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Existence of weak periodic solution for quasilinear parabolic problem with nonlinear boundary conditions

Abdelwahab Elaassri, Kaoutar Lamrini Uahabi, Abderrahim Charkaoui, Nour Eddine Alaa, and Salim Mesbahi

Abstract. In this work, we show the existence of weak periodic solutions of a quasilinear parabolic problem with arbitrary growth nonlinearity in gradient and nonlinear boundary conditions. The objective will be achieved by reformulating the problem in abstract form and applying some results and techniques of functional analysis such as the method of truncation associated with Schauder fixed point theorem.

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1. Introduction

In this work we are interesting in the existence of weak periodic solution for the following quasilinear problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + G(t,x,u,\nabla u) = f & \text{in } \ Q_T \\ u(0,.) \ = \ u(T,.) & \text{in } \ \Omega \\ -\dfrac{\partial u}{\partial \nu} = \beta(t,x)u + g(t,x,u) & \text{on } \ \Sigma_T \end{cases} \tag{1}$$

where Ω is an open regular bounded subset of $\mathbb{R}^N$, $N \geqslant 1$, with smooth boundary $\partial\Omega$, $T > 0$ is the period, $Q_T =]0,T[\times\Omega$, $\Sigma_T =]0,T[\times\partial\Omega$, ν denote the unit normal vector to the boundary $\partial\Omega$, $-\Delta$ denotes the Laplacian operator with nonlinear boundary conditions, G is a caratheodory and f is a measurable function.

Many phenomena of biology, chemistry and physics are modelled by equations and systems of the form (1). Before stating the main result of this work, it is worth mentioning that several mathematicians have dealt with this type of problem in Dirichlet or Neumann boundary conditions using various analytical and numerical techniques and methods, under different hypotheses depending on the situation , see for example, Amman [5], Badii [6], [7], Browder [9], Nakao [16], Pao [17], Sun [21], Wang [22], [23], Zhang [24].

Concerning the existence of classical solutions, complete results were obtained by Amann in [5]. In [10], Duel and Hess have demonstrated the existence of at least one periodic solution of a very general second order nonlinear parabolic boundary value problem under the assumption that a lower solution ϕ and an upper solution

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ψ with $\phi \leq \psi$ are known. In [17], Pao studied the existence and stability of periodic solutions for a nonlinear parabolic system under nonlinear boundary conditions. The approach to the problem is by the method of upper and lower solutions and its associated monotone iterations. The objective of the study of Alaa and Igurnane in [3] is to show the existence of a weak periodic solutions for some quasi-linear parabolic equations with data measures having critical growth non-linearity on the gradient and Dirichlet boundary condition. In [6], Badii considered the case where G is independent of the gradient, he proved the existence of a weak periodic solutions for nonlinear parabolic equations with the Robin periodic boundary condition under normal general assumptions on the nonlinearities.

The objective of this paper is to present an existence result of at least one weak periodic solution to the problem (1) in the case where G depends on the gradient, applying techniques based on the truncation method joint with the Schauder fixed point theorem.

Our work is presented as follows. In Section 2, we choose the functional framework in which periodic solutions are sought and give the definition of the weak periodic solution. In Section 3, we demonstrate the existence of periodic solutions for an abstract problem in the case when the nonlinearities are bounded by using Schauder fixed point theorem to a nonlinear operator equation. Section 4 is devoted to study the existence when the nonlinearities has critical growth with respect to the gradient.

2. Statement of the main result

In this section we establish our main result, first of all we introduce necessary assumptions.

2.1. Assumptions. Let us now introduce the hypotheses which we assume throughout this section. We consider that

A_1) f is a periodic function such that $f \in L^2(Q_T)$.

A_2) β is a periodic positive continuous and bounded function such that

$$0 < \beta_0 \leqslant \beta(t,x) \leqslant \beta_1, \ \ \forall (t,x) \in \Sigma_T.$$

A_3) $g : \Sigma_T \times \mathbb{R} \to \mathbb{R}$ is a Caratheodory function periodic in time, $s \mapsto g(t,x,s)$ is nondecreasing with respect to s for a.e. $(t,x) \in \Sigma_T$ and

$$g(t,x,s)s \geqslant 0$$

$$\mid g(t,x,s) \mid \leqslant \xi(t,x) + \mid s \mid$$

where $\xi \in L^2(\Sigma_T)$.

A_4) $G : Q_T \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Caratheodory function periodic with respect to t, such that

$$G(t,x,s,r) \in L^1(Q_T), \ \ \forall s \in \mathbb{R}, \forall r \in \mathbb{R}^N \text{ and } a.e.\, (t,x) \in Q_T.$$

A_5) G satisfies the following sign assumption

$$G(t,x,s,r)s \geqslant 0, \ \ \forall s \in \mathbb{R}, \forall r \in \mathbb{R}^N \text{ and } a.e.\, (t,x) \in Q_T.$$

A_6) $\mid G(t,x,s,r) \mid \leq \mu(\mid s \mid)\bigg(H(t,x) + d\,\|r\|^{\alpha}\bigg)$, $\forall r \in \mathbb{R}^N$, $\forall s \in \mathbb{R}$, a.e.$(t,x) \in Q_T$, with $1 \leq \alpha < 2$, $H \in L^1(Q_T)$, $d > 0$ and $\mu : \mathbb{R}^+ \to \mathbb{R}^+$ is a nondecreasing continuous function.

2.2. Notations and definitions. In this section we present our functional framework for the periodic solutions of problem, we set

$$\mathcal{V} := L^2(0,T;H^1(\Omega)) \quad \text{and} \quad \mathcal{V}^* := L^2(0,T;(H^1(\Omega))^*)$$

we denote by $(H^1(\Omega))^*$ the topological dual space of $H^1(\Omega)$ and $< .,. >$ present the duality pairing between $\mathcal{V}$ and $\mathcal{V}^*$, the standard norm of $L^2(0,T;H^1(\Omega))$ is defined by

$$\| u \|_{L^2(0,T;H^1(\Omega))} := \left(\int_{Q_T} \mid \nabla u(t,x) \mid^2 dt\,dx + \int_{Q_T} \mid u(t,x) \mid^2 dt\,dx \right)^{\frac{1}{2}}$$

Through this paper we equipped $\mathcal{V}$ with the norm

$$\| u \|_{\mathcal{V}} := \left(\int_{Q_T} \mid \nabla u(t,x) \mid^2 dt\,dx + \int_{\Sigma_T} \beta(t,\sigma) \mid u(t,\sigma) \mid^2 \, dt\,d\sigma \right)^{\frac{1}{2}}$$

which is equivalent to the standard norm of $L^2(0,T;H^1(\Omega))$, we denote by u the trace of u on Σ_T. Let us define the set

$$\mathcal{W}\,(0,T) = \left\{ u \in \mathcal{V} \mid \frac{\partial u}{\partial t} \in \mathcal{V}^* \text{ and } u\,(0) = u\,(T) \right\}$$

equipped with the norm

$$\|u\|_{\mathcal{W}(0,T)} = \|u\|_{\mathcal{V}} + \left\| \frac{\partial u}{\partial t} \right\|_{\mathcal{V}^*}$$

It is necessary to clarify in which sense we want to solve the problem (1), for which we introduce the notion of weak periodic solution.

Definition 2.1. A function u is said to be a weak periodic solution of the problem (1), if it satisfies

$$u \in \mathcal{V},\; G(t,x,u,\nabla u) \in L^1(Q_T),\; uG(t,x,u,\nabla u) \in L^1(Q_T)$$

and $\forall \phi \in \mathcal{W}(0,T) \cap L^\infty(Q_T)$

$$- < u, \frac{\partial \phi}{\partial t} > + \int_{Q_T} \nabla u \nabla \phi + \int_{Q_T} G(t,x,u,\nabla u)\phi + \int_{\Sigma_T} \beta(t,x) u\phi + \int_{\Sigma_T} g(t,x,u)\phi = \int_{Q_T} f\phi.$$

This enables us to state the first result of the existence of a weak periodic solution.

3. An existence result when G is bounded

Theorem 3.1. *Under hypotheses (A_1)-(A_5) and assuming that there exists a nonnegative function $h \in L^2(Q_T)$ such as for almost $(t,x) \in Q_T$ and all $s \in \mathbb{R}$, $r \in \mathbb{R}^N$,*

$$\mid G(t,x,s,r) \mid \leqslant h(t,x). \tag{2}$$

Then the problem (1) admits a weak periodic solution $u \in \mathcal{W}(0,T)$.

3.1. Proof of Theorem 3.1. The existence of weak solution to (1) will be based on the research of fixed points for the following mapping

$$\begin{aligned} \Psi : \mathcal{V} &\longrightarrow \mathcal{V} \\ w &\longmapsto \Psi(w) = u \end{aligned}$$

where u is the unique weak periodic solution of

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + G\left(t, x, w, \nabla w\right) = f & \text{in } Q_T \\ u\left(0\right) = u\left(T\right) & \text{in } \Omega \\ -\dfrac{\partial u}{\partial t} = \beta\left(t, x\right) u + g\left(t, x, u\right) & \text{on } \Sigma_T \end{cases} \tag{3}$$

The existence and the uniqueness of $u \in \mathcal{W}(0,T) \subset \mathcal{V}$ weak periodic solution of (3) is a direct consequence of Badii's theorem in [6] and satisfy $\forall \varphi \in \mathcal{W}(0,T)$

$$\begin{aligned} -< u, \frac{\partial \varphi}{\partial t} > + \int_{Q_T} \nabla u \nabla \varphi + \int_{\Sigma_T} g\left(t, x, u\right) \varphi + \int_{\Sigma_T} \beta\left(t, x\right) u \varphi \\ + \int_{Q_T} G\left(t, x, w, \nabla w\right) \varphi = \int_{Q_T} f \varphi \end{aligned} \tag{4}$$

Then the mapping is well defined, to prove the existence of a fixed point of Ψ, we will pass by three steps.

3.1.1. *Continuity of* Ψ. To show the continuity of Ψ, we will prove some very important estimates and convergences. Let $w_n \in \mathcal{V}$ be a sequence converge strongly to w in $\mathcal{V}$. Moreover, let u_n denote the weak periodic solution of the problem

$$\begin{aligned} -< u_n, \frac{\partial \varphi}{\partial t} > + \int_{Q_T} \nabla u_n \nabla \varphi + \int_{\Sigma_T} g\left(t, x, u_n\right) \varphi + \int_{\Sigma_T} \beta\left(t, x\right) u_n \varphi \\ + \int_{Q_T} G\left(t, x, w_n, \nabla w_n\right) \varphi = \int_{Q_T} f \varphi \end{aligned} \tag{5}$$

Setting $\varphi = u_n$ as a test function in (5), we have

$$\begin{aligned} -< u_n, \frac{\partial u_n}{\partial t} > + \int_{Q_T} \left|\nabla u_n\right|^2 + \int_{\Sigma_T} g\left(t, x, u_n\right) u_n + \int_{\Sigma_T} \beta\left(t, x\right) u_n^2 \\ + \int_{Q_T} G\left(t, x, w_n, \nabla w_n\right) u_n = \int_{Q_T} f u_n \end{aligned} \tag{6}$$

Conditions (A_2), (A_3), (A_5) the periodicity and the Young's inequality give us,

$$\begin{aligned} \int_{Q_T} \mid \nabla u_n \mid^2 + \int_{\Sigma_T} \beta(t,x) \mid u_n \mid^2 &\leqslant \int_{Q_T} \mid f u_n \mid, \\ &\leqslant C(\epsilon) \parallel f \parallel^2_{L^2(Q_T)} + \epsilon \parallel u_n \parallel^2_{L^2(Q_T)} \\ &\leqslant C(\epsilon) \parallel f \parallel^2_{L^2(Q_T)} + \epsilon C(T,\Omega) \parallel u_n \parallel^2_{\mathcal{V}} \end{aligned}$$

we choose ϵ small enough to obtain the following classical energy estimate

$$\left(\int_{Q_T} \left|\nabla u_n\left(t, x\right)\right|^2 + \int_{\Sigma_T} \beta\left(t, x\right) \left|u_n\right|^2 \right) \leq C \tag{7}$$

where the positive real constant C is independent of n. From (5) and the energy estimate (7) we get that $(\frac{\partial u_n}{\partial t})$ is bounded in the $\mathcal{V}^*$ norm. This provided the boundedness of u_n in the norm of the set $\mathcal{W}(0,T)$, i.e.

$$\|u_n\|_{\mathcal{W}(0,T)} \leq C \text{ , for all } n \in \mathbb{N}$$

Thus, we can select a subsequence, still denoted by u_n such that

$$u_n \rightharpoonup u \text{ weakly in } \mathcal{V} \text{ as } n \to +\infty$$

By Aubin's Theorem [21], the sequence u_n is precompact in $L^2(Q_T)$ therefore,

$$u_n \longrightarrow u \text{ in } L^2(Q_T) \text{ and a.e. in } Q_T$$

Furthermore, according to the trace theorem, see Morrey [15], theorem 3.4.1, one has

$$u_n \longrightarrow u \text{ in } L^2(\Sigma_T) \text{ and a.e. in } \Sigma_T$$

To conclude we prove that the sequence ∇u_n converges strongly to ∇u in $L^2(Q_T)$. From (6), one has

$$\begin{aligned}\int_{Q_T} |\nabla u_n|^2 = \int_{Q_T} f u_n - \int_{\Sigma_T} g(t,x,u_n)\, u_n - \int_{\Sigma_T} \beta(t,x)\, u_n^2 \\ - \int_{Q_T} G(t,x,w_n,\nabla w_n)\, u_n\end{aligned}$$

since w_n converge strongly in $\mathcal{V}$ and u_n converge strongly in $L^2(Q_T)$, we can pass to the limit in the last inequality, so

$$\begin{aligned}\lim_{n\mapsto+\infty} \int_{Q_T} |\nabla u_n|^2 = \int_{Q_T} f u - \int_{\Sigma_T} g(t,x,u)\, u - \int_{\Sigma_T} \beta(t,x)\, u^2 \\ - \int_{Q_T} G(t,x,w,\nabla w)\, u \end{aligned} \tag{8}$$

Moreover, setting $\varphi = u$ as a test function in (5), we obtain

$$\begin{aligned}- < u_n, \frac{\partial u}{\partial t} > + \int_{Q_T} \nabla u_n \nabla u = \int_{Q_T} f u - \int_{\Sigma_T} g(t,x,u_n)\, u - \int_{\Sigma_T} \beta(t,x)\, u_n u \\ - \int_{Q_T} G(t,x,w_n,\nabla w_n)\, u\end{aligned}$$

Taking the limit on $n \to +\infty$, we have

$$\begin{aligned}\int_{Q_T} |\nabla u|^2 = \int_{Q_T} f u - \int_{\Sigma_T} g(t,x,u)\, u - \int_{\Sigma_T} \beta(t,x)\, u^2 \\ - \int_{Q_T} G(t,x,w,\nabla w)\, u \end{aligned} \tag{9}$$

By comparing (8) and (9), we obtain

$$\lim_{n\to+\infty} \int_{Q_T} |\nabla u_n|^2 = \int_{Q_T} |\nabla u|^2$$

which gives us the result indicated. Consequently, the mapping Ψ is continuous.

3.1.2. ***Compactness of*** Ψ. Let (w_n) be a bounded sequence in $\mathcal{V}$ and we denote $u_n = \Psi(w_n)$, by the same reasoning of the first step, we have (up to a subsequence)

$$\begin{aligned}
w_n &\rightharpoonup w \text{ weakly in } \mathcal{V}, \\
u_n &\rightharpoonup u \text{ weakly in } \mathcal{V}, \\
\frac{\partial u_n}{\partial t} &\rightharpoonup \frac{\partial u}{\partial t} \text{ weakly in } \mathcal{V}^*, \\
u_n &\to u \text{ strongly in } L^2(Q_T) \text{ and a.e in } Q_T, \\
u_n &\to u \text{ strongly in } L^2(\Sigma_T) \text{ and a.e in } \Sigma_T
\end{aligned}$$

to get the compactness of Ψ, it suffices to prove the strong convergence of (∇u_n) in $L^2(Q_T)$, noting that the difficulty is presented by the absence of the almost every where convergence of (∇w_n) in Q_T, but we can overcome this problem. Observing that

$$\int_{Q_T} \mid \nabla u_n - \nabla u \mid^2 = \int_{Q_T} \nabla u_n(\nabla u_n - \nabla u) - \int_{Q_T} \nabla u(\nabla u_n - \nabla u),$$

thanks to the weak convergence of (u_n) in $\mathcal{V}$, we have $\lim\limits_{n \mapsto +\infty} \int_{Q_T} \nabla u(\nabla u_n - \nabla u) = 0$, on the other hand, setting $\varphi = u$ as a test function in the equation satisfied by (u_n), it follows

$$\begin{aligned}
\int_{Q_T} \nabla u_n(\nabla u_n - \nabla u) = \int_{Q_T} f(u_n - u) - < \frac{\partial u_n}{\partial t}, u_n - u > - \int_{\Sigma_T} g\,(t, x, u_n)\,(u_n - u) \\
- \int_{\Sigma_T} \beta\,(t, x)\, u_n(u_n - u) - \int_{Q_T} G\,(t, x, w_n, \nabla w_n)\,(u_n - u)
\end{aligned} \tag{10}$$

By using (2), we have

$$\int_{Q_T} \mid G\,(t, x, w_n, \nabla w_n)\,(u_n - u) \mid \leqslant \| h \|_{L^2(Q_T)} \| u_n - u \|_{L^2(Q_T)}$$

then

$$\lim_{n \mapsto +\infty} \int_{Q_T} \mid G\,(t, x, w_n, \nabla w_n)\,(u_n - u) \mid = 0.$$

On the other hand, the periodicity and the weak convergence of $(\frac{\partial u_n}{\partial t})$ in $\mathcal{V}^*$ yields

$$\lim_{n \mapsto +\infty} < \frac{\partial u_n}{\partial t}, u_n - u > = - \lim_{n \mapsto +\infty} < \frac{\partial u_n}{\partial t}, u > = < \frac{\partial u}{\partial t}, u > = 0.$$

Now, we can pass to the limit in (10) to get

$$\lim_{n \mapsto +\infty} \int_{Q_T} \nabla u_n(\nabla u_n - \nabla u) = 0$$

So, we obtain the strong convergence of (∇u_n) in $L^2(Q_T)$, this allows us to get the compactness of Ψ.

3.1.3. Ψ ***send the ball of*** $\mathcal{V}$ ***of*** R ***radius to itself.*** In this step, we get the existence of a constant $R > 0$ such that $\Psi(\mathcal{B}(0,R)) \subset \mathcal{B}(0,R)$ where $\mathcal{B}(0,R)$ is the ball of $\mathcal{V}$ with radius R. Let $w \in \mathcal{V}$ and $u = \Psi(w)$, by taking u as test function in the equation satisfied by u, it is easy to prove the following estimate

$$\| u \|_{\mathcal{V}} \leqslant \left(\| h \|_{L^2(Q_T)} + \| f \|_{L^2(Q_T)} \right) := R.$$

Then by a simple application of Schauder's fixed point theorem there exists $u \in \mathcal{V}$ such that $u = \Psi(u)$ i.e u solves (1).

4. Existence result when G has critical growth nonlinearities with respect to the gradient

In this section, we will study the existence of weak periodic solution of (1) when the nonlinearities G has critical growth with respect to the gradient, which can be satisfied by the assumption (A_6).

Theorem 4.1. *Assume that* (A_1)-(A_6) *hold. Then* (1) *has a weak periodic solution* u *in the sense of the definition 2.1.*

4.1. Proof of Theorem 4.1. To prove Theorem 4.1, we will use the method of truncation combined with the result of Theorem 3.1, the difficulty come back to prove the almost everywhere convergence of the gradient of the approximate solution under nonlinear boundary condition.

4.1.1. ***Approximating problem.*** For all $n \geqslant 1$, we approximate G as follows

$$G_n(t,x,s,r) = \frac{G(t,x,s,r)}{1 + \frac{1}{n} \mid G(t,x,s,r) \mid},$$

we remark that G_n satisfies the following properties

$$G_n(t,x,s,r)s \geqslant 0, \text{ and } \mid G_n(t,x,s,r) \mid \leqslant n.$$

We define the approximate problem of (1) by

$$\begin{cases} u_n \in \mathcal{W}(0,T) \\ - < u_n, \frac{\partial \phi}{\partial t} > + \displaystyle\int_{Q_T} \nabla u_n \nabla \phi + \int_{Q_T} G_n(t,x,u_n,\nabla u_n)\phi \\ \quad + \displaystyle\int_{\Sigma_T} \beta(t,x) u_n \phi + \int_{\Sigma_T} g(t,x,u_n)\phi = \int_{Q_T} f\phi \\ \forall \phi \in \mathcal{W}(0,T). \end{cases} \tag{11}$$

Since G_n is bounded then we can applied the result of the Theorem 3.1 to obtain the existence of u_n a weak periodic solution of the approximate problem (11). We want to pass to the limit in (11), for which we need to prove some a priori estimates, we start by defining the truncation function that will be used after, for all $k > 0$, $T_k(s) = \min\{k, \max\{s, -k\}\}$.

Lemma 4.2. *Assume that* (A_1)-(A_6) *hold, then*

i) *There exists a constant* C_1 *independent of* n *such that*

$$\|u_n\|_{\mathcal{V}} \leq C_1.$$

ii) *There exists a constant C_2 depending only on T, Ω and $\| f \|_{L^2(Q_T)}$ such that*

$$\int_{Q_T} G_n(t,x,u_n,\nabla u_n)u_n \le C_2.$$

iii) *There exists a constant C_3 independent of n such that*

$$\int_{Q_T} | G_n(t,x,u_n,\nabla u_n) | \le C_3.$$

Proof. i) By taking $\phi = u_n$ as a test function in (11), we get

$$-< u_n, \frac{\partial u_n}{\partial t} > + \int_{Q_T} | \nabla u_n |^2 + \int_{Q_T} G_n(t,x,u_n,\nabla u_n)u_n + \int_{\Sigma_T} \beta(t,x) | u_n |^2 + \int_{\Sigma_T} g(t,x,u_n)u_n = \int_{Q_T} f u_n$$

Thanks to $(A_3), (A_5)$ and the periodicity, we have

$$\int_{Q_T} | \nabla u_n |^2 + \int_{\Sigma_T} \beta(t,x) | u_n |^2 \leqslant \int_{Q_T} | f u_n |$$

Using Young's inequality, one has

$$\begin{aligned} \| u_n \|_{\mathcal{V}}^2 &\leqslant C(\epsilon) \| f \|_{L^2(Q_T)}^2 + \epsilon \| u_n \|_{L^2(Q_T)}^2 \\ &\leqslant C(\epsilon) \| f \|_{L^2(Q_T)}^2 + \epsilon C(T,\Omega) \| u_n \|_{\mathcal{V}}^2 \end{aligned}$$

we choose ϵ small enough to obtain (u_n) bounded in $\mathcal{V}$.

ii) Setting $\phi = u_n$ as test function in (11), we obtain

$$\int_{Q_T} G_n(t,x,u_n,\nabla u_n)u_n + \int_{\Sigma_T} g(t,x,u_n)u_n = - \| u_n \|_{\mathcal{V}}^2 + < u_n, \frac{\partial u_n}{\partial t} > + \int_{Q_T} f u_n$$

we use the periodicity of u_n, (i) and the Hölder's inequality, we get

$$\int_{Q_T} G_n(t,x,u_n,\nabla u_n)u_n + \int_{\Sigma_T} g(t,x,u_n)u_n \leqslant \| f \|_{L^2(Q_T)} \| u_n \|_{L^2(Q_T)} + C_1$$

From the hypothesis (A_2) and the result of (i), we have

$$\int_{Q_T} G_n(t,x,u_n,\nabla u_n)u_n \leqslant C_2$$

iii) We remark that,

$$\int_{Q_T} | G_n(t,x,u_n,\nabla u_n) | \, dx\, dt = \int_{Q_T \cap [|u_n| \geqslant 1]} | G_n(t,x,u_n,\nabla u_n) | \, dx\, dt + \int_{Q_T \cap [|u_n| \leqslant 1]} | G_n(t,x,u_n,\nabla u_n) | \, dx\, dt$$

From (ii), we have

$$\int_{Q_T \cap [|u_n| \geqslant 1]} | G_n(t,x,u_n,\nabla u_n) | \, dx\, dt \leqslant \int_{Q_T} G_n(t,x,u_n,\nabla u_n)u_n dx\, dt \leqslant C_2$$

To estimate the second integral, we use (A_6) which gives

$$\int_{Q_T\cap[|u_n|\leqslant 1]} |\, G_n(t,x,u_n,\nabla u_n)\,|\, dx\, dt \leqslant \mu(1)\Big[\int_{Q_T} H(t,x)dx\, dt + d\,|\,\nabla u_n\,|^{\alpha}\, dx\, dt\Big]$$
$$\leqslant \mu(1)\Big[\,\|\, H\,\|_{L^1(Q_T)} + d\,|\,Q_T\,|^{\frac{2-\alpha}{2}}\|\,\nabla u_n\,\|^{\frac{\alpha}{2}}_{L^2(Q_T)}\Big]$$
$$\leqslant \mu(1)\Big[\,\|\, H\,\|_{L^1(Q_T)} + C\Big]$$

Consequently $G_n(t,x,u_n,\nabla u_n)$ is bounded in $L^1(Q_T)$.

□

Since u_n is bounded in $\mathcal{V}$ and G_n is bounded in $L^1(Q_T)$, it follows that $\frac{\partial u_n}{\partial t}$ is bounded in $\mathcal{V}^* + L^1(Q_T)$ and from the compactness result of Aubin-Simon [20], we obtain (u_n) is relatively compact in $L^2(Q_T)$, so we can extract a subsequence still denoted by (u_n) for simplicity, such that

$$u_n \to u \text{ strongly in } L^2(Q_T) \text{ and a.e. in } Q_T$$

On the other hand, the trace theorem [15] yields

$$u_n \to u \text{ strongly in } L^2(\Sigma_T) \text{ and a.e. in } \Sigma_T$$

In order to pass to the limit in the approximate problem (11), we need to prove the almost everywhere convergence of the sequence (∇u_n). To do this, we adapt the technic presented by [19] to our case, this method consists to prove that (∇u_n) is a Cauchy sequence in measure, which will give that (∇u_n) converges almost everywhere to ∇u (up to a subsequence).

Lemma 4.3. *The sequence (∇u_n) converges almost everywhere to ∇u in Q_T.*

Proof. We shall prove that (∇u_n) is a Cauchy sequence in measure namely

$\forall \delta > 0$, $\forall \epsilon > 0$, $\exists N_0$ such that $\forall m, n \geqslant N_0$ $\operatorname{meas}\{(t,x),\ |\,(\nabla u_n - \nabla u_m)(t,x)\,| \geqslant \delta\} \leqslant \epsilon$.

Let $\delta > 0$, $\epsilon > 0$ and for $k > 0$, $\eta > 0$, we have the following inclusion

$$\{(t,x),\ |\,(\nabla u_n - \nabla u_m)(t,x)\,| \geqslant \delta\} \subset \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$$

where,

$\Gamma_1 = \{(t,x),\ |\,\nabla u_n\,| \geqslant k\}$, $\Gamma_2 = \{(t,x),\ |\,\nabla u_m\,| \geqslant k\}$, $\Gamma_3 = \{(t,x),\ |\,u_n - u_m\,| \geqslant \eta\}$,
$\Gamma_4 = \{(t,x),\ |\,\nabla u_n - \nabla u_m\,| \geqslant \delta,\ |\,\nabla u_n\,| \leqslant k,\ |\,\nabla u_m\,| \leqslant k,\ |\,u_n - u_m\,| \leqslant \eta\}$.

For meas (Γ_1) and meas (Γ_2), we have

$$k \operatorname{meas}(\Gamma_1) \leqslant \int_{\Gamma_1} |\,\nabla u_n\,| \leqslant \int_{Q_T} |\,\nabla u_n\,| \leqslant |\,Q_T\,|^{\frac{1}{2}} \|\,\nabla u_n\,\|_{L^2(Q_T)}$$

Since (∇u_n) is bounded in $L^2(Q_T)$, then for k large enough, we have

$$\operatorname{meas}(\Gamma_1) \leqslant \frac{C_1}{k} \leqslant \epsilon$$

and by the same manner, we get

$$\operatorname{meas}(\Gamma_2) \leqslant \frac{C_1}{k} \leqslant \epsilon$$

Now, we fix k such that $\text{meas}(\Gamma_1) \leqslant \epsilon$ and $\text{meas}(\Gamma_2) \leqslant \epsilon$ for all $m, n \in \mathbb{N}$.
To bound $\text{meas}(\Gamma_3)$, one has

$$\eta \ \text{meas}(\Gamma_3) \leqslant \int\limits_{\Gamma_3} \mid u_n - u_m \mid \leqslant \int\limits_{Q_T} \mid u_n - u_m \mid \leqslant \mid Q_T \mid^{\frac{1}{2}} \| u_n - u_m \|_{L^2(Q_T)}$$

we recall that (u_n) converge strongly in $L^2(Q_T)$ (up to a subsequence), hence (u_n) is a Cauchy sequence in $L^2(Q_T)$ then for a given η, there exists $N_0 \in \mathbb{N}$ such that for $m, n \geqslant 0$, we have

$$\text{meas}(\Gamma_3) \leqslant \epsilon. \tag{12}$$

It remains to bound $\text{meas}(\Gamma_4)$, and to choose η, to do this, we will use the equation satisfied by u_n and the monotonicity of the operator $-\Delta$.
As well known $\mid (\nabla u_1 - \nabla u_2)(t, x) \mid^2 > 0$ for $\nabla u_1 \neq \nabla u_2$. It is easily verified that the set $(\nabla u_1, \nabla u_2)$ such that $\mid \nabla u_1 \mid \leqslant k$, $\mid \nabla u_1 \mid \leqslant k$ and $\mid \nabla u_1 - \nabla u_2 \mid \geqslant \delta$ is compact and the map is continuous

$$\begin{aligned} Q_T \times \mathbb{R}^N &\to \mathbb{R}^N \\ (t, x, \zeta) &\mapsto \zeta(t, x) \end{aligned}$$

is continuous for almost all $(t, x) \in Q_T$, then $\mid (\nabla u_1 - \nabla u_2)(t, x) \mid^2 > 0$ achieved its minimum on this compact which can be noted by $\gamma(t, x)$ and satisfies $\gamma(t, x) > 0$ a.e $\in Q_T$. Thanks to the property $\gamma(t, x) > 0$ a.e, then to bound $\text{meas}(\Gamma_4)$, it suffices to prove the existence of $\epsilon' > 0$ such that

$$\int\limits_{\Gamma_4} \gamma(t, x) \, dx \, dt \leqslant \epsilon'$$

By using the definition of Γ_4 and γ, we get

$$\begin{aligned} \int\limits_{\Gamma_4} \gamma(t, x) \, dx \, dt &\leqslant \int\limits_{\Gamma_4} (\nabla u_n - \nabla u_m)(\nabla u_n - \nabla u_m) 1_{\{|u_n - u_m| \leqslant \eta\}} \, dx \, dt \\ &\leqslant \int\limits_{Q_T} \mid \nabla T_\eta(u_n - u_m) \mid^2 \, dx \, dt \end{aligned}$$

To estimate the last quantity, we use the equation satisfied by $(u_n - u_m)$, which can be written as

$$\begin{aligned} < \frac{\partial(u_n - u_m)}{\partial t}, \phi > &+ \int_{Q_T} \nabla(u_n - u_m) \nabla \phi + \int_{\Sigma_T} \beta(t, x)(u_n - u_m)\phi \\ &+ \int_{Q_T} (G_n(t, x, u_n, \nabla u_n) - G_m(t, x, u_m, \nabla u_m))\phi \\ &+ \int_{\Sigma_T} (g(t, x, u_n) - g(t, x, u_m))\phi = 0. \end{aligned}$$

By choosing $\phi = T_\eta(u_n - u_m) \in \mathcal{V} \cap L^\infty(Q_T)$ and we use the periodicity of $(u_n - u_m)$, we get

$$\int_{Q_T} \mid \nabla T_\eta(u_n - u_m) \mid^2 + \int_{\Sigma_T} \beta(t,x)(u_n - u_m) T_\eta(u_n - u_m)$$
$$\leqslant \eta \int_{Q_T} \mid G_n(t,x,u_n,\nabla u_n) - G_m(t,x,u_m,\nabla u_m) \mid + \eta \int_{\Sigma_T} \mid g(t,x,u_n) - g(t,x,u_m) \mid .$$

We combine the fact that $sT_\eta(s) \geqslant 0$ with the assumption (A_3) and the result (iii) of Lemma 4.2, we get

$$\begin{aligned}\int_{Q_T} \mid \nabla T_\eta(u_n - u_m) \mid^2 &\leqslant 2\eta C_3 + \eta \int_{\Sigma_T} 2\Big(\xi(t,x) + \mid u_n \mid + \mid u_m \mid\Big) \\ &\leqslant 2\eta C_3 + 2\eta \mid \Sigma_T \mid^{\frac{1}{2}} \Big(\parallel \xi \parallel_{L^2(\Sigma_T)} + \parallel u_n \parallel_{L^2(\Sigma_T)} + \parallel u_m \parallel_{L^2(\Sigma_T)} \Big) \\ &\leqslant \eta C,\end{aligned}$$

where C is a constant independent of n and m. Then, for η small enough, we obtain $\int\limits_{\Gamma_4} \gamma \leqslant \epsilon'$ which implies $\text{meas}(\Gamma_4) \leqslant 4\epsilon$.
Now η is fixed and from (12) we obtain the existence of N_0 such that $\text{meas}(\Gamma_4) \leqslant \epsilon$, furthermore, we conclude that for all $m, n \geqslant N_0$, one has

$$\text{meas}\{(t,x), \mid (\nabla u_n - \nabla u_m)(t,x) \mid \geqslant \delta\} \leqslant 4\epsilon,$$

so the sequence (∇u_n) converge in measure to ∇u, which gives the existence of a subsequence of (∇u_n) still denoted by (∇u_n) for simplicity, such that (∇u_n) converge to ∇u a.e. in Q_T. □

Lemma 4.4. *The nonlinearities $G_n(t,x,u_n,\nabla u_n)$ converges strongly to $G(t,x,u,\nabla u)$ in $L^1(Q_T)$.*

Proof. Since (∇u_n) converges almost everywhere to ∇u, then

$$G_n(t,x,u_n,\nabla u_n) \to G(t,x,u,\nabla u) \text{ a.e in } Q_T$$

hence it suffices to show that $G_n(t,x,u_n,\nabla u_n)$ is equi-integrable in $L^1(Q_T)$ namely

$$\forall \varepsilon > 0, \exists \delta > 0, \forall E \subset Q_T, \text{ if } |E| < \delta \text{ then } \int\limits_E \mid G_n(t,x,u_n,\nabla u_n) \mid \, dx\, dt \leq \varepsilon$$

Let E be a measurable subset of Q_T, $\varepsilon > 0$, and $k > 0$. We have

$$\int\limits_E \mid G_n(t,x,u_n,\nabla u_n) \mid \, dx\, dt = I_1 + I_2.$$

Where

$$I_1 = \int\limits_{E \cap [|u_n| > k]} \mid G_n(t,x,u_n,\nabla u_n) \mid \, dx\, dt,$$

and

$$I_2 = \int\limits_{E \cap [|u_n| \leq k]} \mid G_n(t,x,u_n,\nabla u_n) \mid \, dx\, dt.$$

For I_1 we use (ii) of Lemma 4.2 to obtain the following inequality

$$I_1 \leq \frac{1}{k} \int\limits_{Q_T} u_n G_n(t,x,u_n,\nabla u_n)\, dx\, dt \leq \frac{C_2}{k}$$

Then, we deduce the existence of $k^* > 0$, such that, for all $k \geqslant k^*$, we have

$$I_1 \leq \frac{\epsilon}{3}.$$

To treat the second integral I_2 we use the assumption (A_6), we obtain for all $k \geq k^*$

$$I_2 \leq \mu(k)\bigg(\int\limits_E (H(t,x) + d \mid \nabla u_n \mid^\alpha)\, dx\, dt \bigg).$$

Since $H \in L^1(Q_T)$, then H is equi-integrable in $L^1(Q_T)$, there exists $\delta > 0$, such that, if $\mid E \mid \leq \delta$, one has

$$\int\limits_E H(t,x)\, dx\, dt \leq \frac{\epsilon}{3}.$$

On the other hand, Hölder's inequality yields

$$d \int\limits_E \mid \nabla u_n \mid^\alpha \leqslant d \mid E \mid^{\frac{2-\alpha}{2}} \bigg(\int\limits_E \mid \nabla u_n \mid^2 \bigg)^{\frac{\alpha}{2}} \leqslant C \mid E \mid^{\frac{2-\alpha}{2}}$$

the last inequality is obtained by using the boundedness of (u_n) in $\mathcal{V}$, moreover $1 \leqslant \alpha < 2$ then $0 < 2-\alpha \leqslant 1$. Consequently, we can choose $\mid E \mid \leqslant (\frac{\epsilon}{3C})^{\frac{2}{2-\alpha}}$ to get $d \int\limits_E \mid \nabla u_n \mid^\alpha \leq \frac{\varepsilon}{3}$. Finally, by choosing $\delta^* = \inf(\delta, (\frac{\epsilon}{3C})^{\frac{2}{2-\alpha}})$, if $\mid E \mid \leq \delta^*$, we obtain

$$\int\limits_E \mid G_n(t,x,u_n,\nabla u_n) \mid \, dx\, dt \leq \varepsilon.$$

$\square$

References

[1] N. Alaa, M. Pierre, Weak solutions for some quasi-linear elliptic equations with data measures, *SIAM J. Math. Anal.* **24** (1993), 23–35.

[2] N. Alaa, I. Mounir, Global existence for some quasilinear parabolic Reaction-Diffusion systems with mass control and critical growth with respect to the gradient, *Journal of Mathematical Analysis and Application* **253** (2001), 532–557.

[3] N. Alaa, M. Iguernane, Weak periodic solutions of some quasilinear parabolic equations with data measure, *J. of Inequalities in Pure and Applied Mathematics* **3** (2002), no. 3, Article 46.

[4] N. Alaa, F. Aqel, Periodic solution for some parabolic degenerate equation withcritical growth with respect to the gradient, *Annals of the University of Craiova Mathematics and Computer Science Series* **42** (2015), no. 1, 13–26.

[5] H. Amman, *Periodic solutions of semilinear parabolic equations*, Nonlinear Analysis Academic Press, New York, 1978, 1-29.

[6] M. Badii, Periodic solutions for a nonlinear parabolic equation with nonlinear boundary conditions, *Rend. Sem. Mat. Univ. Pol. Torino* **67** (2009), no. 3, 341-349.

[7] M. Badii, Periodic solutions for a class of degenerate evolution problems, *Nonlinear Anal.* **44** (2001), no. 4, 499-508.

[8] V. Barbu, *Nonlinear semigroups and differential equations in Banach spaces*, Noordhoff International Publishing, Leiden, 1976.

[9] F.E. Browder, Nonlinear maximal monotone operators in Banach space, *Math. Ann.* **175** (1968), 89–113.
[10] J. Deuel, P. Hesse, Nonlinear parabolic boundary value problems with upper and lower solutions, *Israel Journal of Mathematics* **29** (1978), 92–104.
[11] Ju.S. Kolesov, Periodic solutions of quasilinear parabolic equations of second order, *Trans. Moscow Math. Soc.* **21** (1970), 114–146.
[12] G. Kouadri, A. Charkaoui, S. Med Said, N. Alaa, Modeling and mathematical analysis of quench-heat conduction in metallic materials, *J. Adv. Math. Stud.* **11** (2018), no. 2, 316–330.
[13] M.A. Krasnoselskii, *Topological methods in the theory of nonlinear integral equations*, Pergamon Press, New York, 1964.
[14] J.L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod, Paris, 1969.
[15] C.B. JR. Morrey, *Multiple integrals in the calculus of variation*, Springer Verlag, New York, 1966.
[16] M. Nakao, Y. Ohara, Gradient estimates of periodic solutions for some quasilinear parabolic equations, *J. Math. Anal. Appl.* **204** (1996), no. 3, 868–883.
[17] C.V. Pao, Periodic solutions of parabolic systems with nonlinear boundary conditions, *J. Math. Anal. Appl.* **234** (1999), 695–716.
[18] A. Porretta, Existence results for nonlinear parabolic equations via strong convergence of truncations, *Annali di Mathematica Pura ed Applicata* **177** (1989), no. 1, 143–172.
[19] A. Prignet, Existence and uniqueness of "entropy" solutions of parabolic problems with L1 data, *Nonlin. Anal. TMA* **28** (1997), 1943–1954.
[20] J. Simon, Compact sets in $L^p(O, T; B)$, *Annali di Mathematica Pura ed Applicata* **146** (1987), no. 1, 65–96.
[21] J.B. Sun, J.X. Yin, Y.F. Wang, Asymptotic bounds of solutions for a periodic doubly degenerate parabolic equation, *Nonlinear Anal. TMA*, **74** (2011), 2415–2424.
[22] Y.F. Wang, J.X. Yin, Z.Q. Wu, Periodic solutions of porous medium equations with weakly nonlinear sources, *Communications in Mathematical Research*, **16** (2000), no. 4, 475–483.
[23] Y.F.Wang, J.X. Yin, Z.Q. Wu, Periodic solutions of evolution p-Laplacian equations with nonlinear sources, *J. Math. Anal. Appl.* **219** (1998), no. 1, 76–96.
[24] D. Zhang, J. Sun, B. Wu, Periodic solutions of a porous medium equation, *Elect. J. of Qualitative Theory of Diff. Eq.* **2011** (2011), no. 42, 1–7.

(Abdelwahab Elaassri, Kaoutar Lamrini Uahabi) Laboratory MASI, Multidisiplinary Faculty of Nador, University Mohammed first, Selouane, Nador - 62702, Morocco
E-mail address: `elaassri_abdelwahab@yahoo.fr, lamrinika@yahoo.fr`

(Abderrahim Charkaoui,Nour Eddine Alaa) Laboratory LAMAI, Faculty of Science and Technology, University Cadi Ayyad, B.P. 549, Street Abdelkarim Elkhattabi, Marrakech - 40000, Morocco
E-mail address: `n.alaa@uca.ac.ma, charkaoui.abderrahim92@gmail.com`

(Salim Mesbahi) Department of Mathematics, Faculty of Science, Ferhat Abbas University, Campus El-Bez, Setif, 19000, Algeria
E-mail address: `salimbra@gmail.com`