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Subspace iteration method for generalized singular values

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Abstract. It's well known that the Singular Values Decomposition (SVD) is useful in many applications such as low rank approximation, data reductions, identification of the best approximation of the original data points using fewer dimensions. It's also a useful tool for computation of eigenvalues of matrix A^TA without explicitly forming the matrix product. The Generalized Singular Values Decomposition (GSVD) of the pair (A, B) is also a useful tool for computation of the generalized eigenvalues of the symmetric pencil $A^TA-\lambda B^TB$. The generalized singular values of the pair (A, B) are nothing but the square roots of generalized eigenvalues of the symmetric eigenproblem $A^TAv - \lambda B^TBv = 0$. The novelty of this work is the method that computes the largest generalized singular values and vectors using iterative subspace-like method. Numerical examples show the effectiveness of the presented method.

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1. Introduction

The generalized singular value decomposition (GSVD) is a powerful and useful tool in Linear Algebra. It is a further generalization of singular value decomposition (SVD), it was introduced by Paige and Saunders [12] which is extension of the quotient singular value decomposition [2]. The main idea of the GSVD is to decompose a rectangular matrix pair (A, B) into the following two decompositions:

$$A = U\Sigma_A Z^{-1} \text{ and } B = V\Sigma_B Z^{-1}, \tag{1}$$

where A and B are (m, p) and (n, p) matrices, respectively, U and V are orthogonal matrices and Z invertible and Σ_A and Σ_B are positive diagonal matrices (see Theorem 6.6.1 [6]). If B is square nonsingular the GSVD gives the SVD of AB^{-1}:

$$AB^{-1} = U\Sigma_A\Sigma_B^{-1}V^T.$$

Even if B is non square matrix, we obtain the SVD of AB^+, where B^+ is the pseudo-inverse of a matrix B. Note that the case when the matrix B is the identity matrix has been studied in [1]. In fact we use the GSVD method to approximate eigenvalues of generalized symmetric problem

$$A^TAv - \lambda B^TBv = 0.$$

The decomposition (1) is based on the following theorem

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Theorem 1.1. [6] *Consider the matrix*

$$Q = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}, \; Q_1 \in \mathbb{R}^{m\times p}, Q_2 \in \mathbb{R},^{n\times p}$$

where $m \geq p$ and $n \geq p$. If the columns of Q are orthonormal, then there exist orthogonal matrices $U_1 \in \mathbb{R}^{m\times m}, U_2 \in \mathbb{R}^{n\times n}$, and $V_1 \in \mathbb{R}^{p\times p}$ such that

$$\begin{bmatrix} U_1 & 0 \\ 0 & U_2 \end{bmatrix}^T \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} V = \begin{bmatrix} C \\ S \end{bmatrix},$$

where

$$C = diag(cos(\theta_1), \ldots, cos(\theta_p)) \in \mathbb{R}^{m\times p}, \; S = diag(sin(\theta_1), \ldots, sin(\theta_p)) \in \mathbb{R}^{n\times p},$$

and

$$0 \leq \theta_1 \leq \theta_2 \leq \cdots \leq \theta_p \leq \frac{\pi}{2}.$$

In this paper we discuss the numerical solution of the generalized symmetric eigenvalue problem $A^T A - \lambda B^T B$ (matrix pencil), by using the GSVD method. For details on generalized eigenvalue problem see [1, 3, 4, 5, 6, 7, 9, 10, 15, 16]. And for details on generalized singular values one can see references [2, 4, 6, 7, 8, 11, 12, 13, 14]. This paper is organized as follows. In section 2, we give QR-like method for GSVD. The Golub-Kahan method for GSVD is presented in section 3. Section 4 is dedicated to an Iterative subspace method for computing generalized singular values. The proposed method gives the largest in magnitude generalized singular values. In section 5, we present some numerical results that compare the proposed approach to results obtained by Matlab function *gsvd* and at the end we give some concluding remarks.

2. The QR-like method for GSVD

Our main goal in this section is to give an iterative algorithm that compute the generalized singular value decomposition (GSVD) of the pair (A, B), where A and B are (m, p) and (n, p) matrices, respectively. The approach is based on a QR-Francis like method using both QR and RQ factorizations.

Algorithm 1 upper-bidiagonal/upper-triangular reduction

Input: $A \in \mathbf{R}^{m\times p}$, $B \in \mathbf{R}^{n\times p}$ $(p \leq min(m, n))$
Output : upper-bidiagonal/upper-triangular matrix pair reduction.
Initialization : $A^{(0)} = A$ *and* $B^{(0)} = B$;
For $k = 1 : p$
  (1) Set $u_a = A(:, k)$; $u_b = B(:, k)$;
  (2) Compute appropriate Householder matrices $H_a^{(k)}$ and $H_b^{(k)}$ corresponding to u_a and u_b ;
  (3) $A^{(k)} = H_a^{(k)} A^{(k-1)}$; $B^{(k)} = H_b^{(k)} B^{(k-1)}$;
  (4) **If** $k < p - 1$
    Set $u_a = A(k, :)$; and compute an appropriate Householder matrix $H_a^{(k)}$ correspond of u_a ;
    $A^{(k)} = A^{(k-1)} H_a^{(k)}$; $B^{(k)} = B^{(k-1)} H_a^{(k)}$;
    EndIf
EndFor

We begin by reducing the matrix pair (A, B) to an upper-bidiagonal/upper-triangular form (see Algorithm 1) or reducing the matrix pair (A, B) to an upper-triangular/lower-bidiagonal (see Algorithm 2).

Algorithm 2 upper triangular/lower-Bidiagonal reduction

Input: $A \in \mathbf{R}^{m\times p}$, $B \in \mathbf{R}^{n\times p}$ and $(p \leq min(m, n))$
Output : upper triangular/lower-bidiagonal matrix pair reduction.
Initialization : $A^{(0)} = A$ *and* $B^{(0)} = B$;
For $k = 1 : p$
(1) **If** $k < p$
Set $u_b = B^{(k-1)}(k, :)$; and compute the Householder matrix $H_b^{(k)}$ correspond of u_b
$A^{(k)} = H_b^{(k)} A^{(k-1)}$; $B^{(k)} = H_b^{(k)} B^{(k-1)}$; $V^{(k)} = H_b^{(k)} V^{(k-1)}$;
EndIf
(2) Construct the upper triangularity of A
Set $u_a = A^{(k-1)}(:, k)$; and compute the Householder matrix $H_a^{(k)}$ correspond of u_a
$A^{(k)} = A^{(k-1)} H_a^{(k)}$;
Construct the lower bidiagonalization of B
(3) **If** $k < m$
Set $u_b = B^{(k-1)}(k, :)$ and compute the Householder matrix $H_b^{(k)}$ correspond of u_b
$B^{(k)} = B^{(k-1)} H_b^{(k)}$; $Q^{(k)} = Q^{(k-1)} H_b^{(k)}$;
EndIf
EndFor

Let us now present the algorithm to compute the GSVD of (A, B). In the following we give an (m, p) matrix A and a (n, p) matrix B (here n, $m \geq p$) and we compute orthogonal matrices P, Q, V and upper triangular matrix R of suitable sizes such that $A = (P^T \Sigma_A V)R$ and $B = (Q^T \Sigma_B V)R$, where Σ_A and Σ_B are diagonal with positive diagonal entries. We first begin by computing the QR factorization of the augmented matrix $\begin{pmatrix} A \\ B \end{pmatrix}$; $\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} R$. Where, $Q_1 \in \mathbb{R}^{m\times p}$, $Q_2 \in \mathbb{R}^{n\times p}$ are such that $Q_1^T Q_1 + Q_2^T Q_2 = I_p$ and $R \in \mathbb{R}^{p\times p}$ is upper triangular. We set $A^{(0)} = Q_1$ and $B^{(0)} = Q_2$. Now, $A^{(0)}$ and $B^{(0)}$ are such that $A^{(0)T} A^{(0)} + B^{(0)T} B^{(0)} = I_p$. We use Algorithm 2 to reduce the matrix pair $(A^{(0)}, B^{(0)})$ to upper triangular/lower-bidiagonal form. In the second step, we generate a sequence of orthogonally equivalent matrices pairs $(A^{(0)}, B^{(0)}) \leftarrow (A^{(1)}, B^{(1)})$, $(A^{(2)}, B^{(2)})$, ... that converge to diagonal equivalent matrix pair (A^{∞}, B^{∞}). To take advantage of the triangular/lower-bidiagonal structure of the matrix pair $(A^{(0)}, B^{(0)})$, only Givens rotations are used. The upper triangular/lower-bidiagonal structure is preserved at each step k. The above method is summarized in the following algorithm:

Algorithm 3 Generalized SVD QR-Francis like Algorithm

Input : $A \in \mathbf{R}^{m\times p}$, $B \in \mathbf{R}^{n\times p}$ (n, $m \geq p$)
Output : Orthogonal matrices P, Q, , V and upper triangular matrix R of suitable sizes such that $A = (P^T \Sigma_A V)R$ and $B = (Q^T \Sigma_B V)R$ where Σ_A and Σ_B are diagonal.
1. Initialization : $k = 0$; $P = I_n$; $Q = I_m$ and $V = I_p$;
2. Compute the QR factorisation of the matrix $\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} R.$
3. Set $A^{(0)} = Q_1$ and $B^{(0)} = Q_2$.
4. Reduce $A^{(0)}$ to an lower bi-diagonal matrix and $B^{(0)}$ to an upper triangular one using Algorithm 2.
5. Reduce iteratively $A^{(0)}$ and $B^{(0)}$ to diagonal matrices $A^{(\infty)}$. and $B^{(\infty)}$
For $k = 1, 2, ...$ **until convergence**
(a) **If** $n > p$
  For $i = 1 : p$
  • Compute $V_{i+1,i}$ the Givens rotations that annihilate component $A^{(k-1)}(i+1, i)$;
  $A^{(k-1)} \leftarrow V_{i+1,i}A^{(k-1)}$;
  End
 Else
  For $j = 1 : p-1$
  • Compute $V_{j+1,j}$ the Givens rotations that annihilate component $A^{(k-1)}(j+1, j)$;
  $A^{(k-1)} \leftarrow V_{j+1,j}A^{(k-1)}$;
  End
 End
(c) Update, $A^{(k-1)} \longleftarrow {A^{(k-1)}}^T$ and $B^{(k-1)} \longleftarrow {B^{(k-1)}}^T$.
(d) **For** $i = 1 : p-1$
  • Compute $V_{i,i+1}$ the Givens rotations that annihilate component $A^{(k-1)}(i, i+1)$;
  $A^{(k-1)} \leftarrow V_{i,i+1}A^{(k-1)}$; $B^{(k-1)} \leftarrow V_{i,i+1}B^{(k-1)}$;
  End
(e) **If** $m > p$
  For $j = 1 : p$
  • Compute $U_{j,j+1}$ the Givens rotations that annihilate component $B^{(k-1)}(j, j+1)$;
  $B^{(k-1)} \leftarrow B^{(k-1)}U_{j,j+1}$;
  End
 Else
  For $i = 1 : p-1$
  • Compute $U_{i,i+1}$ the Givens rotations that annihilate component $B^{(k)}(i, i+1)$;
  $B^{(k-1)} \leftarrow B^{(k-1)}U_{i,i+1}$;
  End
 EndIf
(f) Update, $A^{(k)} \longleftarrow {A^{(k-1)}}^T$ and $B^{(k)} \longleftarrow {B^{(k-1)}}^T$.
EndFor
6. **Return** $\Sigma_A = A^{(\infty)}$ and $\Sigma_B = B^{(\infty)}$.

3. Golub-Kahan for GSVD

We describe a generalization of Golub-Kahan bi-diagonalization for computing the generalized singular values of two matrices A and B. The approach consists in two steps: first, reducing matrices A and B to an upper triangular and upper bi-diagonal form, respectively, and thereafter, applying Givens rotations from both right and left sides to iteratively obtain the diagonal matrices $A^{(\infty)}$ and $B^{(\infty)}$. At each step k, the upper bidiagonal form of $B^{(k-1)}$ is transformed to lower bidiagonal form. The triangularity of $A^{(k-1)}$ is destroyed. We restore the upper bidiagonal form of $B^{(k-1)}$ and the triangularity of $A^{(k-1)}$, and we repeat the process until convergence.

The above method is summarized in the following algorithm:

Algorithm 4 GK-GSVD

Input: $A \in \mathbf{R}^{m\times p}$, $B \in \mathbf{R}^{n\times p}$ ($n,\, m \geq p$)
Output : Orthogonal matrices P, Q, , V and upper triangular matrix R of suitable sizes such that $A = (P\Sigma_A V^T)R$ and $B = (Q\Sigma_B V^T)R$ where Σ_A and Σ_B are diagonal.
1. **Initialization :** $k := 0$;
2. Compute the QR factorisation of the matrix $\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} R.$
3. Set $A^{(0)} = Q_1$ and $B^{(0)} = Q_2$.
4. Reduce $A^{(0)}$ to upper bidiagonal matrices and $B^{(0)}$ to upper triangular one using Algorithm 1.
5. **For** $k = 1, 2, ...$ **until convergence**
 (a) transforming $B^{(k-1)}$ to lower bi-diagonal matrix. $A^{(k-1)}$ became tri-diagonal ;
  For $i = 1 : p-1$
  • Compute $V_{i,i+1}^{(k-1)}$ the Givens rotations that annihilate component $B^{(k-1)}(i, i+1)$;
  $B^{(k-1)} \leftarrow B^{(k-1)}V_{i,i+1}^{(k-1)}$; $A^{(k-1)} \leftarrow A^{(k-1)}V_{i,i+1}^{(k-1)}$; $V^{(k-1)} \leftarrow V^{(k-1)}V_{i,i+1}^{(k-1)}$
  EndFor
 (b) Restore $A^{(k-1)}$ and $B^{(k-1)}$ to upper bi-diagonal form
  For $j = 1 : p-1$
  • Compute $V_{i+1,j}^{(k-1)}$ the Givens rotations that annihilate component $A^{(k-1)}(j, j+1)$;
  $A^{(k-1)} \leftarrow P_{j,j+1}^{(k-1)}A$; $P^{(k-1)} \leftarrow P_{j,j+1}^{(k-1)}P^{(k-1)}$;
  • Compute $G_{j+1,j}^{(k-1)}$ the Givens rotations that annihilate component $B^{(k-1)}(j, j+1)$;
  $B^{(k-1)} \leftarrow G_{j+1,j}^{(k-1)}B^{(k-1)}$; $Q^{(k-1)} \leftarrow Q_{j,j+1}^{(k-1)}Q^{(k-1)}$;
  EndFor
 Update $A^{(k)} \leftarrow A^{(k-1)}$, $B^{(k)} \leftarrow B^{(k-1)}$
6. **Return** $\Sigma_A = A^{(\infty)}$ and $\Sigma_B = B^{(\infty)}$.

4. Iterative subspace method for computing generalized singular values

Our main goal in this section is to give an iterative algorithm that computes the s largest generalized singular values and the left and right corresponding generalized singular vectors using an iterative subspace iteration. The approach is based on the technique of the power method. We compute the following incomplete GSVD decompositions: $AZ = U\Sigma_A$ and $BZ = W\Sigma_B$ where Σ_A and Σ_B are (s, s) diagonal matrices, P, Q, and V orthogonal matrices and $Z = R^{-1}V^T$. The triangular matrix R is obtained from the QR factorization $\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} R$. The advantage of this approach is that we use only s vectors. This method can be used to solve the generalized eigenvalue problem of the following form

$$A^T A - \lambda B^T B = 0.$$

From a block-vector $X^{(0)} \in \mathbf{R}^{p\times s}$, we construct the block-orthonormal vector sequences $U^{(k)} \in \mathbf{R}^{m\times s}$, $V^{(k)} \in \mathbf{R}^{n\times s}$ and $Z^{(k)} \in \mathbf{R}^{s\times p}$ that converges respectively to the s associated first left-A and left-B generalized singular vectors, and the right generalized singular vectors. The algorithm below presents pseudocode for the method.

Algorithm 5 BGP-GSVD

Input: $A \in \mathbf{R}^{m\times p}$, $B \in \mathbf{R}^{n\times p}$
Output : Orthogonal matrices U, W, V and upper triangular matrix R of suitable sizes such that $AZ = U\Sigma_A$ and $BZ = W\Sigma_B$ where Σ_A and Σ_B are diagonal and $Z = R^{-1}V$.
1. **Initialization :** $k := 0$; and $V^{(0)} \in \mathbf{R}^{p\times s}$
2. Compute the QR factorization of the matrix $\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} R$;
3. Set $A^{(0)} = Q_1$ and $B^{(0)} = Q_2$;
4. **For** $k = 1, 2, ...$ **until convergence**
   (a) $AV^{(k-1)} = QS$ (**QR** factorization), set $U^{(k)} = Q(:, 1 : s)$;
   (b) $A^TU^{(k)} = Y\Sigma_A^{(k)}$ (**QR** factorization), set $Y \leftarrow Y(:, 1 : s)$;
   (c) $Y^TB^T = TP$ (**RQ** factorization), $P \leftarrow P^T; W^{(k)} = P(m-s+1 : m, :)$;
   (d) $PB^T = \Sigma_B^{(k)}H$ (**RQ** factorization),
   (e) $V^{(k)} = H(:, p-s+1 : p)$;
5. **Return** $U = U^{(\infty)}$, $W = W^{(\infty)}$, $\Sigma_A = \Sigma_A^{(\infty)}$, $\Sigma_B = \Sigma_B^{(\infty)}$ and $Z = R^{-1}V^{(\infty)}$.

We set $diag(\Sigma_A) = [\sigma_1, \sigma_2, ..., \sigma_s]^T$ and $diag(\Sigma_B) = [\sigma'_1, \sigma'_2, ..., \sigma'_s]^T$. The s largest generalized singular values of matrix AB^+ are $\dfrac{\sigma_i}{\sigma'_i}$, $i = 1, ..., s$.

4.1. Convergence. We begin by the QR factorization of matrix $\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} R_0$ and we set $A = Q_1$ and $B = Q_2$. Now $A^TA + B^TB = I$. Let q be an integer such that $r = qs$ where $r = min(rank(A), rank(B))$, for simplicity we give the proof when and $rank(A) \geq rank(B))$ suppose that $\sigma_1 \geq \ldots \geq \sigma_{\mathbf{s}} > \sigma_{\mathbf{s+1}} \geq \ldots \geq \sigma_{qs} > 0$ and $0 < \sigma'_1 \leq \ldots \leq \sigma'_{\mathbf{s}} < \sigma'_{s+1} \leq \ldots \leq \sigma'_{qs}$ are the generalized singular values of the matrix pair (A, B). We set

$$\Sigma_A = \begin{bmatrix} \Sigma_{A,1} & & & \\ & \Sigma_{A,2} & & \\ & & \ddots & \\ & & & \Sigma_{A,q} \end{bmatrix} \text{ and } \Sigma_B = \begin{bmatrix} \Sigma_{B,1} & & & \\ & \Sigma_{B,2} & & \\ & & \ddots & \\ & & & \Sigma_{B,q} \end{bmatrix}$$

where $\Sigma_{A,i}$ and $\Sigma_{B,i}$ are the diagonal matrices with nonzero, monotonically decreasing, creasing diagonal $\sigma_{(i-1)s+1} \geq \sigma_{(i-1)s+2} \geq \ldots \geq \sigma_{is} > 0$ and $0 < \sigma'_{(i-1)s+1} \leq \sigma'_{(i-1)s+2} \leq \ldots \leq \sigma'_{is}$ respectively. We can write A, B, B^+ and $(B^T)^+$ as

$$A = \sum_{i=1}^{q} U_i\Sigma_{A,i}V_i^T; B = \sum_{i=1}^{q} W_i\Sigma_{B,i}V_i^T,$$

$$B^+ = \sum_{i=1}^{q} V_i\Sigma_{B,i}^+W_i^T \text{ and } (B^T)^+ = \sum_{i=1}^{q} W_i\Sigma_{B,i}^+V_i^T$$

where U_i, W_i and V_i are the orthogonal matrices whose columns are respectively the corresponding left and right singular vectors. Let $V^{(0)} \in \mathbf{R}^{m\times s}$, $V^{(0)} = \sum_{i=1}^{q} V_iX_i +$

$V^{(0)*}$, where $span\left(V^{(0)*}\right) \subseteq span\left\{v_{r+1}, v_{r+2}, \cdots, v_m\right\} = \ker\{A\}$. We have

$$W^{(0)} = AV^{(0)} = U_1\Sigma_{A,1}X_1 + \sum_{\mathbf{i=2}}^{q} U_i\Sigma_{A,i}X_i.$$

Suppose that the component $X_1 = I_s$, then

$$AV^{(0)} = U^{(1)}\mathbf{R}_1 \,(\mathbf{QR} \text{ factorization}) = U_1\Sigma_{A,1} + \sum_{\mathbf{i=2}}^{q} U_i\Sigma_{A,i}X_i$$

$U_1^T U^{(1)}\mathbf{R}_1 = \Sigma_{A,1}$ that prove $\mathbf{R}_1$ is non singular and then

$$U^{(1)} = U_1\Sigma_{A,1}\mathbf{R}_1^{-1} + \sum_{\mathbf{i=2}}^{q} U_i\Sigma_{A,i}X_i\mathbf{R}_1^{-1}$$

and

$$A^T U^{(1)} = S^{(1)}\mathbf{R}_2(\mathbf{QR} \text{ factorization}) = V_1\Sigma_{A,1}^2\mathbf{R}_1^{-1} + \sum_{\mathbf{i=2}}^{q} V_i\Sigma_{A,i}^2X_i\mathbf{R}_1^{-1}$$

$V_1^T S^{(1)}\mathbf{R}_2 = \Sigma_{A,1}^2\mathbf{R}_1^{-1}$, $\mathbf{R}_2$ is non singular

$$S^{(1)} = V_1\Sigma_{A,1}^2\mathbf{R}_1^{-1}\mathbf{R}_2^{-1} + \sum_{\mathbf{i=2}}^{q} V_i\Sigma_{A,i}^2X_i\mathbf{R}_1^{-1}\mathbf{R}_2^{-1}$$

then,

$$\begin{aligned}(B^T)^+S^{(1)} &= W^{(1)}\mathbf{R}_3(\mathbf{QR} \text{ factorization})\\ &= W_1\Sigma_{A,1}^2\Sigma_{B,1}^{-1}\mathbf{R}_1^{-1}\mathbf{R}_2^{-1} + \sum_{\mathbf{i=2}}^{q} W_i\Sigma_{A,i}^2\Sigma_{B,i}^{-1}X_i\mathbf{R}_3^{-1}\end{aligned}$$

$W_1^T W^{(1)}\mathbf{R}_3 = \Sigma_{A,1}^2\Sigma_{B,i}^{-1}\mathbf{R}_1^{-1}\mathbf{R}_2^{-1}$, $\mathbf{R}_3$ is non singular

$$W^{(1)} = W_1\Sigma_{A,1}^2\Sigma_{B,1}^{-1}\mathbf{R}_1^{-1}\mathbf{R}_2^{-1}\mathbf{R}_3^{-1} + \sum_{\mathbf{i=2}}^{q} W_i\Sigma_{A,i}^2\Sigma_{B,i}^{-1}X_i\mathbf{R}_1^{-1}\mathbf{R}_2^{-1}\mathbf{R}_3^{-1}.$$

We have,

$$\begin{aligned}B^+W^{(1)} &= V^{(1)}\mathbf{R}_4(\mathbf{QR} \text{ factorization})\\ &= V_1\Sigma_{A,1}^2\Sigma_{B,1}^{-2}\mathbf{R}_1^{-1}\mathbf{R}_2^{-1}\mathbf{R}_3^{-1} + \sum_{\mathbf{i=2}}^{q} W_i\Sigma_{A,i}^2\Sigma_{B,i}^{-2}X_i\mathbf{R}_3^{-1}\end{aligned}$$

$V_1^T V^{(1)}\mathbf{R}_4 = \Sigma_{A,1}^2\Sigma_{B,1}^{-2}\mathbf{R}_1^{-1}\mathbf{R}_3^{-1}\mathbf{R}_2^{-1}$, $\mathbf{R}_4$ is non singular

$$V^{(1)} = V_1\Sigma_{A,1}^2\Sigma_{B,1}^{-2}\mathbf{R}_1^{-1}\mathbf{R}_2^{-1}\mathbf{R}_3^{-1}\mathbf{R}_4^{-1} + \sum_{\mathbf{i=2}}^{q} V_i\Sigma_{A,i}^2\Sigma_{B,i}^{-2}X_i\mathbf{R}_1^{-1}\mathbf{R}_2^{-1}\mathbf{R}_3^{-1}\mathbf{R}_4^{-1}$$

and so on, if we note $\mathbf{N}_t = \mathbf{R}_1^{-1}\mathbf{R}_2^{-1}\cdots\mathbf{R}_t^{-1}$, at step k we have

$$\begin{aligned}AV^{(k-1)} &= U^{(k)}\mathbf{R}_{2k-1}(\mathbf{QR} \text{ factorization})\\ &= U_1\Sigma_{A,1}^{2k-1}\Sigma_{B,1}^{-2k+2}\mathbf{N}_{2(k-1)} + \sum_{\mathbf{i=2}}^{q} U_i\Sigma_{A,i}^{2k-1}\Sigma_{B,i}^{-2k+2}X_i\mathbf{N}_{2(k-1)}\end{aligned}$$

$$U^{(k)} = U_1 \Sigma_{A,1}^{2k-1} \Sigma_{B,1}^{-2k+2} \mathbf{N}_{2k-1} + \sum_{\mathbf{i=2}}^{q} U_i \Sigma_{A,i}^{2k-1} \Sigma_{B,i}^{-2k+2} X_i \mathbf{N}_{2k-1}$$

and

$$\begin{aligned} A^T U^{(k)} &= S^{(k)} \mathbf{R}_{2k} (\mathbf{QR} \text{ factorization}) \\ &= V_1 \Sigma_{A,1}^{2k} \Sigma_{B,1}^{-2k+2} \mathbf{N}_{2k-1} + \sum_{\mathbf{i=2}}^{q} V_i \Sigma_{A,i}^{2k} \Sigma_{B,i}^{-2k+2} X_i \mathbf{N}_{2k-1} \\ V_1^T S^{(k)} \mathbf{R}_{2k} &= \Sigma_{A,1}^{2k} \Sigma_{B,1}^{-2k+2} \mathbf{N}_{2k-1},\ \mathbf{R}_{2k} \text{ is non singular} \\ S^{(k)} &= V_1 \Sigma_{A,1}^{2k} \Sigma_{B,1}^{-2k+2} \mathbf{N}_{2k} + \sum_{\mathbf{i=2}}^{q} V_i \Sigma_{A,i}^{2k} \Sigma_{B,i}^{-2k+2} X_i \mathbf{N}_{2k} \end{aligned}$$

then,

$$\begin{aligned} (B^T)^+ S^{(k)} &= W^{(k)} \mathbf{R}_{2k+1} (\mathbf{QR} \text{ factorization}) \\ &= W_1 \Sigma_{A,1}^{2k} \Sigma_{B,1}^{-2k+1} \mathbf{N}_{2k} + \sum_{\mathbf{i=2}}^{q} W_i \Sigma_{A,i}^{2} \Sigma_{B,i}^{-2k+1} X_i \mathbf{N}_{2k} \\ W_1^T W^{(k)} \mathbf{R}_{2k+1} &= \Sigma_{A,1}^{2k} \Sigma_{B,1}^{-2k+1} \mathbf{N}_{2k}, \text{ is non singular} \\ W^{(k)} &= W_1 \Sigma_{A,1}^{2k} \Sigma_{B,1}^{-2k+1} \mathbf{N}_{2k+1} + \sum_{\mathbf{i=2}}^{q} W_i \Sigma_{A,i}^{2k} \Sigma_{B,i}^{-2k+1} X_i \mathbf{N}_{2k+1} \end{aligned}$$

We have,

$$\begin{aligned} (B)^+ W^{(k)} &= V^{(k)} \mathbf{R}_{2k+2} (\mathbf{QR} \text{ factorization}) \\ &= V_1 \Sigma_{A,1}^{2k} \Sigma_{B,1}^{-2k} \mathbf{N}_{2k+1} + \sum_{\mathbf{i=2}}^{q} V_i \Sigma_{A,i}^{2k} \Sigma_{B,i}^{-2k} X_i \mathbf{N}_{2k+1} \\ V_1^T V^{(k)} \mathbf{R}_{2k+2} &= \Sigma_{A,1}^{2k} \Sigma_{B,1}^{-2k} \mathbf{N}_{2k+1}, \text{ is non singular} \\ V^{(k)} &= V_1 \Sigma_{A,1}^{2k} \Sigma_{B,1}^{-2k} \mathbf{N}_{2k+2} + \sum_{\mathbf{i=2}}^{q} V_i \Sigma_{A,i}^{2k} \Sigma_{B,i}^{-2k} X_i \mathbf{N}_{2k+2} \end{aligned}$$

$U^{(k)}$, $V^{(k)}$ and $W^{(k)}$ are orthogonal matrices, then

$$I_s = \left(U^{(k)}\right)^T U^{(k)} = \mathbf{N}_{2k-1}^T \Sigma_{B,1}^{-4k+4} \Sigma_{A,1}^{4k-2} \mathbf{N}_{2k-1} + \sum_{\mathbf{i=2}}^{q} \mathbf{N}_{2k-1}^T X_i^T \Sigma_{B,i}^{-4k+4} \Sigma_{A,i}^{4k-2} X_i \mathbf{N}_{2k-1}$$

$$I_s = \left(W^{(k)}\right)^T W^{(k)} = \mathbf{N}_{2k+1}^T \Sigma_{B,1}^{-4k+2} \Sigma_{A,1}^{4k} \mathbf{N}_{2k+1} + \sum_{\mathbf{i=2}}^{q} \mathbf{N}_{2k+1}^T X_i^T \Sigma_{B,i}^{-4k+2} \Sigma_{A,i}^{4k} X_i \mathbf{N}_{2k+1}$$

and

$$I_s = \left(V^{(k)}\right)^T V^{(k)} = \mathbf{N}_{2k+2}^T \Sigma_{B,1}^{-4k} \Sigma_{A,1}^{4k} \mathbf{N}_{2k+2} + \sum_{\mathbf{i=2}}^{q} \mathbf{N}_{2k+2}^T X_i^T \Sigma_{B,i}^{-4k} \Sigma_{A,i}^{4k} X_i \mathbf{N}_{2k+2}$$

by left and right-factoring, we obtain

$$I_s = \mathbf{N}_{2k-1}^T \Sigma_{B,1}^{-2k+2} \Sigma_{A,1}^{2k-1} \Big(I_s + \\ + \sum_{\mathbf{i=2}}^{q} \Sigma_{B,1}^{2k-2} \Sigma_{A,1}^{-2k+1} X_i^T \Sigma_{B,i}^{-4k+4} \Sigma_{A,i}^{4k-2} X_i \Sigma_{B,1}^{2k-2} \Sigma_{A,1}^{-2k+1} \Big) \Sigma_{A,1}^{2k-1} \Sigma_{B,1}^{-2k+2} \mathbf{N}_{2k-1}$$

$$I_s = \mathbf{N}_{2k+1}^T \Sigma_{B,1}^{-2k+1} \Sigma_{A,1}^{2k} \Big(I_s + \\ + \sum_{\mathbf{i=2}}^{q} \Sigma_{B,1}^{2k+1} \Sigma_{A,1}^{-2k} X_i^T \Sigma_{B,i}^{-4k+2} \Sigma_{A,i}^{4k} X_i \Sigma_{B,1}^{2k-1} \Sigma_{A,1}^{-2k} \Big) \Sigma_{A,1}^{2k-1} \Sigma_{B,1}^{-2k+1} \mathbf{N}_{2k+1}$$

$$I_s = \mathbf{N}_{2k+2}^T \Sigma_{B,1}^{-2k} \Sigma_{A,1}^{2k} \left(I_s + \sum_{\mathbf{i=2}}^{q} \Sigma_{B,1}^{2k} \Sigma_{A,1}^{-2k} X_i^T \Sigma_{B,i}^{-4k} \Sigma_{A,i}^{4k} X_i \Sigma_{B,1}^{2k} \Sigma_{A,1}^{-2k} \right) \Sigma_{A,1}^{2k} \Sigma_{B,1}^{-2k} \mathbf{N}_{2k+2}$$

Since $\|\Sigma_{A,1}\| = \sigma_1$, $\left\|\Sigma_{A,1}^{-1}\right\| = \dfrac{1}{\sigma_s}$, $\left\|\Sigma_{B,1}^{-1}\right\| = \dfrac{1}{\sigma'_{(i-1)s+1}}$ and $\|\Sigma_{A,i}\| = \sigma_{(i-1)s+1}$, $\|\Sigma_{B,i}\| = \sigma'_{is}$ then,

$$\begin{aligned} \left\|\Sigma_{A,1}^{-p} \Sigma_{B,1}^{p} X_i^T \Sigma_{A,i}^{2p} \Sigma_{B,1}^{-2p} X_i \Sigma_{B,1}^{p} \Sigma_{A,1}^{-p}\right\| &\leq \|\Sigma_{A,i}\|^{2p} \left\|\Sigma_{A,1}^{-1}\right\|^{2p} \|\Sigma_{B,i}\|^{2p} \left\|\Sigma_{B,1}^{-1}\right\|^{2p} \|X_i\|^2 \\ &\leq \left(\frac{\sigma_{(i-1)s+1}}{\sigma_s}\right)^{2p} \left(\frac{\sigma'_s}{\sigma'_{(i-1)s+1}}\right)^{2p} \|X_i\|^2 \underset{p\to+\infty}{\longrightarrow} 0 \end{aligned}$$

Thus

$$\lim_{p\to\infty} \left(\mathbf{N}_p^T \Sigma_{B,1}^{-p} \Sigma_{A,1}^{p}\right) \left(\Sigma_{A,1}^{p} \Sigma_{B,1}^{-p} \mathbf{N}_p\right) = \lim_{p\to\infty} \left(\Sigma_{A,1}^{p} \Sigma_{B,1}^{-p} \mathbf{N}_p\right)^T \left(\Sigma_{A,1}^{p} \Sigma_{B,1}^{-p} \mathbf{N}_p\right) = I_s.$$

Moreover, the matrix $\Sigma_{A,1}^{p} \Sigma_{B,1}^{-p} \mathbf{N}_p$ is triangular with positive diagonal entries, then $\lim\limits_{p\to\infty} \Sigma_{A,1}^{p} \Sigma_{B,1}^{-p} \mathbf{N}_p = \lim\limits_{p\to\infty} \mathbf{N}_p^{-1} \Sigma_{B,1}^{p} \Sigma_{A,1}^{-p} = I_s$. Otherwise

$$\begin{aligned} & A^T U^{(k)} \left(\mathbf{N}_{2k-1}^{-1} \Sigma_{B,1}^{2k-2} \Sigma_{A,1}^{-2k+1}\right) \Sigma_{A,1}^{-1} \\ &\qquad = V_1 + \sum_{\mathbf{i=2}}^{q} V_i \Sigma_{A,i}^{2k} \Sigma_{B,i}^{-2k+2} X_i \mathbf{N}_{2k-1}^{-1} \mathbf{N}_{2k-1} \Sigma_{B,1}^{2k-2} \Sigma_{A,1}^{-2k} \underset{k\to+\infty}{\longrightarrow} V_1 \end{aligned}$$

and

$$\begin{aligned} & A V^{(k)} \left(\mathbf{N}_{2k+2}^{-1} \Sigma_{B,1}^{2k} \Sigma_{A,1}^{-(2k+1)}\right) \\ &\qquad = U_1 + \sum_{\mathbf{i=2}}^{q} U_i \Sigma_{A,i}^{2k+1} \Sigma_{B,i}^{-2k} X_i \mathbf{N}_{2k-1} \mathbf{N}_{2k+1}^{-1} \Sigma_{B,1}^{2k} \Sigma_{A,1}^{(-2k-1)} \underset{k\to+\infty}{\longrightarrow} U_1 \end{aligned}$$

$$\begin{aligned} & (B^T)^+ S^{(k)} \left(\mathbf{N}_{2k}^{-1} \Sigma_{B,1}^{2k-1} \Sigma_{A,1}^{-(2k)}\right) \\ &\qquad = W_1 + \sum_{\mathbf{i=2}}^{q} W_i \Sigma_{A,i}^{2k} \Sigma_{B,i}^{-2k+1} X_i \mathbf{N}_{2k}^{-1}) \left(\mathbf{N}_{2k}^{-1} \Sigma_{B,1}^{2k-1} \Sigma_{A,1}^{-2k}\right) \end{aligned}$$

$$(B^T)^+S^{(k)}\mathbf{R}_{2k+1}^{-1}\mathbf{R}_{2k+1}\left(\mathbf{N}_{2k}^{-1}\Sigma_{B,1}^{2k-1}\Sigma_{A,1}^{-(2k)}\right)$$
$$= W_1 + \sum_{\mathbf{i=2}}^{q} W_i\Sigma_{A,i}^{2k}\Sigma_{B,i}^{-2k+1}X_i\mathbf{N}_{2k}^{-1})\left(\mathbf{N}_{2k}^{-1}\Sigma_{B,1}^{2k-1}\Sigma_{A,1}^{-2k}\right)$$
$$(B^T)^+S^{(k)}\mathbf{R}_{2k+1}^{-1}\left(\mathbf{N}_{2k+1}^{-1}\Sigma_{B,1}^{2k-1}\Sigma_{A,1}^{-(2k)}\right)$$
$$= W_1 + \sum_{\mathbf{i=2}}^{q} W_i\Sigma_{A,i}^{2k}\Sigma_{B,i}^{-2k+1}X_i\mathbf{N}_{2k}^{-1})\left(\mathbf{N}_{2k}^{-1}\Sigma_{B,1}^{2k-1}\Sigma_{A,1}^{-2k}\right)\underset{p\to+\infty}{\longrightarrow} W_1$$

And we have,

$$\mathbf{R}_{2k+1}\left(\mathbf{N}_{2k}^{-1}\Sigma_{B,1}^{2k-1}\Sigma_{A,1}^{-(2k)}\right) = \mathbf{R}_{2k+1}\left(\mathbf{N}_{2k}^{-1}\Sigma_{B,1}^{2k}\Sigma_{A,1}^{-(2k)}\right)\Sigma_{B,1}^{-1}\longrightarrow\Sigma_{B,1}^{-1}$$
$$(B)^+W^{(k)}\left(\mathbf{N}_{2k+1}^{-1}\Sigma_{B,1}^{2k}\Sigma_{A,1}^{-2k}\right) = V_1 + \sum_{\mathbf{i=2}}^{q} V_i\Sigma_{A,i}^{2k}\Sigma_{B,i}^{-2k}X_i\mathbf{N}_{2k+1}\mathbf{N}_{2k+1}^{-1}\Sigma_{B,1}^{2k}\Sigma_{A,1}^{-2k}$$
$$(B)^+W^{(k)}\mathbf{R}_{2k+2}^{-1}\left(\mathbf{N}_{2k+2}^{-1}\Sigma_{B,1}^{2k}\Sigma_{A,1}^{-2k}\right) = V_1 + \sum_{\mathbf{i=2}}^{q} V_i\Sigma_{A,i}^{2k}\Sigma_{B,i}^{-2k}X_i\mathbf{N}_{2k+1}\mathbf{N}_{2k+1}^{-1}\Sigma_{B,1}^{2k}\Sigma_{A,1}^{-2k}$$
$$\underset{k\to+\infty}{\longrightarrow} V_1$$

and

$$\left(\mathbf{N}_{2k+2}^{-1}\Sigma_{B,1}^{2k}\Sigma_{A,1}^{-2k}\right) = \mathbf{R}_{2k+2}\mathbf{N}_{2k+2}^{-1}\Sigma_{B,1}^{2k-1}\Sigma_{A,1}^{-2k}\Sigma_{B,1}^{-1}\longrightarrow\Sigma_{B,1}^{-1}$$

That implies that $\lim_{k\to+\infty} A^TU^{(k)} = V_1\Sigma_{A,1}$, $\lim_{k\to+\infty} AV^{(k)} = U_1\Sigma_{A,1}$, $\lim_{k\to+\infty}(B^T)^+S^{(k)}$ $= W_1\Sigma_{B,1}^{-1}$ and $\lim_{k\to+\infty}(B)^+W^{(k)} = V_1\Sigma_{B,1}^{-1}$

5. Numerical examples

In this section we compared the numerical results obtained by Algorithm 3, Algorithm 4 and Algorithm 5 with *gsvd* Matlab function in terms of the relative error. All of the reported numerical experiments were performed on Matlab version R2016a. $A \in \mathbf{R}^{m\times p}$ and $B \in \mathbf{R}^{n\times p}$ are rectangular matrices defined as $A = U\Sigma_A Z^{-1}$ and $B = W\Sigma_B Z^{-1}$, where $U$and W are random orthogonal matrices, Z non singular and $r = rank(B)$. We give below relative errors occurred when computing the generalized eigenvalues of the pair (A^TA, B^TB). We compare the numerical results obtained for different sizes. For the first test we take $m = 1500$, $n = 500$ and $r = p = 100$, the figure 1 gives the corresponding relative errors of eigenvalues of the pair (A^TA, B^TB) computed by Algorithm 3 and the one computed by *gsvd* Matlab function. For the second test we take $m = 1000$, $n = 100$, $p = 50$ and $r = 10$, we plot the relative errors of eigenvalues of the pair (A^TA, B^TB) computed by Algorithm 4 and the one computed by *gsvd* Matlab function in the figure 2 . Finally we take $m = 1000$, $n = 800$, $p = 80$, $r = q = 6$ and we plot the relative errors of eigenvalues of the pair (A^TA, B^TB) computed by Algorithm 5 and the one computed by *gsvd* Matlab function in the figure 3.

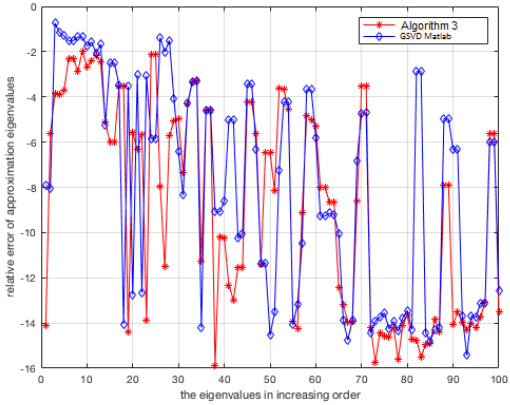

FIGURE 1. Relative error of gsvd matlab function and Algorithm 3.

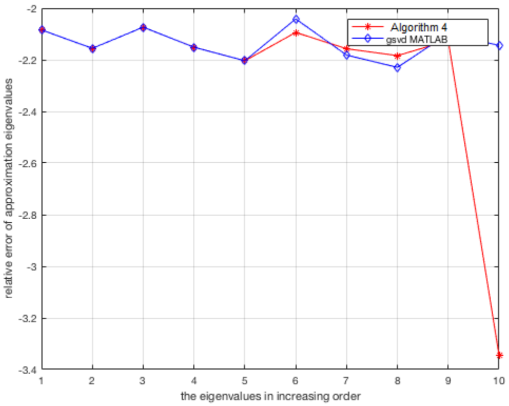

FIGURE 2. Relative error of gsvd matlab function and Algorithm 4.

6. Conclusion

We have presented a generalization of the subspace iteration method (BPG-GSVD) to compute the s-largest in magnitude generalized eigenvalues of the matrix pencil $A^TA - \lambda B^TB$. Two others generalization are presented, the first one is a generalization of the well-known Francis-QR method and the second one is a generalization of the Golub-Kahan method to compute the GSVD decomposition.

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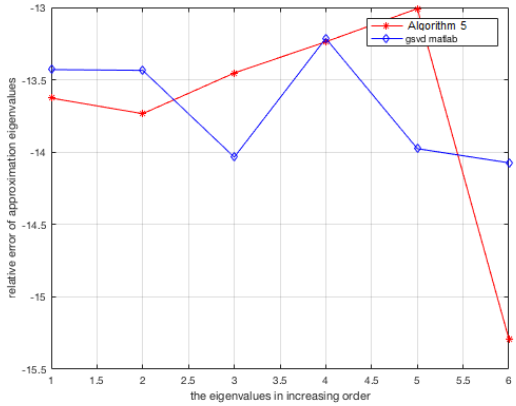

FIGURE 3. Relative error of gsvd matlab function and Algorithm 5.

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