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Existence results of weak periodic solution for some quasilinear parabolic problem with L^1 data

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Abstract. The goal of this paper is to study the existence of a weak periodic solution for a quasilinear parabolic equation with arbitrary growth non linearity with respect to the gradient and Dirichlet boundary condition. The existence of at least one weak periodic solution is proved under the presence of a weak periodic super solution.

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1. Introduction

The existence of time periodic solutions of ordinary, functional differential equations, semilinear and quasilinear parabolic equations under linear and nonlinear boundary conditions has been extensively due to its importance. Condence degree theory, Schauder fixed point theorem, bifurcation theory are often used to prove the existence of periodic solutions. But, to the best of our knowledge, only a few papers used the method of sub- and super-solution to show the existence of weak periodic solutions for semilinear parabolic equation with singular nonlinearities.

At the same time, the problem of existence of periodic solutions for boundary value problems has been attracted great interests of scientists and several papers has been published. However, most of the works concentrated on the quasilinear problems and a lots of results have been reported ([3, 4, 5, 6, 11, 12, 16]) and the refernces therein.

In this work we are interested in the existence of weak periodic solution for the following quasilinear problem

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} - \Delta u + G(t,x,\nabla u) = f & \text{in } Q_T \\ u(0,.) = u(T,.) & \text{in } \Omega \\ u(t,x) = 0 & \text{on } \Sigma_T, \end{cases} \tag{1}$$

where Ω is an open regular bounded subset of $\mathbb{R}^N$, $N \geqslant 1$, with smooth boundary $\partial\Omega$, $T > 0$ is the period, $Q_T =]0,T[\times\Omega$, $\Sigma_T =]0,T[\times\partial\Omega$, G is a caratheodory and f is a nonnegative measurable function belonging in $L^1(Q_T)$.

In [5], Amann has solved the problem (1) when f is regular enough and G have sub-quadratic growth with respect to the gradient, namely

$$| G(t,x,u,\nabla u) | \leqslant c(| u |)(| \nabla u |^2 + 1).$$

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He used the technique of sub- and super-solutions and fixed point theorem in Banach spaces.

The paper [11] of Hess and Deuel, was concerned about the problem (1) with f belonging in $L^2(Q_T)$ and the growth of the nonlinearities G with respect to the gradient is sub-linear, namely

$$\mid G(t,x,u,\nabla u) \mid \leqslant k(t,x) + c \mid \nabla u \mid .$$

They obtained the existence of weak periodic solution $L^2(0,T;H_0^1(\Omega)) \cap \mathcal{C}([0,T],L^2(\Omega))$, by using the method of sub- and super-solution.

Alaa and Iguernane [3] have presented an existence result of a weak periodic solution of (1) in $L^2(0,T;H_0^1(\Omega)) \cap \mathcal{C}([0,T],L^2(\Omega))$ when the nonlinearities are negative, they obtained the existence by the technique of sub and super solution.

We have presented the main result of this paper in the following manner. In section 2 we start by necessaries hypothesis and we define the notion of weak periodic solution of (1) in order to enunciate the main theorem of our work. Section 3 is devoted to prove the main theorem presented in section 2. Finally, in section 4 we generalise the result of section 2 to a class of a quasilinear periodic problem.

2. The main result

In this section we establish our main result, first of all we introduce necessaries assumptions.

2.1. Assumptions. Let us now introduce the hypothesis which we assume throughout this section. We consider that

$$f \in L^1(Q_T), f \geqslant 0, \tag{2}$$

$$G(t,x,r) \in L^1(Q_T) \text{ for all } r \in \mathbb{R}^N \text{and } a.e. \, (t,x) \in \mathbb{R}^N, \tag{3}$$

$$G : [0,T] \times \Omega \times \mathbb{R}^N \to [0,+\infty[\text{ a Caratheodory function}, \tag{4}$$

$$G(t,x,0) = \min\{G(t,x,r), r \in \mathbb{R}^N\} = 0, \tag{5}$$

$$G(t,x,s,r) \leq K(t,x) + d \left\| r \right\|^p , \tag{6}$$

for all $p \in [1, \frac{N+2}{N+1}[, r \in \mathbb{R}^N$ and a.e. $(t,x) \in Q_T$, with $K \in L^1(Q_T)$ and $d > 0$. We have to clarify in which sense we want to solve the problem (1), for which we introduce the notion of weak periodic solution.

Definition 2.1. A function u is said to be a weak periodic solution of the problem (1), if satisfies

$$\begin{cases} u \in L^1(0,T;W_0^{1,1}(\Omega)) \cap \mathcal{C}([0,T],L^1(\Omega)), & \\ G(t,x,\nabla u) \ \in L^1(Q_T) & \\ \dfrac{\partial u}{\partial t} - \Delta u + G(t,x,\nabla u) = f & \text{in } \mathcal{D}'(Q_T) \\ u(0,.) \ = \ u(T,.) & \text{in } L^1(\Omega) \end{cases} \tag{7}$$

Definition 2.2. We call weak periodic super-solution (resp. sub-solution) of (1) a function u satisfying (7) with " = " replaced by " $\geq$ " (resp. " $\leq$ ").

Remark 2.1. In (7) $u(0,.) = u(T,.)$ in $L^1(\Omega)$ means that for all $\phi \in L^\infty(\Omega)$,

$$\lim_{s \mapsto 0} \int_{Q_T} (u(T-s,x) - u(s,x))\phi(x) = 0.$$

This enables us to state the main result of this section.

Theorem 2.1. *Assume that* (2)-(6) *hold and there exists* w *a weak super-solution of (1). Then the problem (1) has a weak periodic solution satisfying* $0 \le u \le w$ *in* Q_T.

3. Proof of the main result

Set $X_T = L^p(0,T; W_0^{1,p}(\Omega))$, where $1 \le p < \frac{N+2}{N+1}$.

3.1. Approximating problem. We approximate G as follows

$$G_n(t,x,r) = \frac{G(t,x,r)}{1 + \dfrac{1}{n} \mid G(t,x,r) \mid} 1_{[w \leqslant n]},$$

We remark that G_n satisfies the following properties

$$0 \leqslant G_n \leqslant G, \quad \text{and} \quad \mid G_n \mid \leqslant n.$$

Since $f \in L^1(Q_T)^+$, we can construct a sequence $h_n \in C_0^2(Q_T)$, such that

$$h_n \geqslant 0, \qquad \| h_n \|_{L^1(Q_T)} \le \| f \|_{L^1(Q_T)},$$

and h_n converges to f in $L^1(Q_T)^+$, we denote

$$f_n = h_n 1_{[w \le n]}, \qquad w_n = \min(w,n).$$

We define the approximate problem of (1) by

$$\begin{cases} u_n \in L^2(0,T; H_0^1(\Omega)) \cap \mathcal{C}([0,T], L^2(\Omega)), & \\ \dfrac{\partial u_n}{\partial t} - \Delta u_n + G_n(t,x,\nabla u_n) = f_n & \text{in } \mathcal{D}'(Q_T) \\ u_n(0,.) \; = \; u_n(T,.) & \text{in } L^2(\Omega) \end{cases} \tag{8}$$

Since G_n is bounded and $f_n \in L^\infty(Q_T)$, then the classical result of [11] can be applied to obtain the existence of $u_n \in L^2(0,T;H_0^1(\Omega)) \cap \mathcal{C}([0,T], L^2(\Omega)) \cap L^\infty(Q_T)$ weak periodic solution of (8) such that

$$0 \le u_n \le w_n \le w.$$

We want to pass to the limit in (8), for this we need to prove the following lemmas.

Lemma 3.1. i) *There exists a constant* C *depending on* $\| f \|_{L^1(Q_T)}$ *such that*

$$\int_{Q_T} \mid G_n(t,x,\nabla u_n) \mid \le C.$$

ii) *There exists a constant* C *depending on* p, T, Ω *such that*

$$\| u_n \|_{X_T} \le C \Big[2 \| f \|_{L^1(Q_T)} + \| w(0) \|_{L^1(\Omega)} \Big].$$

Proof. i) Integrate the equation satisfied by u_n over Q_T, we get

$$\int_{Q_T} \frac{\partial u_n}{\partial t} - \int_{Q_T} \Delta u_n + \int_{Q_T} G_n(t,x,\nabla u_n) = \int_{Q_T} f_n(t,x),$$

since $u_n(0,.) = u_n(T,.)$ in Ω and $G_n \geq 0$, we have

$$\begin{aligned}\int_{Q_T} \mid G_n(t,x,\nabla u_n) \mid dxdt &\leqslant \int_{Q_T} f_n(t,x)dxdt,\\ &\leqslant \parallel f \parallel_{L^1(Q_T)} .\end{aligned}$$

ii) Furthermore, by [8] we have

$$\begin{aligned}\parallel u_n \parallel_{X_T} &\leq C(p,\Omega)\Big[\parallel f_n \parallel_{L^1(Q_T)} + \parallel G_n(\nabla u_n) \parallel_{L^1(Q_T)} + \parallel u_n(T) \parallel_{L^1(\Omega)} \Big]\\ &\leq C(p,\Omega)\Big[2 \parallel f \parallel_{L^1(Q_T)} + \parallel w(0) \parallel_{L^1(\Omega)} \Big].\end{aligned}$$

□

According to the classical result of [7], the application $(u_n(0),\xi_n) \longmapsto u_n$ is compact from $L^1(\Omega)\times L^1(Q_T)$ into $L^1(0,T;W_0^{1,1}(\Omega))$, where $\xi_n(t,x) = f_n(t,x) - G_n(t,x,\nabla u_n)$. Then, we can extract a subsequence of u_n, still denoted by u_n for simplicity, such that

$$\begin{aligned} u_n &\longrightarrow u \text{ in } L^1(0,T;W_0^{1,1}(\Omega))\\ (u_n,\nabla u_n) &\longrightarrow (u,\nabla u) \text{ a.e. in } Q_T.\end{aligned}$$

To ensure that u is a solution of problem (1), it remains to show that u_n converges to u strongly in X_T.
To do this, we write for $m,n \geq 1$ and $0<\gamma<1$,

$$\int_{Q_T} \mid \nabla u_n - \nabla u_m \mid^p dxdt \leq \Big(\int_{Q_T} \mid \nabla u_n - \nabla u_m \mid\Big)^{\gamma}\Big(\int_{Q_T} \mid \nabla u_n - \nabla u_m \mid^{\frac{p-\gamma}{1-\gamma}}\Big)^{1-\gamma}. \tag{9}$$

Choose γ such that $\frac{p-\gamma}{1-\gamma} = q \in [1, \frac{N+2}{N+1}[$, then (9) gives desired result.
Thanks to the assumption (6), we deduce that

$$G_n(t,x,\nabla u_n) \longrightarrow G(t,x,\nabla u) \text{ in } L^1(Q_T).$$

On the other hand,

$$u_n(T) = S(T)u_n(0) + \int_0^T S(T-s)\xi_n(s,.)ds,$$

where $S(t)$ is the semigroup of contractions in $L^1(\Omega)$ generated by the operator $-\Delta$ with Dirichlet boundary condition on $\partial\Omega$. Since $u_n(0,.) = u_n(T,.)$ in $L^1(\Omega)$, we have for all $\phi \in L^\infty(\Omega)$

$$\begin{aligned}\lim_{n\to+\infty} \int_\Omega u_n(0,x)\phi(x)dx = &\lim_{n\to+\infty} \int_\Omega S(T)u_n(0,x)\phi(x)dx\\ &+ \lim_{n\to+\infty} \int_\Omega \int_0^T S(T-s)\xi_n(s,x)\phi(x)dsdx,\end{aligned}$$

Since $S(t)$ is continuous in $L^1(\Omega)$ and $\xi_n \to \xi$ strongly in $L^1(Q_T)$, then

$$\begin{aligned}\int_\Omega u(0,x)\phi(x)dx &= \int_\Omega S(T)u(0,x)\phi(x)dx + \int_\Omega \int_0^T S(T-s)\xi(s,x)dsdx,\\ &= \int_\Omega u(T,x)\phi(x)dx.\end{aligned}$$

Then $u(0,.) = u(T,.)$ in $L^1(\Omega)$. Which ends the proof.

4. Application

The objective of this section is to prove the existence of a weak periodic solution for the following quasilinear parabolic problem

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} - \Delta u + G(t,x,\nabla u) = F(t,x,u) + \mu & \text{in } Q_T \\ u(0,.) = u(T,.) & \text{in } \Omega \\ u(t,x) = 0 & \text{on } \Sigma_T \end{cases} \tag{10}$$

where Ω is an open regular bounded subset of $\mathbb{R}^N$, $N \geqslant 1$, with smooth boundary $\partial\Omega$, $T > 0$ is the period, $Q_T =]0,T[\times\Omega$, $\Sigma_T =]0,T[\times\partial\Omega$, F and G are assumed to be Caratheodory functions and μ is a nonnegative measurable function belonging in $L^1(Q_T)$.

4.1. Assumptions. Let us now introduce the hypothesis which we assume throughout this section. We consider that

$$\mu \in L^1(Q_T), \mu \geqslant 0, \tag{11}$$

$$F : [0,T] \times \Omega \times \mathbb{R} \to [0,+\infty[\text{ a Caratheodory function,} \tag{12}$$

$$F(t,x,s) \in L^1(Q_T),\ F \text{ is nondecreasing with respect to } s, \tag{13}$$

$$G : [0,T] \times \Omega \times \mathbb{R}^N \to [0,+\infty[\text{ a Caratheodory function,} \tag{14}$$

$$G(t,x,r) \leq H(t,x) + d\,\|r\|^2, \text{ for all } r \in \mathbb{R}^N \text{ and } a.e.(t,x) \in Q_T, \tag{15}$$

with $H \in L^1(Q_T)$ and $d > 0$. The notion of weak periodic solution is presented here to clarify in which sense we want to solve the problem (10).

Definition 4.1. A function u is said to be a weak periodic solution of the problem (10), if satisfies

$$\begin{cases} u \in L^1(0,T;W_0^{1,1}(\Omega)) \cap \mathcal{C}([0,T],L^1(\Omega)), \\ G(t,x,\nabla u),\ F(t,x,u) \in L^1(Q_T) \\ \dfrac{\partial u}{\partial t} - \Delta u + G(t,x,\nabla u) = F(t,x,u) + \mu & \text{in } \mathcal{D}'(Q_T) \\ u(0,.) = u(T,.) & \text{in } L^1(\Omega) \end{cases} \tag{16}$$

Definition 4.2. We call weak periodic super-solution (resp. sub-solution) of (10) a function u satisfying (16) with " $=$ " replaced by " $\geq$ " (resp. " $\leq$ ").

Under some assumptions and thanks to the result of Theorem 2.1 we can prove the following theorem.

Theorem 4.1. *Suppose that the hypotheses* (11)-(15) *hold and assume that there exists* $\widehat{w}$ *such that,*

$$\begin{cases} \widehat{w} \in L^1(0,T;W_0^{1,1}(\Omega)) \cap \mathcal{C}([0,T],L^1(\Omega)), \\ F(t,x,\widehat{w}) \ \in L^1(Q_T) \\ \dfrac{\partial \widehat{w}}{\partial t} - \Delta \widehat{w} = F(t,x,\widehat{w}) + \mu & \text{in } \mathcal{D}'(Q_T), \\ \widehat{w}(0,.) \ = \ \widehat{w}(T,.) & \text{in } L^1(\Omega). \end{cases}$$

Then (10) *has a weak periodic solution* u *such that* $0 \leqslant u \leqslant \widehat{w}$.

5. Proof of Theorem 4.1

We consider the sequence defined by $u_0 = \widehat{w}$ and for $n \geqslant 1$ u_n is the solution of the problem

$$\begin{cases} u_n \in L^1(0,T;W_0^{1,1}(\Omega)) \cap \mathcal{C}([0,T],L^1(\Omega)), \\ \dfrac{\partial u_n}{\partial t} - \Delta u_n + G_n(t,x,\nabla u_n) = F(t,x,u_{n-1}) + \mu & \text{in } \mathcal{D}'(Q_T) \\ u_n(0,.) \ = \ u_n(T,.) & \text{in } L^1(\Omega) \end{cases} \tag{17}$$

where

$$G_n(t,x,r) = \frac{G(t,x,r)}{1 + \dfrac{1}{n} \mid G(t,x,r) \mid},$$

G_n satisfies the following properties

$$0 \leqslant G_n \leqslant G_{n+1} \leqslant G, \quad \text{and} \quad \mid G_n \mid \leqslant n.$$

By using Theorem 2.1 combined with an induction argument we prove the existence of u_n solution of the problem (17) such that

$$0 \leqslant u_n \leqslant u_{n-1} \leqslant \widehat{w}. \tag{18}$$

5.1. A priori estimates. Before giving the lemmas that will be useful for the proof of Theorem 4.1, let us define the truncation function $T_k \in C^2$ for all real positive number k by,

$$\begin{aligned} T_k(s) = s \quad & \text{if} \quad 0 \leqslant s \leqslant k, \\ T_k(s) \leqslant k+1 \quad & \text{if} \quad s \geqslant k, \\ 0 \leqslant T_k'(s) \leqslant 1 \quad & \text{if} \quad s \geqslant 0, \\ T_k'(s) = 0 \quad & \text{if} \quad s \geqslant k+1, \\ 0 \leqslant -T_k''(s) \leqslant C(k). \end{aligned}$$

For example, the function T_k can be defined as

$$\begin{aligned} T_k(s) = s \quad & \text{in} \quad [0,k], \\ T_k(s) = \frac{1}{2}(s-k)^4 - (s-k)^3 + s \quad & \text{in} \quad [k,k+1], \\ T_k(s) = \frac{1}{2}(k+1) \quad & \text{for} \quad s > k+1. \end{aligned}$$

Then, we pose

$$S_k(v) = \int_0^v T_k(s)\, ds.$$

Lemma 5.1. i) *There exists a constant C deponding on $\| \mu \|_{L^1(Q_T)}$ and $\| F(\widehat{w}) \|_{L^1(Q_T)}$, such that*

$$\int_{Q_T} | G_n(t,x,\nabla u_n) | \, dxdt \leqslant C.$$

ii)

$$\lim_{k\mapsto+\infty} \sup_n \int_{[u_n>k]} | G_n(t,x,\nabla u_n) | \, dxdt = 0.$$

Proof. i) Integrating the equation satisfied by u_n over Q_T,

$$\int_{Q_T} \frac{\partial u_n}{\partial t} - \int_{Q_T} \Delta u_n + \int_{Q_T} G_n(t,x,\nabla u_n) = \int_{Q_T} F(t,x,u_{n-1}) + \int_{Q_T} \mu,$$

since $u_n(0,.) = u_n(T,.)$ in $L^1(\Omega)$ and by using the assumptions on F we get

$$\int_{Q_T} | G_n(t,x,\nabla u_n) | \leqslant \int_{Q_T} F(t,x,\widehat{w}) + \int_{Q_T} \mu.$$

ii) Multiplying the equation satisfied by u_n by the truncated function $T_k(u_n)$ and integrating on Q_T, we obtain

$$\int_{Q_T} \frac{\partial S_k(u_n)}{\partial t} + \int_{Q_T} | \nabla T_k(u_n) |^2 + \int_{Q_T} G_n(t,x,\nabla u_n) T_k(u_n) = \int_{Q_T} F(t,x,u_{n-1}) T_k(u_n) + \int_{Q_T} \mu T_k(u_n),$$

thanks to the hypothesis on F and since u_n is periodic, we get

$$\int_{Q_T} G_n(t,x,\nabla u_n) T_k(u_n) \leqslant \int_{Q_T} F(t,x,\widehat{w}) T_k(u_n) + \int_{Q_T} \mu T_k(u_n),$$

then for every $0 < M < k$ we have

$$k \int_{[u_n>k]} G_n(t,x,\nabla u_n) \leqslant k \int_{Q_T \cap [u_n>M]} \Big(F(t,x,\widehat{w}) + \mu \Big) + M \int_{Q_T \cap [u_n \leqslant M]} \Big(F(t,x,\widehat{w}) + \mu \Big),$$

consequently

$$\int_{[u_n>k]} G_n(t,x,\nabla u_n) \leqslant \int_{Q_T} \Big(F(t,x,\widehat{w}) + \mu \Big) \chi_{[u_n>M]} + \frac{M}{k} \int_{Q_T} \Big(F(t,x,\widehat{w}) + \mu \Big).$$

To conclude the desired result, it suffies to show that

$$\lim_{k\mapsto+\infty} \sup_n \int_{Q_T} \Big(F(t,x,\widehat{w}) + \mu \Big) \chi_{[u_n>M]} = 0,$$

to do this we remark,

$$| [u_n > M] | \leqslant \frac{1}{M} \| u_n \|_{L^1(Q_T)} \leqslant \frac{1}{M} \| \widehat{w} \|_{L^1(Q_T)}$$

which implies

$$\lim_{M\mapsto+\infty} \sup_n | [u_n > M] | = 0.$$

Since $(F(t,x,\widehat{w})+\mu)\in L^1(Q_T)$, we have that for each $\epsilon>0$ there exists δ such that for all mesurable $E\subset Q_T$

$$\mid E\mid<\delta,\quad \int_E\Big(F(t,x,\widehat{w})+\mu\Big)\leqslant\frac{\epsilon}{2},$$

according to the previous result, we obtain that for each $\epsilon>0$, there exists M_ϵ such that for all $M\geqslant M_\epsilon$

$$\sup_n\left(\int_{Q_T}(F(t,x,\widehat{w})+\mu)\chi_{[u_n>M]}\right)\leqslant\frac{\epsilon}{2},$$

choosing $M=M_\epsilon$ and letting k tend to infinity, we obtain

$$\lim_{k\mapsto+\infty}\sup_n\left(\int_{[u_n>k]}G_n(t,x,\nabla u_n)\right)=0.$$

$\square$

Lemma 5.2. *Let u_n be the sequence defined as above. Then*
i) *u_n converges to u strongly in $L^1(0,T;W_0^{1,1}(Q_T))$.*

ii) $\parallel T_k(u_n)\parallel_{L^2(0,T;H_0^1)}\leqslant k\Big[\parallel F(\widehat{w})\parallel_{L^1(Q_T)}+\parallel\mu\parallel_{L^1(Q_T)}\Big]$

Proof. (i) We set

$$\eta_n=F(t,x,u_{n-1})+\mu-G_n(t,x,\nabla u_n)$$

thanks to the result (i) of Lemma 5.1, (13) and (17) we have η_n bounded in $L^1(Q_T)$ and according to [7], the application

$$\begin{array}{rcl} L^1(\Omega)\times L^1(Q_T) & \longrightarrow & L^1(0,T;W_0^{1,1}(Q_T))\\ (u_n(0),\eta_n) & \longmapsto & u_n \end{array}$$

is compact. Then, we can extract a subsequence of u_n, still denoted by u_n for simplicity, such that

$$\begin{array}{rcl} u_n & \longrightarrow & u \text{ in } L^1(0,T;W_0^{1,1}(\Omega))\\ (u_n,\nabla u_n) & \longrightarrow & (u,\nabla u)\ \text{ a.e. in } Q_T \end{array}$$

(ii) Multiplying by $T_k(u_n)$ the equation satisfies by u_n, we obtain

$$\int_{Q_T}\frac{\partial S_k(u_n)}{\partial t}+\int_{Q_T}\mid\nabla T_k(u_n)\mid^2+\int_{Q_T}G_n(t,x,\nabla u_n)T_k(u_n)=\int_{Q_T}F(t,x,u_{n-1})T_k(u_n)$$
$$+\int_{Q_T}\mu T_k(u_n),$$

the periodicity implies,

$$\int_{Q_T}\frac{\partial S_k(u_n)}{\partial t}=0.$$

We use (14) and (18), we get

$$\int_{Q_T}G_n(t,x,\nabla u_n)T_k(u_n)\geqslant 0,$$

and (13), (18) imply

$$\int_{Q_T} \mid \nabla T_k(u_n) \mid^2 \leqslant k\Big[\int_{Q_T} F(t,x,\widehat{w}) + \int_{Q_T} \mu\Big].$$

□

Lemma 5.3. *Let u_n be the sequence defined as above. Then*

$$T_k(u_n) \text{ converges to } T_k(u) \text{ strongly in } L^2(0,T;H_0^1(\Omega)).$$

Proof. For the proof of this lemma, we consider $v_{n,k} = T_k(\widehat{w} - u_n)$, $v_k = T_k(\widehat{w} - u)$ and $v_{n,k,h} = (T_k(\widehat{w} - u_n))_h$, where σ_h denotes the steklov regularization defined by

$$\sigma_h(t,x) = \frac{1}{h}\int_t^{t+h} \sigma(s,x)ds.$$

According to (18), we have

$$0 \leqslant v_{n,k,h} \leqslant v_{k,h},$$

To prove $T_k(u_n)$ converges strongly to $T_k(u)$ in $L^2(0,T;H_0^1(\Omega))$, it suffices to prove that

$$\lim_{n\to\infty}\int_{Q_T} \|\nabla v_{n,k}\|^2 dxdt \leq \int_{Q_T} \|\nabla v_k\|^2 dxdt.$$

For $h > 0$, we have

$$\begin{aligned}
\lim_{n\mapsto+\infty}\int_{Q_T} \|\nabla v_{n,k}\|^2 dxdt &= \lim_{h\to 0}\lim_{n\mapsto+\infty}\int_{Q_{T-h}} \|\nabla v_{n,k,h}\|^2 dxdt \\
&= \lim_{h\to 0}\lim_{n\mapsto+\infty}\int_0^{T-h} < v_{n,k,h}, -\Delta v_{n,k,h} > dxdt \\
&\leq \lim_{h\to 0}\lim_{n\mapsto+\infty}\int_0^{T-h} < v_{n,k,h}, \frac{\partial v_{n,k,h}}{\partial t} - \Delta v_{n,k,h} > dxdt,
\end{aligned}$$

we remark that,

$$\frac{\partial v_{n,k,h}}{\partial t} - \Delta v_{n,k,h} \geqslant 0,$$

then

$$\begin{aligned}
\lim_{n\mapsto+\infty}\int_{Q_T} \|\nabla v_{n,k}\|^2 dxdt &\leqslant \lim_{h\to 0}\lim_{n\mapsto+\infty}\int_0^{T-h} < v_{k,h}, \frac{\partial v_{n,k,h}}{\partial t} - \Delta v_{n,k,h} > dxdt, \\
&\leqslant \lim_{h\to 0}\lim_{n\mapsto+\infty}\Big[\int_0^{T-h} < v_{k,h}, \frac{\partial v_{n,k,h}}{\partial t} > dt \\
&\qquad + \int_{Q_{T-h}} \nabla v_{n,k,h}\nabla v_{k,h} dxdt\Big].
\end{aligned}$$

Since $v_{n,k}$ converges to v_k weakly in $L^2(0,T;H_0^1(\Omega))$, then $v_{n,k,h}$ converges to $v_{k,h}$ weakly in $L^2(0,T;H_0^1(\Omega))$, we obtain

$$\begin{aligned}
\lim_{n\mapsto+\infty}\int\limits_{Q_T}\|\nabla v_{n,k}\|^2\,dxdt &\leqslant \lim_{h\to 0}\Bigg[\int\limits_0^{T-h} < v_{k,h},\frac{\partial v_{k,h}}{\partial t} > dt + \int\limits_{Q_{T-h}}\|\nabla v_{k,h}\|^2\,dxdt\Bigg],\\
&\leqslant \lim_{h\to 0}\Bigg[\frac{1}{2}\int\limits_\Omega [v_{k,h}^2]_0^{T-h}dx + \int\limits_{Q_{T-h}}\|\nabla v_{k,h}\|^2\,dxdt\Bigg],\\
&\leqslant \int\limits_{Q_T}\|\nabla v_k\|^2\,dxdt.
\end{aligned}$$

$\square$

5.2. Passing to the Limit. According to Lemma 5.2 there exist a measurable function $u \in L^1(0,T;W_0^{1,1}(\Omega))$ and a subsequence still denoted by (u_n) for simplicity, such that

$$\begin{aligned}
u_n &\longrightarrow u \text{ in } L^1(0,T;W_0^{1,1}(\Omega))\\
(u_n,\nabla u_n) &\longrightarrow (u,\nabla u) \text{ a.e. in } Q_T
\end{aligned}$$

then,

$$F(t,x,u_{n-1}) \longrightarrow F(t,x,u) \text{ a.e. in } Q_T,$$

thanks to Lebesgue theorem, we have

$$F(t,x,u_{n-1}) \longrightarrow F(t,x,u) \text{ in } L^1(Q_T).$$

By the previous result of Lemmas 5.1 and 5.2

$$G_n(t,x,\nabla u_n) \longrightarrow G(t,x,\nabla u) \text{ a.e. in } Q_T, \tag{19}$$

it remains to show that

$$G_n(t,x,u_n,\nabla u_n) \to G(t,x,u,\nabla u) \text{ in } L^1(Q_T),$$

using (19) it suffices to prove that $G_n(t,x,\nabla u_n)$ is equi-integrable in $L^1(Q_T)$ namely

$$\forall\varepsilon > 0,\ \ \exists\,\delta > 0,\ \forall\, E\subset Q_T,\ \text{ if }\ |E|<\delta\ \text{ then } \int\limits_E G_n(t,x,\nabla u_n)\,dx\,dt \le \varepsilon.$$

Let E be a measurable subset of Q_T, $\varepsilon > 0$, and $k>0$. We have

$$\int\limits_E G_n(t,x,\nabla u_n)\,dx\,dt = I_1 + I_2$$

where

$$I_1 = \int\limits_{E\cap[u_n>k]} G_n(t,x,\nabla u_n)\,dx\,dt,$$

and

$$I_2 = \int\limits_{E\cap[u_n\le k]} G_n(t,x,\nabla u_n)\,dx\,dt.$$

The first integral I_1 verifies the following inequality

$$I_1 \leq \int\limits_{[u_n > k]} G_n(t, x, \nabla u_n)\, dx\, dt,$$

we obtain from the Lemma 5.1 the existence of $k^* > 0$, such that, for all $k \geqslant k^*$, we have

$$I_1 \leq \frac{\epsilon}{3}.$$

Concerning I_2 we use the assumption (15), we obtain for all $k \geq k^*$

$$I_2 \leq \int\limits_E (H(t, x) + d \mid \nabla T_k(u_n) \mid^2)\, dx\, dt.$$

Since $H \in L^1(Q_T)$, then H is equi-integrable in $L^1(Q_T)$, there exists $\delta_1 > 0$, such that, if $\mid E \mid \leq \delta_1$, then

$$\int\limits_E H(t, x)\, dx\, dt \leq \frac{\epsilon}{3}.$$

We have also from the Lemma 5.3 the sequence $(\mid \nabla T_k(u_n) \mid^2)_n$ is equi-integrable in $L^1(Q_T)$, which implies the existence of $\delta_2 > 0$, such that, if $\mid E \mid \leq \delta_2$, we have

$$d \int\limits_E \mid \nabla T_k(u_n) \mid^2 \; dx\, dt \leq \frac{\epsilon}{3}.$$

Finally, by choosing $\delta^* = \inf(\delta_1, \delta_2)$, if $\mid E \mid \leq \delta^*$, we obtain

$$\int\limits_E G_n(t, x, \nabla u_n)\, dx\, dt \;\leq\; \varepsilon.$$

On the other hand,

$$u_n(T) = S(T)u_n(0) + \int_0^T S(T - s)\eta_n(s, .)ds,$$

where

$$\eta_n(t, x) = F(t, x, u_{n-1}) + \mu(t, x) - G_n(t, x, \nabla u_n).$$

Since $u_n(0, .) \;=\; u_n(T, .)$ in $L^1(\Omega)$, we have for all $\phi \in L^\infty(\Omega)$

$$\begin{aligned}\lim_{n\to+\infty} \int_\Omega u_n(0, x)\phi(x)dx = \lim_{n\to+\infty} \int_\Omega S(T)u_n(0, x)\phi(x)dx \\ + \lim_{n\to+\infty} \int_\Omega \int_0^T S(T - s)\eta_n(s, x)\phi(x)dsdx,\end{aligned}$$

Since $S(t)$ is continuous in $L^1(\Omega)$ and $\eta_n \to \eta$ strongly in $L^1(Q_T)$, then

$$\begin{aligned}\int_\Omega u(0, x)\phi(x)dx \;&=\; \int_\Omega S(T)u(0, x)\phi(x)dx + \int_\Omega \int_0^T S(T - s)\eta(s, x)dsdx, \\ &=\; \int_\Omega u(T, x)\phi(x)dx.\end{aligned}$$

Then $u(0, .) = u(T, .)$ in $L^1(\Omega)$. Which ends the proof.

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