,v)} > 0$ such that

$$\pi_{s_{i+d_w^{(l,v)}}} = \alpha_w^{(l,v)} \pi_{s_i} \text{ (direct proportionality)}, \forall i \in \langle r \rangle \text{ with } s_i \in K_{(v-1)b_l+1}^{(l+1)},$$

which, using vectors, is equivalent to

$$(\pi_{s_i})_{i \in \langle r \rangle,\ s_i \in K_{w+1}^{(l+1)}} = \alpha_w^{(l,v)} (\pi_{s_i})_{i \in \langle r \rangle,\ s_i \in K_{(v-1)b_l+1}^{(l+1)}},$$

where

$$d_w^{(l,v)} = \left| K_{(v-1)b_l+1}^{(l+1)} \right| + \left| K_{(v-1)b_l+2}^{(l+1)} \right| + ... + \left| K_w^{(l+1)} \right|,$$

$\forall l \in \langle t \rangle$, $\forall v \in \langle u_l \rangle$, $\forall w \in D_{v,b_l}$.

Definition 1.6. (Based on Definition 3.1 from [14].) The probability distribution $\pi = (\pi_{s_i})_{i \in \langle r \rangle}$ having the above property (direct proportionality) we call the *wavy probability distribution* (*with respect to the order relation* $\leqq$ *and partitions* Δ_1, Δ_2, ..., Δ_{t+1}).

The wavy probability distributions of first type and those of second type from [10] are, according to Definition 1.6, wavy probability distributions (see also Example 3.1 in [14]). Below we give another simple example of wavy probability distribution.

Example 1.1. Let $S = \langle 9 \rangle$. Let $\leqq = \leq$. Let

$$\pi = \left(\frac{a}{Z}, \frac{a^3}{Z}, \frac{a^4}{Z}, \frac{a^3}{Z}, \frac{a^5}{Z}, \frac{a^6}{Z}, \frac{a^{10}}{Z}, \frac{a^{12}}{Z}, \frac{a^{13}}{Z} \right),$$

a probability distribution on S, where $a > 0$ and

$$Z = a + a^3 + a^4 + a^3 + a^5 + a^6 + a^{10} + a^{12} + a^{13}$$

(the normalization constant). Let

$$\Delta_1 = (S) = (\langle 9 \rangle),$$

$$\Delta_2 = (\{1,2,3\}, \{4,5,6\}, \{7,8,9\}),$$

$$\Delta_3 = (\{i\})_{i \in \langle 9 \rangle}$$

$$(\Delta_1 \succ \Delta_2 \succ \Delta_3;\ \ |\{1,2,3\}| = |\{4,5,6\}| = |\{7,8,9\}| = 3).$$

First, we consider Δ_1 and Δ_2. We have

$$\pi_4 = a^2 \pi_1,\ \pi_5 = a^2 \pi_2,\ \pi_6 = a^2 \pi_3,$$

which, using vectors, is equivalent to

$$(\pi_4, \pi_5, \pi_6) = a^2 (\pi_1, \pi_2, \pi_3)$$

(the proportionality factor is a^2), and

$$\pi_7 = a^9 \pi_1,\ \pi_8 = a^9 \pi_2,\ \pi_9 = a^9 \pi_3,$$

which is equivalent to

$$(\pi_7, \pi_8, \pi_9) = a^9 (\pi_1, \pi_2, \pi_3)$$

(the proportionality factor is a^9). Second, we consider Δ_2 and Δ_3. We have

$$\pi_2 = a^2 \pi_1$$

(here, we do not use vectors anymore; the proportionality factor is a^2),

$$\pi_3 = a^3 \pi_1$$

(the proportionality factor is a^3),

$$\pi_5 = a^2\pi_4,$$

$$\pi_6 = a^3\pi_4$$

(the proportionality factors are a^2 and a^3, respectively),

$$\pi_8 = a^2\pi_7,$$

$$\pi_9 = a^3\pi_7$$

(the proportionality factors are also a^2 and a^3, respectively). Consequently, π is a wavy probability distribution on S (neither of the first type nor of the second type because $S = \langle 9 \rangle$ and, moreover, $\leqq = \leq$).

The next result is another important result.

Theorem 1.4. (Based on Theorem 3.1 from [14].) *Let* $\pi = (\pi_{s_i})_{i\in\langle r\rangle}$ *be a wavy probability distribution* (*on* S) *with respect to the order relation* $\leqq$ *and partitions* Δ_1, Δ_2, ..., Δ_{t+1} — *for* S, $\leqq$, ..., *see Definition* 1.6 *and above this definition. Consider a Markov chain with state space* S *and transition matrix* $P = P_1P_2...P_t$ $(t \geq 1)$, *where* (*we again use the notation from Definition* 1.6 *and above this definition*)

$$(P_l)_{s_{i+d_w^{(l,v)}}\to\xi} = \begin{cases} \dfrac{\pi_{s_{i+d_u^{(l,v)}}}}{\sum\limits_{z\in\{0\}\cup D_{v,b_l}} \pi_{s_{i+d_z^{(l,v)}}}} & \text{if } \xi = s_{i+d_u^{(l,v)}} \text{ for some } u \in \{0\}\cup D_{v,b_l}, \\ 0 & \text{if } \xi \neq s_{i+d_u^{(l,v)}},\ \forall u \in \{0\}\cup D_{v,b_l}, \end{cases}$$

$\forall l \in \langle t\rangle$, $\forall v \in \langle u_l\rangle$, $\forall i \in \langle r\rangle$ *with* $s_i \in K^{(l+1)}_{(v-1)b_l+1}$, $\forall w \in \{0\}\cup D_{v,b_l}$, $\forall \xi \in S$, *setting* $d_0^{(l,v)} = 0$, $\forall l \in \langle t\rangle$, $\forall v \in \langle u_l\rangle$. *Then this chain is a Gibbs sampler in a generalized sense and*

$$P = e'\pi$$

(*therefore, this chain attains its stationarity at time* 1, *its stationary probability distribution* (*limit probability distribution*) *being, obviously,* π).

Proof. It is easy to see that

$$\pi_{s_i}(P_l)_{s_is_j} = \pi_{s_j}(P_l)_{s_js_i},\ \forall l \in \langle t\rangle,\ \forall i,j \in \langle r\rangle.$$

Taking — Theorem 1.3 together with the above equations, definitions of matrices P_l, $l \in \langle t\rangle$, ... suggest to take so —

$$Q_l = P_l,\ \forall l \in \langle t\rangle,$$

we obtain that the above Markov chain is a Gibbs sampler in a generalized sense. For the proof of equation $P = e'\pi$ — this equation follows from Theorems 1.1 and 1.2 —, see the proof of Theorem 3.1 in [14]. □

Theorem 1.4 leads to the next result.

Theorem 1.5. (Based on Theorem 3.2 from [14].) *Let* $\pi = (\pi_{s_i})_{i\in\langle r\rangle}$ *be a wavy probability distribution* (*on* S) *with respect to the order relation* $\leqq$ *and partitions* Δ_1, Δ_2, ..., Δ_{t+1} — *for* S, $\leqq$, ..., *see Definition* 1.6 *and above this definition. Suppose that*

$$\pi_{s_i} = \frac{\nu_{s_i}}{Z},\ \forall i \in \langle r\rangle,$$

where

$$Z = \sum_{i \in \langle r \rangle} \nu_{s_i},$$

Z is the normalization constant (π is a positive probability distribution, so, $\nu_{s_i} \in \mathbb{R}^+$, $\forall i \in \langle r \rangle$, and, as a result, $Z \in \mathbb{R}^+$). Then

$$Z = \nu_{s_1} \prod_{l \in \langle t \rangle} \left(1 + \sum_{w \in D_{1,b_l}} \alpha_w^{(l,1)} \right).$$

Proof. See the proof of Theorem 3.2 from [14]. □

2. Basic things, II

In this section, we present the Ewens distribution, Ewens sampling formula, and, in connection with these, some basic things on permutations.

We begin with some basic things on permutations in connection with the Ewens distribution and Ewens sampling formula.

Consider the group $(\mathbb{S}_n, \circ)$, where $\mathbb{S}_n$ is the set of permutations of order n $(n \geq 1)$ and $\circ$ is the usual composition of functions. $(u_1, u_2, ..., u_k)$ is a cycle of length k, where k, u_1, u_2, ..., $u_k \in \langle n \rangle$, $u_s \neq u_t, \forall s, t \in \langle k \rangle$, $s \neq t$; (u_1) is a degenerate (improper) cycle and (u_1, u_2) is a transposition. Set $(u) =$Id, $\forall u \in \langle n \rangle$, where (u) is a degenerate cycle, $\forall u \in \langle n \rangle$, and Id is the identity permutation.

Setting $(u, u) =$Id, $\forall u \in \langle n \rangle$, we have the following result.

Theorem 2.1. (Similar to Theorem 2.1 from [11].) *Let $n \geq 2$. Let*

$$\mathbb{E}_{n,l} = \{(1, i_1) \circ (2, i_2) \circ ... \circ (l, i_l) \circ \sigma_l \mid i_1, i_2, ..., i_l \in \langle n \rangle,\ 1 \leq i_1 \leq n,$$
$$2 \leq i_2 \leq n, ...,\ l \leq i_l \leq n,\ \sigma_l \in \mathbb{S}_n,\ \sigma_l(v) = v,\ \forall v \in \langle l \rangle\},\ \forall l \in \langle n-1 \rangle.$$

Then

$$\mathbb{E}_{n,l} = \mathbb{S}_n,\ \forall l \in \langle n-1 \rangle.$$

Proof. (Similar to the proof of Theorem 2.1 from [11].) Let $l \in \langle n-1 \rangle$. Since $(\mathbb{S}_n, \circ)$ is a group, we have $\mathbb{E}_{n,l} \subseteq \mathbb{S}_n$. Therefore, $|\mathbb{E}_{n,l}| \leq |\mathbb{S}_n| = n!$. To finish the proof, we show that $|\mathbb{E}_{n,l}| = n!$.

The number of permutations $\sigma_l \in \mathbb{S}_n$ with $\sigma_l(v) = v$, $\forall v \in \langle l \rangle$, is equal to $(n-l)!$. Since $1 \leq i_1 \leq n$, $2 \leq i_2 \leq n$, ..., $l \leq i_l \leq n$, it follows that $|\mathbb{E}_{n,l}|$ is at most equal to

$$n(n-1)...(n-l+1)[(n-l)!] = n!.$$

We show that

$$(1, i_1) \circ (2, i_2) \circ ... \circ (l, i_l) \circ \sigma_l = (1, j_1) \circ (2, j_2) \circ ... \circ (l, j_l) \circ \tau_l$$

if and only if

$$i_k = j_k,\ \forall k \in \langle l \rangle,\ \text{and } \sigma_l = \tau_l,$$

where i_1, j_1, i_2, j_2, ..., i_l, $j_l \in \langle n \rangle$, $1 \leq i_1$, $j_1 \leq n$, $2 \leq i_2$, $j_2 \leq n$, ..., $l \leq i_l$, $j_l \leq n$, $\sigma_l, \tau_l \in \mathbb{S}_n$, $\sigma_l(v) = \tau_l(v) = v$, $\forall v \in \langle l \rangle$.

"$\Longleftarrow$" Obvious.

"$\Longrightarrow$" We have

$$[(1, i_1) \circ (2, i_2) \circ ... \circ (l, i_l) \circ \sigma_l](1) = [(1, j_1) \circ (2, j_2) \circ ... \circ (l, j_l) \circ \tau_l](1).$$

Therefore,

$$i_1 = j_1.$$

Since $i_1 = j_1$, we have

$$(2, i_2) \circ ... \circ (l, i_l) \circ \sigma_l = (2, j_2) \circ ... \circ (l, j_l) \circ \tau_l.$$

It follows that

$$[(2, i_2) \circ ... \circ (l, i_l) \circ \sigma_l] (2) = [(2, j_2) \circ ... \circ (l, j_l) \circ \tau_l] (2) .$$

Therefore,

$$i_2 = j_2.$$

Proceeding in this way, we obtain

$$i_1 = j_1, \ i_2 = j_2, \ ..., \ i_l = j_l,$$

and, as a result of these equations,

$$\sigma_l = \tau_l.$$

We conclude that

$$|\mathbb{E}_{n,l}| = n!.$$

□

Theorem 2.1 says that we can work with $\mathbb{E}_{n,l}$ instead of $\mathbb{S}_n$, $\forall l \in \langle n-1 \rangle$ (this fact will be used in Section 3 (Theorem 3.1, ...) and Section 4 (Theorem 4.1, ...)).

Let $\psi = (u_1, u_2, ..., u_k)$ be a (proper or not) cycle ($\psi \in \mathbb{S}_n$, $1 \leq k \leq n$). We call u_1, u_2, ..., u_k the *cyclic elements of* (*cycle*) ψ. *E.g.*, the cyclic elements of cycle

$$(1, 2, 4) = \begin{pmatrix} 1234 \\ 2431 \end{pmatrix} = (2431) \in \mathbb{S}_4$$

are 1, 2, 4 while the cyclic element (this is not a proper cyclic element) of cycle

$$(2) = \begin{pmatrix} 1234 \\ 1234 \end{pmatrix} = (1234) = \mathrm{Id} \in \mathbb{S}_4$$

is 2 (not 1, 3, or 4). We call $\{u_1, u_2, ..., u_k\}$ the *set* (or *orbit*) *of cyclic elements of* (*cycle*) ψ.

Let $N(\sigma)$ be the number of pair-wise disjoint cycles of permutation σ, where $\sigma \in \mathbb{S}_n$. *E.g.*, $N(\mathrm{Id}) = n$ because Id$= (1) \circ (2) \circ ... \circ (n)$ ((1), (2), ..., (n) are degenerate cycles).

Theorem 2.2. *Let* $n \geq 2$. *Then*

$$N((1, i_1) \circ (2, i_2) \circ ... \circ (l-1, i_{l-1}) \circ (l, j) \circ \sigma_l) =$$

$$= \begin{cases} N((1, i_1) \circ (2, i_2) \circ ... \circ (l-1, i_{l-1}) \circ (l, k) \circ \sigma_l) & \text{if } j = k = l \text{ or } j, k > l, \\ N((1, i_1) \circ (2, i_2) \circ ... \circ (l-1, i_{l-1}) \circ (l, k) \circ \sigma_l) + 1 & \text{if } j = l, k > l, \\ N((1, i_1) \circ (2, i_2) \circ ... \circ (l-1, i_{l-1}) \circ (l, k) \circ \sigma_l) - 1 & \text{if } j > l, k = l, \end{cases}$$

$\forall l \in \langle n-1 \rangle$, $\forall i_1, i_2, ..., i_{l-1}, j, k \in \langle n \rangle$, $1 \le i_1 \le n$, $2 \le i_2 \le n, ..., l-1 \le i_{l-1} \le n$, $l \le j$, $k \le n$, $\forall \sigma_l \in \mathbb{S}_n$, $\sigma_l(v) = v$, $\forall v \in \langle l \rangle$ ($(1, i_1) \circ (2, i_2) \circ ... \circ (l-1, i_{l-1})$, *etc. vanish when* $l = 1$).

Proof. Case 1. $j = k = l$ or j, $k > l$.

Subcase 1.1. $j = k = l$. Obvious ($(l, j) = (l, k) =$Id).

Subcase 1.2. j, $k > l$. Since $\sigma_l(v) = v$, $\forall v \in \langle l \rangle$, $\exists u \in \langle n-l \rangle$, $\exists \gamma_1, \gamma_2, ..., \gamma_u \in \mathbb{S}_n$, $\gamma_1, \gamma_2, ..., \gamma_u$ are pair-wise disjoint cycles and $\lfloor \gamma_w \rceil \ge 1$, $\forall w \in \langle u \rangle$, where $\lfloor \gamma_w \rceil$ is the length of cycle γ_w, $\forall w \in \langle u \rangle$, such that (the cycles of length 1 are not omitted)

$$\sigma_l = (1) \circ (2) \circ ... \circ (l) \circ \gamma_1 \circ \gamma_2 \circ ... \circ \gamma_u.$$

Since j, $k > l$, $\exists s, t \in \langle u \rangle$ such that j is a cyclic element of γ_s and k is a cyclic element of γ_t. It follows that

$$(l, j) \circ \sigma_l = (1) \circ (2) \circ ... \circ (l-1) \circ \xi_1^{(1)} \circ \xi_2^{(1)} \circ ... \circ \xi_u^{(1)},$$

where

$$\xi_z^{(1)} = \begin{cases} \gamma_z & \text{if } z \neq s, \\ \text{the cycle whose set of cyclic elements} & \\ \text{contains } l, \ j, \text{ and the cyclic elements of } \gamma_s & \text{if } z = s, \end{cases}$$

$\forall z \in \langle u \rangle$ (obviously, $\xi_1^{(1)}$, $\xi_2^{(1)}$, ..., $\xi_u^{(1)}$ are pair-wise disjoint cycles), and

$$(l, k) \circ \sigma_l = (1) \circ (2) \circ ... \circ (l-1) \circ \varphi_1^{(1)} \circ \varphi_2^{(1)} \circ ... \circ \varphi_u^{(1)},$$

where

$$\varphi_z^{(1)} = \begin{cases} \gamma_z & \text{if } z \neq t, \\ \text{the cycle whose set of cyclic elements} & \\ \text{contains } l, \ k, \text{ and the cyclic elements of } \gamma_t & \text{if } z = t, \end{cases}$$

$\forall z \in \langle u \rangle$. Consequently,

$$N((l, j) \circ \sigma_l) = N((l, k) \circ \sigma_l)$$

Further, we consider the permutations

$$(l-1, i_{l-1}) \circ (l, j) \circ \sigma_l \text{ and } (l-1, i_{l-1}) \circ (l, k) \circ \sigma_l.$$

If $i_{l-1} = l-1$, from (recall that $(x, x) =$Id, $\forall x \in \langle n \rangle$)

$$N((l-1, i_{l-1}) \circ (l, j) \circ \sigma_l) = N((l, j) \circ \sigma_l) =$$
$$= N((l, k) \circ \sigma_l) = N((l-1, i_{l-1}) \circ (l, k) \circ \sigma_l),$$

we obtain

$$N((l-1, i_{l-1}) \circ (l, j) \circ \sigma_l) = N((l-1, i_{l-1}) \circ (l, k) \circ \sigma_l).$$

If $i_{l-1} > l-1$, then $\exists s, t \in \langle u \rangle$ such that i_{l-1} is a cyclic element of $\xi_s^{(1)}$ and, on the other hand, i_{l-1} is a cyclic element of $\varphi_t^{(1)}$. It follows that

$$(l-1, i_{l-1}) \circ (l, j) \circ \sigma_l = (1) \circ (2) \circ ... \circ (l-2) \circ \xi_1^{(2)} \circ \xi_2^{(2)} \circ ... \circ \xi_u^{(2)},$$

where

$$\xi_z^{(2)} = \begin{cases} \xi_z^{(1)} & \text{if } z \neq s, \\ \text{the cycle whose set of cyclic elements} & \\ \text{contains } l-1, \ i_{l-1}, \text{ and the cyclic elements of } \xi_s^{(1)} & \text{if } z = s, \end{cases}$$

$\forall z \in \langle u \rangle$, and

$$(l-1, i_{l-1}) \circ (l,k) \circ \sigma_l = (1) \circ (2) \circ ... \circ (l-2) \circ \varphi_1^{(2)} \circ \varphi_2^{(2)} \circ ... \circ \varphi_u^{(2)},$$

where

$$\varphi_z^{(2)} = \begin{cases} \gamma_z^{(1)} & \text{if } z \neq t, \\ \text{the cycle whose set of cyclic elements} & \\ \text{contains } l-1,\ i_{l-1},\ \text{and the cyclic elements of } \gamma_t^{(1)} & \text{if } z = t, \end{cases}$$

$\forall z \in \langle u \rangle$. Consequently, if $i_{l-1} > l-1$, then

$$N\left((l-1, i_{l-1}) \circ (l,j) \circ \sigma_l\right) = N\left((l-1, i_{l-1}) \circ (l,k) \circ \sigma_l\right).$$

Finally, for $i_{l-1} \geq l-1$ ($i_{l-1} = l-1$ or $i_{l-1} > l-1$), we have

$$N\left((l-1, i_{l-1}) \circ (l,j) \circ \sigma_l\right) = N\left((l-1, i_{l-1}) \circ (l,k) \circ \sigma_l\right).$$

Proceeding in this way for

$$(l-2, i_{l-2}) \circ (l-1, i_{l-1}) \circ (l,j) \circ \sigma_l \text{ and } (l-2, i_{l-2}) \circ (l-1, i_{l-1}) \circ (l,k) \circ \sigma_l,$$

for

$$(l-3, i_{l-3}) \circ (l-2, i_{l-2}) \circ (l-1, i_{l-1}) \circ (l,j) \circ \sigma_l \text{ and}$$
$$(l-3, i_{l-3}) \circ (l-2, i_{l-2}) \circ (l-1, i_{l-1}) \circ (l,k) \circ \sigma_l,$$

$$\vdots$$

for

$$(1, i_1) \circ (2, i_2) \circ ... \circ (l-1, i_{l-1}) \circ (l,j) \circ \sigma_l \text{ and}$$
$$(1, i_1) \circ (2, i_2) \circ ... \circ (l-1, i_{l-1}) \circ (l,k) \circ \sigma_l,$$

we obtain (finally)

$$N\left((1, i_1) \circ (2, i_2) \circ ... \circ (l-1, i_{l-1}) \circ (l,j) \circ \sigma_l\right) =$$
$$= N\left((1, i_1) \circ (2, i_2) \circ ... \circ (l-1, i_{l-1}) \circ (l,k) \circ \sigma_l\right).$$

Case 2. $j = l$, $k > l$. In this case, we have $(l,j) = (l,l) = (l)$. Further, we proceed in a way similar to that used in Subcase 1.2 — finally, we obtain

$$N\left((1, i_1) \circ (2, i_2) \circ ... \circ (l-1, i_{l-1}) \circ (l,j) \circ \sigma_l\right) =$$
$$= N\left((1, i_1) \circ (2, i_2) \circ ... \circ (l-1, i_{l-1}) \circ (l,k) \circ \sigma_l\right) + 1.$$

Case 3. $j > l$, $k = l$. Similar to Case 2 — finally, we obtain

$$N\left((1, i_1) \circ (2, i_2) \circ ... \circ (l-1, i_{l-1}) \circ (l,j) \circ \sigma_l\right) =$$
$$= N\left((1, i_1) \circ (2, i_2) \circ ... \circ (l-1, i_{l-1}) \circ (l,k) \circ \sigma_l\right) - 1.$$

□

Recall that $\mathbb{R}^+ = \{x \mid x \in \mathbb{R} \text{ and } x > 0\}$.

Let

$$\pi_\sigma = \frac{\theta^{N(\sigma)}}{Z},\ \forall \sigma \in \mathbb{S}_n,$$

where $\theta \in \mathbb{R}^+$ and

$$Z = \sum_{\sigma \in \mathbb{S}_n} \theta^{N(\sigma)}$$

$(n \geq 1)$. Z is known;

$$Z = \theta(\theta+1)\dots(\theta+n-1)$$

(see also Comment 5 from Section 4 — a new computation method for Z is given there). The probability distribution $\pi = (\pi_\sigma)_{\sigma \in \mathbb{S}_n}$ (on $\mathbb{S}_n$) is called the *Ewens distribution,* see, *e.g.*, [1] and [4]. This probability distribution is called so because, from it, we can obtain the Ewens sampling formula, a formula for a probability distribution on

$$\mathbb{A}_n = \{(m_1, m_2, \dots, m_n) \mid (m_1, m_2, \dots, m_n) \in \mathbb{N}^n \text{ and } m_1 + 2m_2 + \dots + nm_n = n\}$$

$(n \geq 1)$. The Ewens sampling formula is

$$P(\{(m_1, m_2, \dots, m_n)\}) = \frac{n!}{\theta(\theta+1)\dots(\theta+n-1)} \prod_{j=1}^{n} \frac{\theta^{m_j}}{j^{m_j} m_j!},$$

$\forall (m_1, m_2, \dots, m_n) \in \mathbb{A}_n$, where P is the probability on $(\mathbb{A}_n, \mathcal{P}(\mathbb{A}_n))$ ($\mathcal{P}(\mathbb{A}_n)$ is the power set of $\mathbb{A}_n$; $(\mathbb{A}_n, \mathcal{P}(\mathbb{A}_n))$ is a measurable space),

$$P(B) = \begin{cases} \sum\limits_{(m_1, m_2, \dots, m_n) \in B} \frac{n!}{\theta(\theta+1)\dots(\theta+n-1)} \prod\limits_{j=1}^{n} \frac{\theta^{m_j}}{j^{m_j} m_j!} & \text{if } \emptyset \neq B \subseteq \mathbb{A}_n, \\ 0 & \text{if } B = \emptyset, \end{cases}$$

$P(\{(m_1, m_2, \dots, m_n)\})$ is the probability of $\{(m_1, m_2, \dots, m_n)\}$, and $\theta \in \mathbb{R}^+$, see [5], see, *e.g.*, also [2], [6], and [17]. This formula is used in genetics and other fields. Below we derive this formula from the formula of Ewens distribution, $\pi_\sigma = \frac{\theta^{N(\sigma)}}{Z}$, $\forall \sigma \in \mathbb{S}_n$.

Let $\sigma \in \mathbb{S}_n$. σ can be written as a composition of pair-wise disjoint cycles. Let $k_i(\sigma)$ be the number of pair-wise disjoint cycles of length i of σ, where $i \in \langle n \rangle$. The vector $k(\sigma) = (k_1(\sigma), k_2(\sigma), \dots, k_n(\sigma))$ is called the *cycle structure vector of* σ (see, *e.g.*, [1]).

Note that $k(\sigma) \in \mathbb{A}_n$. Let $(m_1, m_2, \dots, m_n) \in \mathbb{A}_n$. We have (see, *e.g.*, also [1]), using the Cauchy formula on permutations (see, *e.g.*, [18]-[19]),

$$P_{\mathbb{S}_n}(\{\sigma \mid \sigma \in \mathbb{S}_n \text{ and } k(\sigma) = (m_1, m_2, \dots, m_n)\}) =$$

$$= \sum_{\sigma \in \mathbb{S}_n,\ k(\sigma) = (m_1, m_2, \dots, m_n)} \pi_\sigma = \sum_{\sigma \in \mathbb{S}_n,\ k(\sigma) = (m_1, m_2, \dots, m_n)} \frac{\theta^{m_1 + m_2 + \dots + m_n}}{Z} =$$

$$= \frac{\theta^{m_1 + m_2 + \dots + m_n}}{Z} \cdot |\{\sigma \mid \sigma \in \mathbb{S}_n \text{ and } k(\sigma) = (m_1, m_2, \dots, m_n)\}| =$$

$$= \frac{\theta^{m_1 + m_2 + \dots + m_n}}{Z} \cdot \left(n! \prod_{j=1}^{n} \frac{1}{j^{m_j} m_j!} \right) = \frac{n!}{\theta(\theta+1)\dots(\theta+n-1)} \prod_{j=1}^{n} \frac{\theta^{m_j}}{j^{m_j} m_j!} =$$

$$= P(\{(m_1, m_2, \dots, m_n)\}),$$

therefore, we obtained the Ewens sampling formula from the formula of Ewens distribution, where — it is obvious or almost obvious — $P_{\mathbb{S}_n}$ is the probability on $(\mathbb{S}_n, \mathcal{P}(\mathbb{S}_n))$,

$$P_{\mathbb{S}_n}(A) = \begin{cases} \sum\limits_{\sigma \in A} \pi_\sigma & \text{if } \emptyset \neq A \subseteq \mathbb{S}_n, \\ 0 & \text{if } A = \emptyset. \end{cases}$$

Above we were forced to use "P" with the subscript $\mathbb{S}_n$. When no confusion can arise, we use "P" for probability.

3. A basic property of Ewens distribution

In this section, we show that the Ewens distribution on $\mathbb{S}_n$ is a wavy probability distribution with respect to an order relation and n partitions which will be specified — recall that the fact that the number of partitions is n is important.

Let $n \geq 2$. Set

$$W_{(i_1,i_2,...,i_l)} = \{(1,i_1) \circ (2,i_2) \circ ... \circ (l,i_l) \circ \sigma_l \mid \sigma_l \in \mathbb{S}_n, \sigma_l(v) = v, \forall v \in \langle l \rangle\},$$

$\forall l \in \langle n-1 \rangle$, $\forall i_1,\ i_2,\ ...,\ i_l \in \langle n \rangle$, $1 \leq i_1 \leq n,\ 2 \leq i_2 \leq n, ...,\ l \leq i_l \leq n$.

Theorem 3.1.

$$\left(W_{(i_1,i_2,...,i_l)}\right)_{\substack{i_1,i_2,...,i_l \in \langle n \rangle \\ 1 \leq i_1 \leq n \\ 2 \leq i_2 \leq n \\ \vdots \\ l \leq i_l \leq n}}$$

is a partition of $\mathbb{S}_n$ $(n \geq 2)$, $\forall l \in \langle n-1 \rangle$.

Proof. We have

$$\bigcup_{\substack{i_1,i_2,...,i_l \in \langle n \rangle \\ 1 \leq i_1 \leq n \\ 2 \leq i_2 \leq n \\ \vdots \\ l \leq i_l \leq n}} W_{(i_1,i_2,...,i_l)} = \mathbb{E}_{n,l} = \mathbb{S}_n,\ \forall l \in \langle n-1 \rangle$$

(see Theorem 2.1).

Now, we show that

$$W_{(i_1,i_2,...,i_l)} \cap W_{(j_1,j_2,...,j_l)} = \emptyset$$

if $\exists u \in \langle l \rangle$ such that $i_u \neq j_u$, where $l \in \langle n-1 \rangle$, $i_1,\ j_1,\ i_2,\ j_2,\ ...,\ i_l,\ j_l \in \langle n \rangle$, $1 \leq i_1,\ j_1 \leq n,\ 2 \leq i_2,\ j_2 \leq n, ...,\ l \leq i_l,\ j_l \leq n$. Suppose that $\exists u \in \langle l \rangle$ with $i_u \neq j_u$ such that

$$W_{(i_1,i_2,...,i_l)} \cap W_{(j_1,j_2,...,j_l)} \neq \emptyset.$$

Let $\omega \in W_{(i_1,i_2,...,i_l)} \cap W_{(j_1,j_2,...,j_l)}$. We have

$$\omega = (1,i_1) \circ (2,i_2) \circ ... \circ (l,i_l) \circ \sigma_l = (1,j_1) \circ (2,j_2) \circ ... \circ (l,j_l) \circ \tau_l,$$

where $\sigma_l, \tau_l \in \mathbb{S}_n$, $\sigma_l(v) = \tau_l(v) = v, \forall v \in \langle l \rangle$. Proceeding as in the proof of Theorem 2.1, we obtain

$$i_1 = j_1,\ i_2 = j_2,\ ...,\ i_l = j_l,\ \sigma_l = \tau_l.$$

Therefore, we obtained a contradiction. □

Set the partitions (this can now be done)

$$\Delta_1 = (\mathbb{S}_n),$$

$$\Delta_{l+1} = \left(W_{(i_1,i_2,...,i_l)}\right)_{\substack{i_1,i_2,...,i_l \in \langle n \rangle \\ 1 \leq i_1 \leq n \\ 2 \leq i_2 \leq n \\ \vdots \\ l \leq i_l \leq n}},$$

$\forall l \in \langle n-1 \rangle$. Obviously, we have $\Delta_n = (\{\sigma\})_{\sigma \in \mathbb{S}_n}$.

Let $n \geq 2$. Set

$$B_n = \{1, 2, ..., n\} \times \{2, 3, ..., n\} \times ... \times \{n-1, n\}.$$

Note that

(1)

$$\mathbb{E}_{n,n-1} = \mathbb{S}_n$$

(by Theorem 2.1, taking $l = n-1$);

(2)

$$\mathbb{E}_{n,n-1} = \bigcup_{(i_1,i_2,...,i_{n-1}) \in B_n} W_{(i_1,i_2,...,i_{n-1})}$$

(by (1) and Theorem 3.1, taking $l = n-1$);

(3)

$$W_{(i_1,i_2,...,i_{n-1})} = \{(1, i_1) \circ (2, i_2) \circ ... \circ (n-1, i_{n-1})\}$$

(σ_l =Id when $l = n-1$).

Let $\sigma, \tau \in \mathbb{S}_n$. By (1)–(3), $\exists\, (i_1, i_2, ..., i_{n-1})\,,\, (j_1, j_2, ..., j_{n-1}) \in B_n$ such that

$$\sigma = (1, i_1) \circ (2, i_2) \circ ... \circ (n-1, i_{n-1}) \text{ and } \tau = (1, j_1) \circ (2, j_2) \circ ... \circ (n-1, j_{n-1}).$$

Set

$$\sigma \overset{E}{\leq} \tau \text{ if } (i_1, i_2, ..., i_{n-1}) \overset{lex}{\leq} (j_1, j_2, ..., j_{n-1}),$$

where $\overset{lex}{\leq}$ is the lexicographic order on B_n.

Theorem 3.2. $\overset{E}{\leq}$ *is an order relation on* $\mathbb{S}_n$ $(n \geq 2)$.

Proof. Obvious (because $\overset{lex}{\leq}$ is an order relation). □

Remark 3.1. Similar to the above definition (construction) of $\overset{E}{\leq}$, we can define (construct) an order relation on $\mathbb{S}_n$ for the Mallows model through Cayley metric (see [11] for this model and other things). Moreover, the above definition of $\overset{E}{\leq}$ gives a suggestion to define an order relation on $\mathbb{S}_n$ for the Mallows model through Kendall metric (see [12] for this model and other things).

Below we give the first main result of this article.

Theorem 3.3. *Let $n \geq 2$. The Ewens distribution on $\mathbb{S}_n$ is a wavy probability distribution with respect to the order relation $\overset{E}{\leq}$ and n partitions above.*

Proof. Since $\sigma_l(v) = v$, $\forall v \in \langle l \rangle$, and $\sigma_{l+1}(v) = v$, $\forall v \in \langle l+1 \rangle$, $\forall l \in \langle n-2 \rangle$, we have

$$W_{(i_1,i_2,...,i_{l+1})} \subset W_{(i_1,i_2,...,i_l)},$$

$\forall l \in \langle n-2 \rangle$, $\forall i_1, i_2, ..., i_{l+1} \in \langle n \rangle$, $1 \leq i_1 \leq n$, $2 \leq i_2 \leq n$, ..., $l+1 \leq i_{l+1} \leq n$. Obviously, $W_{(i_1)} \subset \mathbb{S}_n$, $\forall i_1 \in \langle n \rangle$. Therefore,

$$\Delta_1 \succ \Delta_2 \succ ... \succ \Delta_n.$$

The conditions (c1) and (c2) also hold.

Fix $l \in \langle n-1\rangle$. Consider the partitions Δ_l and Δ_{l+1} (see the definition of wavy probability distribution again). Let $K \in \Delta_l$. We have

$$K = \begin{cases} \mathbb{S}_n & \text{if } l = 1, \\ W_{(i_1, i_2, \ldots, i_{l-1})} & \begin{array}{l} \text{for some } i_1, i_2, ..., i_{l-1} \in \langle n\rangle, \ 1 \le i_1 \le n, \\ 2 \le i_2 \le n, \ ..., \ l-1 \le i_{l-1} \le n \text{ if } l \ne 1. \end{array} \end{cases}$$

Using the order relation $\overset{E}{\le}$, the first subset of K belonging to Δ_{l+1} is $W_{(i_1,i_2,\ldots,i_{l-1},l)}$, the second one is $W_{(i_1,i_2,\ldots,i_{l-1},l+1)}$, ..., the last one is $W_{(i_1,i_2,\ldots,i_{l-1},n)}$ ($i_1, i_2, ..., i_{l-1}$ vanish when $l = 1$). The first element of $W_{(i_1,i_2,\ldots,i_{l-1},l)}$ is

$$(1, i_1) \circ (2, i_2) \circ ... \circ (l-1, i_{l-1}) \circ (l, l) \circ (l+1, l+1) \circ ... \circ (n-1, n-1)$$

(here, $\sigma_l = (l+1, l+1) \circ ... \circ (n-1, n-1)$), the second one is

$$(1, i_1) \circ (2, i_2) \circ ... \circ (l-1, i_{l-1}) \circ (l, l) \circ (l+1, l+2) \circ (l+2, l+2) \circ ... \circ (n-1, n-1)$$

(here, $\sigma_l = ...$), ..., the last one is

$$(1, i_1) \circ (2, i_2) \circ ... \circ (l-1, i_{l-1}) \circ (l, l) \circ (l+1, n) \circ (l+2, n) \circ ... \circ (n-1, n);$$

the first element of $W_{(i_1,i_2,\ldots,i_{l-1},l+1)}$ is

$$(1, i_1) \circ (2, i_2) \circ ... \circ (l-1, i_{l-1}) \circ (l, l+1) \circ (l+1, l+1) \circ ... \circ (n-1, n-1)$$

(here, $\sigma_l = (l+1, l+1) \circ ... \circ (n-1, n-1)$), the second one is

$$(1, i_1) \circ (2, i_2) \circ ... \circ (l-1, i_{l-1}) \circ (l, l+1) \circ (l+1, l+2) \circ (l+2, l+2) \circ ... \circ (n-1, n-1),$$

$$\vdots$$

the last one is

$$(1, i_1) \circ (2, i_2) \circ ... \circ (l-1, i_{l-1}) \circ (l, l+1) \circ (l+1, n) \circ (l+2, n) \circ ... \circ (n-1, n);$$

$$\vdots$$

$$\vdots$$

the first element of $W_{(i_1,i_2,\ldots,i_{l-1},n)}$ is

$$(1, i_1) \circ (2, i_2) \circ ... \circ (l-1, i_{l-1}) \circ (l, n) \circ (l+1, l+1) \circ ... \circ (n-1, n-1)$$

(here, $\sigma_l = (l+1, l+1) \circ ... \circ (n-1, n-1)$), the second one is

$$(1, i_1) \circ (2, i_2) \circ ... \circ (l-1, i_{l-1}) \circ (l, n) \circ (l+1, l+2) \circ (l+2, l+2) \circ ... \circ (n-1, n-1),$$

$$\vdots$$

the last one is

$$(1, i_1) \circ (2, i_2) \circ ... \circ (l-1, i_{l-1}) \circ (l, n) \circ (l+1, n) \circ (l+2, n) \circ ... \circ (n-1, n).$$

So, by Theorem 2.2 we have, using vectors,

$$(\pi_\sigma)_{\sigma \in W_{(i_1,i_2,\ldots,i_l)}} = \frac{1}{\theta} (\pi_\sigma)_{\sigma \in W_{(i_1,i_2,\ldots,i_{l-1},l)}}, \ \forall i_l \in \langle n\rangle, \ l+1 \le i_l \le n$$

(not $l \le i_l \le n$; the proportionality factor is $\frac{1}{\theta}$, $\forall i_l \in \langle n\rangle, \ l+1 \le i_l \le n$). $\square$

Remark 3.2. At present we know six interesting wavy probability distributions: the probability distribution from Application 3.5 in [10], the probability distribution of a random vector with independent components in the finite case, *i.e.*, when the number of components is finite and each component has a finite number of values, at least two values (see [15]), the Mallows model through Cayley metric and that through Kendall metric (see [11]-[12] and Remark 3.1), the Potts model on the tree (see [13] and [15, Remark 5.1]), and the Ewens distribution (see Theorem 3.3).

4. Fast exact sampling, normalization constant, important probabilities

In this section, we present our fast Markovian method for sampling exactly (not approximately) from $\mathbb{S}_n$ according to the Ewens distribution — further, this method leads to a fast exact method for sampling from $\mathbb{A}_n$ according to the Ewens sampling formula. In addition to sampling, for the Ewens distribution, we compute the normalization constant and, by Uniqueness Theorem, certain important probabilities — further, using these probabilities, we give upper bounds for the cumulative distribution function of number of cycles of permutation chosen from $\mathbb{S}_n$ according to the Ewens distribution.

Recall that $e = e(n) = (1, 1, ..., 1) \in \mathbb{R}^n$, $\forall n \geq 1$, and e' is its transpose.

Below we give the second main result of this article — the Gibbs sampler in a generalized sense from this result is constructed using the G method such that Theorem 1.1 can be applied.

Theorem 4.1. *Let $n \geq 2$. Let $\pi = (\pi_\sigma)_{\sigma \in \mathbb{S}_n}$ be the Ewens distribution. Consider a Markov chain with state space $\mathbb{S}_n$ and transition matrix $P = P_1 P_2 ... P_{n-1}$, where P_l, $l \in \langle n-1 \rangle$, are stochastic matrices on $\mathbb{S}_n$,*

$$(P_l)_{(1,i_1)\circ(2,i_2)\circ...\circ(l,i_l)\circ\sigma_l \to \xi} =$$

$$= \begin{cases} \dfrac{\pi_{(1,i_1)\circ(2,i_2)\circ...\circ(l-1,i_{l-1})\circ(l,j)\circ\sigma_l}}{\sum\limits_{l \leq k \leq n} \pi_{(1,i_1)\circ(2,i_2)\circ...\circ(l-1,i_{l-1})\circ(l,k)\circ\sigma_l}} & \text{if } \xi = (1,i_1)\circ(2,i_2)\circ...\circ(l-1,i_{l-1}) \\ & \circ(l,j)\circ\sigma_l \text{ for some } j,\ l \leq j \leq n, \\ 0 & \text{if } \xi \neq (1,i_1)\circ(2,i_2)\circ...\circ(l-1,i_{l-1}) \\ & \circ(l,j)\circ\sigma_l,\ \forall j,\ l \leq j \leq n, \end{cases}$$

$\forall l \in \langle n-1 \rangle$ ($(1,i_1)\circ(2,i_2)\circ...\circ(l-1,i_{l-1})$ vanishes when $l = 1$), $\forall i_1, i_2, ..., i_l \in \langle n \rangle$, $1 \leq i_1 \leq n$, $2 \leq i_2 \leq n, ..., l \leq i_l \leq n$, $\forall \sigma_l \in \mathbb{S}_n$, $\sigma_l(v) = v$, $\forall v \in \langle l \rangle$, $\forall \xi \in \mathbb{S}_n$. Then this chain is a Gibbs sampler in a generalized sense and

$$P = e'\pi$$

(*therefore, this chain attains its stationarity at time* 1, *its stationary probability distribution* (*limit probability distribution*) *being, obviously,* π).

Proof. Theorems 1.4 and 3.3. □

We comment on Theorem 4.1.

1. We can work with the chain with transition matrix P or, equivalently, with the chain with transition matrices $P_1, P_2, ..., P_{n-1}, P_1, P_2, ..., P_{n-1}, ...$ (the former chain is homogeneous while the latter one is nonhomogeneous when $n \geq 3$). We chose the first case. (For finite Markov chain theory, see, *e.g.*, [7].) Any 1-step of the chain with transition matrix P is performed via P_1, P_2, ..., P_{n-1}, *i.e.*, doing $n-1$ transitions: one using P_1, one using P_2, ..., one using P_{n-1}. By Theorem 4.1 the chain with transition matrix P attains its stationarity at time 1 (to attain the stationarity, the chain with transition matrix P makes one step while the other chain makes $n-1$ steps (due to P_1, P_2, ..., P_{n-1})).

2. By Theorem 2.2 we can compute the transition probabilities from Theorem 4.1. We have

$$(P_l)_{(1,i_1)\circ(2,i_2)\circ\ldots\circ(l,i_l)\circ\sigma_l\to\xi} =$$

$$= \begin{cases} \dfrac{\theta^{N\left((1,i_1)\circ(2,i_2)\circ\ldots\circ(l-1,i_{l-1})\circ(l,j)\circ\sigma_l\right)}}{\sum\limits_{l\leq k\leq n}\theta^{N\left((1,i_1)\circ(2,i_2)\circ\ldots\circ(l-1,i_{l-1})\circ(l,k)\circ\sigma_l\right)}} & \text{if } \xi=(1,i_1)\circ(2,i_2)\circ\ldots\circ(l-1,i_{l-1}) \\ & \circ(l,j)\circ\sigma_l \text{ for some } j,\ l\leq j\leq n, \\ 0 & \text{if } \xi\neq(1,i_1)\circ(2,i_2)\circ\ldots\circ(l-1,i_{l-1}) \\ & \circ(l,j)\circ\sigma_l,\ \forall j,\ l\leq j\leq n, \end{cases}$$

$$= \begin{cases} \frac{\theta}{\theta+n-l} & \text{if } \xi=(1,i_1)\circ(2,i_2)\circ\ldots\circ(l-1,i_{l-1})\circ(l,l)\circ\sigma_l, \\ \frac{1}{\theta+n-l} & \text{if } \xi=(1,i_1)\circ(2,i_2)\circ\ldots\circ(l-1,i_{l-1})\circ(l,j)\circ\sigma_l \\ & \text{for some } j,\ l<j\leq n, \\ 0 & \text{if } \xi\neq(1,i_1)\circ(2,i_2)\circ\ldots\circ(l-1,i_{l-1})\circ(l,j)\circ\sigma_l, \\ & \forall j,\ l\leq j\leq n, \end{cases}$$

$\forall l\in\langle n-1\rangle$, $\forall i_1, i_2, ..., i_l\in\langle n\rangle$, $1\leq i_1\leq n$, $2\leq i_2\leq n$, ..., $l\leq i_l\leq n$, $\forall\sigma_l\in\mathbb{S}_n$, $\sigma_l(v)=v$, $\forall v\in\langle l\rangle$, $\forall\xi\in\mathbb{S}_n$, and, as a result,

$$P_l\in G_{\Delta_l,\Delta_{l+1}},\ \forall l\in\langle n-1\rangle$$

— and Theorem 1.1 can be applied!

3. To define transition probabilities of P_l, $l\in\langle n-1\rangle$ fixed, we used states from $\mathbb{E}_{n,l}$. So, using P_l, the chain passes from a state, say, γ of $\mathbb{E}_{n,l}$ to a state, say, δ of $\mathbb{E}_{n,l}$ also. For P_{l+1}, when $l+1\leq n-1$, we need states from $\mathbb{E}_{n,l+1}$, so, when we run the chain, we must rewrite δ using the generators of $\mathbb{E}_{n,l+1}$.

4. There exists a case, a happy case, for which rewriting the states from Comment 3 is not necessary, namely, when $\sigma_l=$Id. So, to avoid rewriting the states, we consider the chain with the initial state Id. Since $P=e'\pi$, we have

$$p_0P^m=\pi,\ \forall m\geq 1, \forall p_0,\ p_0= \text{ initial probability distribution.}$$

So, for the initial probability distribution p_0 with $(p_0)_{\text{Id}}=1$, the above equations hold as well. From Id$=(1,1)\circ$Id$\in\mathbb{E}_{n,1}$ ($\sigma_1=$Id) the chain passes in one of the states

$$\text{Id}=(1,1)=(1,1)\circ\text{Id}\in\mathbb{E}_{n,1},$$

$$(1,2)=(1,2)\circ\text{Id}\in\mathbb{E}_{n,1},$$

$$\vdots$$

$$(1,n)=(1,n)\circ\text{Id}\in\mathbb{E}_{n,1},$$

the transition probabilities being (see Comment 2)

$$\frac{\theta}{\theta+n-1},\ \frac{1}{\theta+n-1},\ \frac{1}{\theta+n-1},\ ...,\ \frac{1}{\theta+n-1},$$

respectively. Suppose that it passed in the state $(1,3)$ (when $n \geq 3$). From $(1,3) = (1,3) \circ (2,2) \circ \mathrm{Id} \in \mathbb{E}_{n,2}$ ($\sigma_2 = \mathrm{Id}$), the chain passes in one of the states

$$(1,3) = (1,3) \circ (2,2) = (1,3) \circ (2,2) \circ \mathrm{Id} \in \mathbb{E}_{n,2},$$

$$(1,3) \circ (2,3) = (1,3) \circ (2,3) \circ \mathrm{Id} \in \mathbb{E}_{n,2},$$

$$\vdots$$

$$(1,3) \circ (2,n) = (1,3) \circ (2,n) \circ \mathrm{Id} \in \mathbb{E}_{n,2},$$

the transition probabilities being

$$\frac{\theta}{\theta+n-2},\ \frac{1}{\theta+n-2},\ \frac{1}{\theta+n-2},\ ...,\ \frac{1}{\theta+n-2},$$

respectively. Suppose that it passed in the state $(1,3) \circ (2,n-1)$. Etc. Therefore, the states are generated proceeding similar to the swapping method, the difference being that, here, we use the probability distributions (see Comment 2 (the 0's do not count))

$$\left(\frac{\theta}{\theta+n-l},\ \frac{1}{\theta+n-l},\ \frac{1}{\theta+n-l},\ ...,\ \frac{1}{\theta+n-l}\right),\ l \in \langle n-1 \rangle,$$

instead of uniform probability distributions. (For the swapping method, see, *e.g.*, [3, pp. 645−646].) The above probability distributions, the former being almost uniform probability distributions — we call them *almost uniform probability distributions* because each of these probability distributions has identical components, excepting at most one of them (all the components are identical when $\theta = 1$) — and the latter, those of swapping method, being uniform probability distributions, are, concerning the implementation, the best ones. To see that this is also true for the (above) almost uniform probability distributions, we split each almost uniform probability distribution into two blocks ($l \in \langle n-1 \rangle$),

$$\left(\frac{\theta}{\theta+n-l}\right),\ \left(\frac{1}{\theta+n-l},\ \frac{1}{\theta+n-l},\ ...,\ \frac{1}{\theta+n-l}\right).$$

If

$$X > \frac{\theta}{\theta+n-l},\ X \sim U(0,1),$$

further, we work with the latter block, which, by normalization, leads to the uniform probability distribution

$$\left(\frac{1}{n-l}, \frac{1}{n-l}, ..., \frac{1}{n-l}\right).$$

Therefore, our exact sampling Markovian method, having $n-1$ steps, is simple and good.

5. We can compute the normalization constant Z. To compute Z, the reader, if he/she wishes, can — using Theorem 4.1, Comment 2, ... — proceed as in [11], [12], or [13], but, here, we will use Theorem 1.5. (Theorem 4.1 is a special case of Theorem

1.4; Theorem 1.4 leads to Theorem 1.5; so, Theorem 4.1 leads to the formula for Z we give here.) Since

$$\pi_{\mathrm{Id}} = \frac{\theta^n}{Z}$$

(Id is the first element of $\mathbb{S}_n$, which is equipped with the order relation $\overset{E}{\leq}$),

$$b_l = \frac{|\Delta_{l+1}|}{|\Delta_l|} = \begin{cases} n & \text{if } l = 1, \\ \frac{n(n-1)\dots(n-l+1)}{n(n-1)\dots(n-l+2)} & \text{if } l \in \langle n-1 \rangle - \{1\} \end{cases}$$

$$= n - l + 1, \ \forall l \in \langle n-1 \rangle,$$

$$D_{1,b_l} = \{1, 2, ..., b_l - 1\}, \ \forall l \in \langle n-1 \rangle,$$

and (see the proof of Theorem 3.3)

$$\alpha_w^{(l,1)} = \frac{1}{\theta}, \ \forall l \in \langle n-1 \rangle, \ \forall w \in D_{1,b_l},$$

we have (by Theorem 1.5)

$$Z = \theta^n \left(1 + \frac{1}{\theta}(n-1)\right)\left(1 + \frac{1}{\theta}(n-2)\right) \dots \left(1 + \frac{1}{\theta}\right) =$$

$$= \frac{\theta^n}{\theta^{n-1}} (\theta + n - 1)(\theta + n - 2) \dots (\theta + 1) = \theta(\theta + 1) \dots (\theta + n - 1).$$

This result is known (see, *e.g.*, [1]) — above we gave a new computation method, a Markovian computation method.

6. By Uniqueness Theorem from [10] (the presentation of this result is too long, so, we do not give it here) we can compute certain important probabilities for the Ewens distribution and, further, as a result, we can give upper bounds for the cumulative distribution function of number of cycles of permutation chosen from $\mathbb{S}_n$ according to the Ewens distribution. Using the Kronecker delta (symbol),

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$$

$\forall i, j \in \langle n \rangle$, Comment 2, and Uniqueness Theorem, we have

$$P\left(W_{(i_1)}\right) = \sum_{\sigma \in W_{(i_1)}} \pi_\sigma = \frac{\theta^{\delta_{1i_1}}}{\theta + n - 1},$$

$\forall i_1 \in \langle n \rangle$ ($P(A) = \sum_{\sigma \in A} \pi_\sigma$, $\forall A$, $\emptyset \neq A \subseteq \mathbb{S}_n$). Note that

$$W_{(i_1)} = \{\sigma \mid \sigma \in \mathbb{S}_n, \ \sigma(1) = i_1\}, \ \forall i_1 \in \langle n \rangle$$

($W_{(i_1)}$ is the set of permutations from $\mathbb{S}_n$, each permutation having the first component equal to i_1). Further, by Uniqueness Theorem, we have

$$\frac{P\left(W_{(i_1,i_2)}\right)}{P\left(W_{(i_1)}\right)} = \frac{\sum_{\sigma \in W_{(i_1,i_2)}} \pi_\sigma}{\sum_{\sigma \in W_{(i_1)}} \pi_\sigma} = \frac{\theta^{\delta_{2i_2}}}{\theta + n - 2},$$

$\forall i_1, i_2 \in \langle n \rangle$, $1 \le i_1 \le n$, $2 \le i_2 \le n$, so,

$$P\left(W_{(i_1,i_2)}\right) = P\left(W_{(i_1)}\right) \cdot \frac{P\left(W_{(i_1,i_2)}\right)}{P\left(W_{(i_1)}\right)} = \frac{\theta^{\delta_{1i_1}}}{\theta+n-1} \cdot \frac{\theta^{\delta_{2i_2}}}{\theta+n-2},$$

$\forall i_1, i_2 \in \langle n \rangle$, $1 \le i_1 \le n$, $2 \le i_2 \le n$. To compute $P\left(W_{(i_1,i_2,i_3)}\right)$, etc., we use (see Uniqueness Theorem, see also Comment 2)

$$\frac{P\left(W_{(i_1,i_2,\dots,i_u)}\right)}{P\left(W_{(i_1,i_2,\dots,i_{u-1})}\right)} = \frac{\sum\limits_{\sigma\in W_{(i_1,i_2,\dots,i_u)}} \pi_\sigma}{\sum\limits_{\sigma\in W_{(i_1,i_2,\dots,i_{u-1})}} \pi_\sigma} = \frac{\theta^{\delta_{ui_u}}}{\theta+n-u},$$

$\forall i_1, i_2, \dots, i_u \in \langle n \rangle$, $1 \le i_1 \le n$, $2 \le i_2 \le n$, ..., $u \le i_u \le n$ $(3 \le u \le n-1)$. We conclude that

$$P\left(W_{(i_1,i_2,\dots,i_l)}\right) = \frac{\theta^{\delta_{1i_1}}}{\theta+n-1} \cdot \frac{\theta^{\delta_{2i_2}}}{\theta+n-2} \cdot \dots \cdot \frac{\theta^{\delta_{li_l}}}{\theta+n-l} =$$

$$= \frac{\theta^{\delta_{1i_1}+\delta_{2i_2}+\dots+\delta_{li_l}}}{(\theta+n-1)(\theta+n-2)\dots(\theta+n-l)},$$

$\forall l \in \langle n-1 \rangle$, $\forall i_1, i_2, \dots, i_l \in \langle n \rangle$, $1 \le i_1 \le n$, $2 \le i_2 \le n$, ..., $l \le i_l \le n$, because

$$P\left(W_{(i_1,i_2,\dots,i_l)}\right) = P\left(W_{(i_1)}\right) \cdot \frac{P\left(W_{(i_1,i_2)}\right)}{P\left(W_{(i_1)}\right)} \cdot \dots \cdot \frac{P\left(W_{(i_1,i_2,\dots,i_l)}\right)}{P\left(W_{(i_1,i_2,\dots,i_{l-1})}\right)},$$

$\forall l \in \langle n-1 \rangle$, $\forall i_1, i_2, \dots, i_l \in \langle n \rangle$, $1 \le i_1 \le n$, $2 \le i_2 \le n$, ..., $l \le i_l \le n$. Now, we give upper bounds for the cumulative distribution function of X, where $X =$ the number of cycles of permutation chosen from $\mathbb{S}_n$ according to the Ewens distribution. We give upper bounds for $P(X \le k)$, $k \in \langle n-1 \rangle$ only ($P(X \le 0) = 0$; $P(X \le n) = 1$) — from these bounds, it can be derived the upper bounds for $P(X \le x)$, $x \in \mathbb{R}$. Note that (the first fact)

$$N(\sigma) \ge 1 + \sum_{k=1}^{l} \delta_{ki_k},$$

$\forall l \in \langle n-1 \rangle$, $\forall i_1, i_2, \dots, i_l \in \langle n \rangle$, $1 \le i_1 \le n$, $2 \le i_2 \le n$, ..., $l \le i_l \le n$, $\forall \sigma \in W_{(i_1,i_2,\dots,i_l)}$ (see the definition of $W_{(i_1,i_2,\dots,i_l)}$; if $\sigma \in W_{(i_1,i_2,\dots,i_l)}$ and $i_k = k$ for some $k \in \langle l \rangle$, then (k) is a cycle of σ). Note also that (the second fact)

$$\min_{\sigma\in W_{(i_1,i_2,\dots,i_{l-1},l)}} N(\sigma) = 1 + \min_{\sigma\in W_{(i_1,i_2,\dots,i_l)}} N(\sigma),$$

$\forall l \in \langle n-1 \rangle$ ($i_1, i_2, \dots, i_{l-1}$ vanish when $l = 1$), $\forall i_1, i_2, \dots, i_l \in \langle n \rangle$, $1 \le i_1 \le n$, $2 \le i_2 \le n$, ..., $l-1 \le i_{l-1} \le n$, $l < i_l \le n$ (not $l \le i_l \le n$), because the Ewens distribution is a wavy probability distribution and $\frac{1}{\theta}$ is the proportionality factor in all cases. Based on the first fact, we have, *e.g.*,

$$P(X > k) \ge P\left(W_{(1,2,\dots,k)}\right), \ \forall k \in \langle n-1 \rangle,$$

so,

$$P(X \le k) \le 1 - P\left(W_{(1,2,\dots,k)}\right) =$$

$$= 1 - \frac{\theta^k}{(\theta+n-1)(\theta+n-2)\dots(\theta+n-k)}, \ \forall k \in \langle n-1 \rangle.$$

We can give bounds for $P(X \leq k)$ better these. *E.g.*, we have

$$P(X > k) \geq P\left(W_{(1,2,...,k)}\right) + \sum_{a=k+1}^{n} P\left(W_{(1,2,...,k-1,a,k+1)}\right) +$$

$$+\sum_{b=2}^{n} P\left(W_{(b,2,3,...,k+1)}\right), \ \forall k \in \langle n-2\rangle$$

$(P(X > n-1) = P(X = n) = \pi_{\mathrm{Id}} = \frac{\theta^n}{Z})$, so,

$$P(X \leq k) \leq 1 - \frac{\theta^k}{(\theta+n-1)(\theta+n-2)...(\theta+n-k)} -$$

$$-(2n-k-1)\cdot\frac{\theta^k}{(\theta+n-1)(\theta+n-2)...(\theta+n-(k+1))} =$$

$$= 1 - \frac{\theta^k}{(\theta+n-1)(\theta+n-2)...(\theta+n-k)}\left(1+\frac{2n-k-1}{\theta+n-k-1}\right), \ \forall k \in \langle n-2\rangle .$$

Using the probabilities $P\left(W_{(i_1,i_2,...,i_l)}\right)$, $l \in \langle n-1\rangle$, $i_1, i_2, ..., i_l \in \langle n\rangle$, $1 \leq i_1 \leq n$, $2 \leq i_2 \leq n$, ..., $l \leq i_l \leq n$, and the facts mentioned above or other facts — the tree of inclusions can also be used, see [10] and, here, Example 5.1 for examples of trees of inclusions —, we can obtain better and better upper bounds for $P(X \leq k)$, even the exact value of $P(X \leq k)$ for some $k \in \langle n-1\rangle$ or, *e.g.*, when n is small, for all $k \in \langle n-1\rangle$.

7. Sampling from $\mathbb{S}_n$ according to the Ewens distribution leads to sampling from $\mathbb{A}_n$ according to the Ewens sampling formula. Indeed, to sample from $\mathbb{A}_n$ according to the Ewens sampling formula, we, based on Section 2, proceed as follows.

Choose a permutation, σ, from $\mathbb{S}_n$ according to the Ewens distribution.
Write σ as a composition of pair-wise disjoint cycles.
Compute $k(\sigma)$.
$k(\sigma)$ is the result of sampling from $\mathbb{A}_n$ according to the Ewens sampling formula.

5. An example

In this section, we give an example to illustrate some things from previous sections.

Example 5.1. Consider the Ewens distribution on $\mathbb{S}_3$. Consider that $\mathbb{S}_3$ is equipped with the order relation $\overset{E}{\leq}$. We have (for cycles of length greater than 1, we use commas ("$(1,1)$", "$(2,2)$", and (this is not used below) "$(3,3)$" also contain commas) while for permutations we do not)

$$(1,1)\circ(2,2) = (1)\circ(2)\circ(3) = (123) = \mathrm{Id},$$

$$(1,1)\circ(2,3) = (1)\circ(2,3) = (132),$$

$$(1,2)\circ(2,2) = (1,2)\circ(3) = (213),$$

$$(1,2)\circ(2,3) = (1,2,3) = (231),$$

$$(1,3)\circ(2,2)=(1,3)\circ(2)=(321)\,,$$
$$(1,3)\circ(2,3)=(1,3,2)=(312)\,,$$

so,

$$\mathrm{Id}=(123)\overset{E}{\leq}(132)\overset{E}{\leq}(213)\overset{E}{\leq}(231)\overset{E}{\leq}(321)\overset{E}{\leq}(312)\,,$$
$$\pi_{(123)}=\frac{\theta^3}{Z},\ \pi_{(132)}=\frac{\theta^2}{Z},\ \pi_{(213)}=\frac{\theta^2}{Z},$$
$$\pi_{(231)}=\frac{\theta}{Z},\ \pi_{(321)}=\frac{\theta^2}{Z},\ \pi_{(312)}=\frac{\theta}{Z},$$
$$W_{(1)}=\{(123)\,,(132)\}\,,\ W_{(2)}=\{(213)\,,(231)\}\,,\ W_{(3)}=\{(321)\,,(312)\}$$
$$(\,|W_{(1)}|=|W_{(2)}|=|W_{(3)}|=2,\ \text{see (c1)}),$$
$$W_{(1,2)}=\{(123)\}\,,\ W_{(1,3)}=\{(132)\}\,,\ W_{(2,2)}=\{(213)\}\,,$$
$$W_{(2,3)}=\{(231)\}\,,\ W_{(3,2)}=\{(321)\}\,,\ W_{(3,3)}=\{(312)\}\,.$$

Since

$$\Delta_1=(\mathbb{S}_3)\,,$$
$$\Delta_2=\left(W_{(1)},W_{(2)},W_{(3)}\right)\,,$$
$$\Delta_3=\left(W_{(1,2)},W_{(1,3)},W_{(2,2)},W_{(2,3)},W_{(3,2)},W_{(3,3)}\right)$$
$$(\Delta_1=(\mathbb{S}_3)\succ\Delta_2\succ\Delta_3=(\{\sigma\})_{\sigma\in\mathbb{S}_3}\,),$$

we have, considering Δ_1 and Δ_2,

$$\left(\pi_{(213)},\pi_{(231)}\right)=\frac{1}{\theta}\left(\pi_{(123)},\pi_{(132)}\right)\,,$$
$$\left(\pi_{(321)},\pi_{(312)}\right)=\frac{1}{\theta}\left(\pi_{(123)},\pi_{(132)}\right)\,,$$

(the proportionality factor is $\frac{1}{\theta}$ in both cases), and, considering Δ_2 and Δ_3 (here, we do not use vectors anymore),

$$\pi_{(132)}=\frac{1}{\theta}\pi_{(123)},\ \pi_{(231)}=\frac{1}{\theta}\pi_{(213)},\ \pi_{(312)}=\frac{1}{\theta}\pi_{(321)}$$

(the proportionality factor is $\frac{1}{\theta}$ in all three cases). Therefore, $\pi=(\pi_\sigma)_{\sigma\in\mathbb{S}_3}$ is a wavy probability distribution with respect to the order relation $\overset{E}{\leq}$ and partitions Δ_1, Δ_2, Δ_3. By Theorem 4.1 or Comment 2 we have

$$P_1=\begin{array}{c}(123)\\(132)\\(213)\\(231)\\(321)\\(312)\end{array}\overset{\begin{array}{cccccc}(123)&(132)&(213)&(231)&(321)&(312)\end{array}}{\begin{pmatrix}\frac{\theta}{\theta+2}&0&\frac{1}{\theta+2}&0&\frac{1}{\theta+2}&0\\0&\frac{\theta}{\theta+2}&0&\frac{1}{\theta+2}&0&\frac{1}{\theta+2}\\\frac{\theta}{\theta+2}&0&\frac{1}{\theta+2}&0&\frac{1}{\theta+2}&0\\0&\frac{\theta}{\theta+2}&0&\frac{1}{\theta+2}&0&\frac{1}{\theta+2}\\\frac{\theta}{\theta+2}&0&\frac{1}{\theta+2}&0&\frac{1}{\theta+2}&0\\0&\frac{\theta}{\theta+2}&0&\frac{1}{\theta+2}&0&\frac{1}{\theta+2}\end{pmatrix}}$$

(the rows and columns of P_1 are labeled using the order relation $\overset{E}{\leq}$) and

$$P_2 = \begin{array}{c} \\ (123) \\ (132) \\ (213) \\ (231) \\ (321) \\ (312) \end{array} \begin{array}{c} \begin{array}{cccccc} (123) & (132) & (213) & (231) & (321) & (312) \end{array} \\ \left(\begin{array}{cccccc} \frac{\theta}{\theta+1} & \frac{1}{\theta+1} & & & & \\ \frac{\theta}{\theta+1} & \frac{1}{\theta+1} & & & & \\ & & \frac{\theta}{\theta+1} & \frac{1}{\theta+1} & & \\ & & \frac{\theta}{\theta+1} & \frac{1}{\theta+1} & & \\ & & & & \frac{\theta}{\theta+1} & \frac{1}{\theta+1} \\ & & & & \frac{\theta}{\theta+1} & \frac{1}{\theta+1} \end{array} \right) \end{array}.$$

It is easy to see — or see Theorem 4.1 — that the chain with transition matrix $P = P_1 P_2$ is a Gibbs sampler in a generalized sense taking $Q_1 = P_1$, $Q_2 = P_2$ (Theorem 1.3 together with the equations

$$\pi_\sigma \left(P_l\right)_{\sigma\tau} = \pi_\tau \left(P_l\right)_{\tau\sigma}, \ \forall l \in \langle 2 \rangle, \ \forall \sigma, \tau \in \mathbb{S}_3$$

(it is easy to prove these equations) and other things (see also the proof of Theorem 1.4) suggest this choice). It is easy to see that $P_1 \in G_{\Delta_1,\Delta_2}$, $P_2 \in G_{\Delta_2,\Delta_3}$. By Theorem 4.1 or direct computation, $P = e'\pi$. Since $\pi_{(123)} = \frac{\theta^3}{Z}$, it is easy to see — or see Comment 5 —, using $P = e'\pi$, that $Z = \theta\left(\theta+1\right)\left(\theta+2\right)$. Obviously, P_2 is a block diagonal matrix and (see Definition 1.4) Δ_2-stable matrix on Δ_2. Moreover, P_2 is a Δ_2-stable matrix. P_1 is a stable matrix both on Δ_1 and on Δ_2. By Uniqueness Theorem from [10], Comment 6, or direct computation we have, using Comment 6,

$$P\left(W_{(1)}\right) = \frac{\theta^{\delta_{11}}}{\theta+2} = \frac{\theta}{\theta+2}, \ P\left(W_{(2)}\right) = \frac{\theta^{\delta_{12}}}{\theta+2} = \frac{1}{\theta+2},$$

$$P\left(W_{(3)}\right) = \frac{\theta^{\delta_{13}}}{\theta+2} = \frac{1}{\theta+2},$$

$$P\left(W_{(1,2)}\right) = \frac{\theta^{\delta_{11}}}{\theta+2} \cdot \frac{\theta^{\delta_{22}}}{\theta+1} = \frac{\theta^2}{\left(\theta+1\right)\left(\theta+2\right)},$$

$$P\left(W_{(1,3)}\right) = \frac{\theta^{\delta_{11}}}{\theta+2} \cdot \frac{\theta^{\delta_{23}}}{\theta+1} = \frac{\theta}{\left(\theta+1\right)\left(\theta+2\right)},$$

$$P\left(W_{(2,2)}\right) = \frac{\theta^{\delta_{12}}}{\theta+2} \cdot \frac{\theta^{\delta_{22}}}{\theta+1} = \frac{\theta}{\left(\theta+1\right)\left(\theta+2\right)},$$

$$P\left(W_{(2,3)}\right) = \frac{\theta^{\delta_{12}}}{\theta+2} \cdot \frac{\theta^{\delta_{23}}}{\theta+1} = \frac{1}{\left(\theta+1\right)\left(\theta+2\right)},$$

$$P\left(W_{(3,2)}\right) = \frac{\theta^{\delta_{13}}}{\theta+2} \cdot \frac{\theta^{\delta_{22}}}{\theta+1} = \frac{\theta}{\left(\theta+1\right)\left(\theta+2\right)},$$

$$P\left(W_{(3,3)}\right) = \frac{\theta^{\delta_{13}}}{\theta+2} \cdot \frac{\theta^{\delta_{23}}}{\theta+1} = \frac{1}{\left(\theta+1\right)\left(\theta+2\right)}.$$

The tree of inclusions is (the tree from here is weighted)

$$\begin{array}{ccccccccc}
 & & & & \mathbb{S}_3 & & & & \\
 & & 2\diagup & & 1\,| & & \diagdown 1 & & \\
 & W_{(1)} & & & W_{(2)} & & & W_{(3)} & \\
 & 3\diagup\ \diagdown 2 & & & 2\diagup\ \diagdown 1 & & & 2\diagup\ \diagdown 1 & \\
W_{(1,2)} & & W_{(1,3)} & W_{(2,2)} & & W_{(2,3)} & W_{(3,2)} & & W_{(3,3)}
\end{array}$$

By direct computation or, in some cases, by Comment 6 (the second fact), $\min\limits_{\sigma\in W_{(3)}} N(\sigma) = 1$, 1 is assigned to the edge $\left[\mathbb{S}_3, W_{(3)}\right]$, etc. This tree gives, *e.g.*, the sets of permutations with at least 2 cycles and number of permutations with 2 cycles; these sets are $W_{(1)}$, $W_{(2,2)}$, $W_{(3,2)}$ and the number of permutations with 2 cycles = the number of permutations with at least 2 cycles − the number of permutations with at least 3 cycles $= \left|W_{(1)}\right| + \left|W_{(2,2)}\right| + \left|W_{(3,2)}\right| - \left|W_{(1,2)}\right| = 2+1+1-1 = 3$. If the initial state of chain is Id (to avoid rewriting the states, we must choose Id as the initial state of chain, see Comment 4), then from this state the chain passes in one of the states (see Theorem 4.1, Comment 2, Comment 4, or P_1) Id$=(1,1)=(123)$, $(1,2)=(213)$, $(1,3)=(321)$, the transition probabilities being (see Theorem 4.1, Comment 2, Comment 4, or P_1)

$$\frac{\theta}{\theta+2},\ \frac{1}{\theta+2},\ \frac{1}{\theta+2},$$

respectively. Suppose that it passed in the state (213). From (213) the chain passes in one of the states (see Theorem 4.1, Comment 2, or P_2) $(213)\circ(2,2)=(213)$, $(213)\circ(2,3)=(231)$, the transition probabilities being (see Theorem 4.1, Comment 2, or P_2)

$$\frac{\theta}{\theta+1},\ \frac{1}{\theta+1},$$

respectively. Suppose that it passed in the state (231). (231) is the state selected from $\mathbb{S}_3$ with our method, having, here, 2 $(3-1=2)$ steps.

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