Annals of the University of Craiova, Mathematics and Computer Science Series
Volume 46(1), 2019, Pages 90–98
ISSN: 1223-6934

Mathematical analysis of a modified Weikert system for image enhancement

Hamza Khalfi, Houda Fahim, and Nour Eddine Alaa

Abstract. This paper tackles the problem of proving global existence for an abstract non-linear Reaction-Diffusion system inspired by the famous Weickert model [16]. We show the existence of a global weak solution to a truncated version of the proposed system using a fixed point approach. Then, we establish some crucial estimates under Quasi-positivity and a Triangular Structure condition on the nonlinearities [3]. On passage to the limit, we recover the original system and prove the existence for only integrable nonlinearities and initial conditions.

2010 Mathematics Subject Classification. Primary 35K57; Secondary 35D30.
Key words and phrases. Reaction diffusion systems, Image Restoration, parabolic systems, nonlinear partial differential equations, Schauder fixed point, duality method.

1. Introduction

Nonlinear anisotropic diffusion is a powerful image processing technique allowing an efficient noise removal without degrading the quality of sharp features. A substantial amount of literature tackles specific problems relevant to anisotropic diffusion and applications to image processing. Numerous papers propose different filters and models and investigate their specific features while describing interesting novel applications [9, 10, 11, 2]. Systems of Partial differential equations (PDEs) are very popular in image processing and have been extensively studied in the literature [3, 6, 5, 7]. They have proven to be fundamental tools for image diffusion and restoration such as The Perona-Malik equation [1], which is one of the first attempts to derive a model that incorporates local information from an image within a PDE framework. Other works followed [12, 13, 9] to enforce the role of these class of models. It is usually difficult to find the mathematical foundation to prove the existence of solution to general models in image processing. Researcher cater their proofs to a specific feature or model.

The goal of the present paper is to lay the mathematical foundation to prove the existence of solution to a class of reaction diffusion systems based on the famous Weickert model for image restoration and enhancement using techniques established in [3, 4] while keeping a certain degree of abstraction. To this end, we propose an abstract system coupling Weickert's structure tensor [16] for image restoration and nonlinear sources for image enhancement that in theory would improve the identification of features such as corners or to measure the local coherence of structures. We would like to remain as general as possible in this exposition so we leave the application of this system for a future work where a comparison of different specific nonlinearties

This paper has been presented at the Conference MOCASIM, Marrakesh, 26-27 November 2018.

will be considered. Our considered model is the following coupled reaction diffusion system :

$$\begin{cases} \partial_t u - \operatorname{div}\left(D\left(J_\rho\left(\nabla u_\sigma\right)\right)\nabla u\right) = f(t,x,u,v) & \text{in } Q_T, \\ \partial_t v - d_v \Delta v = g(t,x,u,v) & \text{in } Q_T, \\ \partial_\nu v = 0, \qquad \left\langle D\left(J_\rho\left(\nabla u_\sigma\right)\right)\nabla u, \nu\right\rangle = 0 & \text{on } \Sigma_T, \\ u(0,\cdot) = u_0(\cdot), \quad v(0,\cdot) = v_0(\cdot) & \text{in } \Omega, \end{cases} \tag{1}$$

which is based on Weickert's notion of the structure tensor $D\left(J_\rho\left(\nabla u_\sigma\right)\right)$ [16]. The domain Ω is smooth and bounded in $\mathbb{R}^n$ and $T \in (0,\infty[$, $Q_T =]0,T[\times\Omega$ and $\Sigma_T =]0,T[\times\partial\Omega$ where $\partial\Omega$ denotes the boundary of Ω. ν is the outward normal to the domain and ∂_ν is the normal derivative. Let $\sigma > 0$, ∇u_σ represents the standard regularization through convolution of ∇u by the Gaussian function.

The nonlinear functions $f, g : Q_T \times \mathbb{R}^2 \to \mathbb{R}$ are measurable and $f(t,x,.), g(t,x,.) : \mathbb{R}^2 \to \mathbb{R}$ are continuous. Moreover the source terms satisfy the quasi-positivity property

$$f(t,x,0,s) \geq 0 \quad \forall s \geq 0 \quad \text{and} \quad g(t,x,r,0) \geq 0 \quad \forall r \geq 0, \tag{2}$$

and a triangular structure

$$(f+g)(t,x,r,s) \leq L_1(r+s+1) \quad \text{and} \quad g(t,x,r,s) \leq L_2(r+s+1), \tag{3}$$

where L_1 and L_2 are positive constant. Furthermore,

$$\sup_{|r|+|s|\leq R} \left(|f(t,x,r,s)| + |g(t,x,r,s)|\right) \in L^1\left(Q_T\right), \tag{4}$$

for $R > 0$. The challenge here is to prove the existence of the solution for positive integrable initial conditions u_0, v_0 while taking into consideration the structure tensor which is defined by

$$J_\rho\left(\nabla u_\sigma\right) = G_\rho * \left(\nabla u_\sigma . \nabla u_\sigma^T\right).$$

The tensor D is taken so that the following conditions are met:

- (C_1) Smoothness: $D \in \mathcal{C}^\infty\left(\mathbb{R}^{2\times 2}; \mathbb{R}^{2\times 2}\right)$.
- (C_2) Symmetry: $d_{12}(J) = d_{21}(J)$ for all symmetric matrices $J \in \mathbb{R}^{2\times 2}$.
- (C_3) Uniform positive definiteness: For all $w \in L^\infty\left(\Omega, \mathbb{R}^2\right)$ with $|w(x)| \leq K$ on Ω, there exists a positive lower bound $\nu(K)$ for the eigenvalues of $D\left(J_\rho(w)\right)$.

Before diving into further analysis, let us first clearly enunciate the definition of a weak solution to the reaction diffusion system:

Definition 1.1. We call (u,v) a weak solution of the system (1) if

- $u, v \in L^1(0,T; W^{1,1}(\Omega)) \cap \mathcal{C}([0,T]; L^1(\Omega))$, $u(0,\cdot) = u_0$ and $v(0,\cdot) = v_0$
- $\forall \phi, \psi \in \mathcal{C}^1(Q_T)$ such that $\phi(\cdot,T) = 0$ and $\psi(\cdot,T) = 0$ we have

$$\int_{Q_T} -u\,\partial_t\phi + \left\langle D\left(J_\rho\left(\nabla u_\sigma\right)\right)\nabla u, \nabla\phi\right\rangle = \int_{Q_T} f(t,x,u,v)\,\phi + \int_\Omega u_0\,\phi(\cdot,0)$$

$$\int_{Q_T} -v\,\partial_t\psi + d_v\,\nabla v\,\nabla\psi = \int_{Q_T} g(t,x,u,v)\,\psi + \int_\Omega v_0\,\psi(\cdot,0), \tag{5}$$

where $f(t,x,u,v), g(t,x,u,v) \in L^1(Q_T)$.

2. Existence result for a truncated non-linearity

First, we truncate f and g using the truncation function $\Psi_n \in \mathcal{C}_c^\infty(\mathbb{R})$, such that $0 \leq \Psi_n \leq 1$ and

$$\Psi_n(r) = \begin{cases} 1 & \text{if } |r| \leq n, \\ 0 & \text{if } |r| \geq n+1. \end{cases} \tag{6}$$

Then, we formulate the approximate problem as follows:

$$\begin{cases} \partial_t u_n - \operatorname{div}\left(D\left(J_\rho\left(\nabla u_{n_\sigma}\right)\right)\nabla u_n\right) = f_n(t,x,u_n,v_n) & \text{in } Q_T, \\ \partial_t v_n - d_v\ \Delta v_n = g_n(t,x,u_n,v_n) & \text{in } Q_T, \\ u(0,\cdot) = u_{0,n}(\cdot), \quad v(0,\cdot) = v_{0,n}(\cdot) & \text{in } \Omega, \end{cases} \tag{7}$$

where $f_n(t,x,u_n,v_n) = \Psi_n(|u_n|+|v_n|)\ f(t,x,u_n,v_n)$ and $g_n(t,x,u_n,v_n) = \Psi_n(|u_n|+|v_n|)\ g(t,x,u_n,v_n)$. We also note that the $u_{n,0}$ and $v_{n,0}$ are sufficiently regular sequences approximating the initial conditions such that $u_{n,0}$,$v_{n,0}$ are positive, square integrable and converge to u_0,v_0 in $L^1(\Omega)$.

Theorem 2.1. *Under the previously stated assumptions, there exists a weak solution (u,v) to the considered system* (7).
Moreover there exists $C(n,\rho,\sigma,T,||u_{0,n}||_{L^2(\Omega)},||v_{0,n}||_{L^2(\Omega)})$ such that

$$||(u,v)||_{L^\infty(0,T;L^2(\Omega))^2} + ||(u,v)||_{L^2(0,T;H^1(\Omega))^2} \leq C. \tag{8}$$

Furthermore, $u_n(t,x) \geq 0$ and $v_n(t,x) \geq 0$ a.e. in Q_T.

We will show the existence of a weak solution by the classical Schauder fixed point theorem. The following sketch of the proof assumes that u_n and v_n are positive which directly comes from the quasi-positivity of the source terms see [2].

Proof. Let n be fixed and $w = (w_1,w_2) \in L^\infty(0,T;L^2(\Omega))^2$ bounded $\|w\| \leq K$. In this paragraph, we denote (u,v) the solution of the linear problem:
$\forall \phi,\psi \in \mathcal{C}^1(Q_T)$ such that $\phi(\cdot,T)=0$ and $\psi(\cdot,T)=0$, we have

$$\int_{Q_T} -u\,\partial_t\phi + \left\langle D\left(J_\rho\left(\nabla w_{1_\sigma}\right)\right)\nabla u, \nabla\phi\right\rangle = \int_{Q_T} f_n(t,x,w_1,w_2)\,\phi + \int_\Omega u_{0,n}\,\phi(\cdot,0)$$
$$\int_{Q_T} -v\,\partial_t\psi + d_v\,\nabla v\,\nabla\psi = \int_{Q_T} g_n(t,x,w_1,w_2)\,\psi + \int_\Omega v_{0,n}\,\psi(\cdot,0). \tag{9}$$

Thanks to the conditions (2-4) on $D\left(J_\rho(\cdot)\right)$ the structure tensor is bounded and satisfies

$$\mu(K)\left|\nabla u\right|^2 \leq \left\langle D\left(J_\rho\left(\nabla w_{1_\sigma}\right)\right)\nabla u, \nabla u\right\rangle.$$

This implies that the differential operators in (9) are continuous and coercive. The standard parabolic theory of PDEs insures that the problem has a unique weak solution. Therefore we can define the application

$$F: \quad W(0,T) \to W(0,T)$$
$$w \mapsto (u,v)$$

where

$$\mathcal{W}(0,T) = \{u,v \in L^2(0,T;H^1) \cap L^\infty(0,T;L^2)\ /$$
$$\partial_t u, \partial_t v \in L^2(0,T;(H^1)'), |w|_{L^\infty(0,T;L^2)} \leq K\}. \tag{10}$$

Now we extract some suitable estimates to construct the functional setting where Schauder fixed point theory is applicable. The following result holds for $0 \leq t \leq T$

$$\frac{1}{2}\int_{\Omega} u^2(t) + \int_{Q_T} \langle D\left(J_\rho\left(\nabla w_{1_\sigma}\right)\right)\nabla u, \nabla u\rangle = \frac{1}{2}\int_{\Omega} {u_{0,n}}^2 + \int_{Q_T} u\ f_n(t,x,w_1,w_2)$$
$$\frac{1}{2}\int_{\Omega} v^2(t) + d_v \int_{Q_T} |\nabla v|^2 = \frac{1}{2}\int_{\Omega} {v_{0,n}}^2 + \int_{Q_T} v\ g_n(t,x,w_1,w_2)\,. \tag{11}$$

Consequently,

$$\int_{\Omega} u^2(t) \leq M_f + \int_{Q_T} u^2 + \int_{\Omega} {u_{0,n}}^2$$
$$\int_{\Omega} v^2(t) \leq M_g + \int_{Q_T} v^2 + \int_{\Omega} {v_{0,n}}^2\,. \tag{12}$$

Using Gronwall's inequality we obtain

$$\int_{Q_T} u^2 \leq (\exp(T)-1)\left(M_f + \int_{\Omega} {u_{0,n}}^2\right)$$
$$\int_{Q_T} v^2 \leq (\exp(T)-1)\left(M_g + \int_{\Omega} {v_{0,n}}^2\right). \tag{13}$$

Substituting the expression above in (12), we obtain the desired result,

$$\sup_{0\leq t\leq T}\int_{\Omega} u^2(t) \leq M_f + (\exp(T)-1)\left(M_f + \int_{\Omega} {u_{0,n}}^2\right) + \int_{\Omega} {u_{0,n}}^2 := C_u$$
$$\sup_{0\leq t\leq T}\int_{\Omega} v^2(t) \leq M_g + (\exp(T)-1)\left(M_g + \int_{\Omega} {v_{0,n}}^2\right) + \int_{\Omega} {v_{0,n}}^2 := C_v\ . \tag{14}$$

Therefore by setting $C_1 = \max(C_u, C_v)$ we get

$$||(u,v)||_{L^\infty(0,T;L^2)^2} \leq C_1. \tag{15}$$

The last inequalities ensure that the norm $||(u,v)||_{L^\infty(0,T;L^2)}$ is bounded independently of w. Using the estimation (12) and (4) we deduce

$$\int_{Q_T} u^2 + |\nabla u|^2 \leq \frac{M_f + \int_{Q_T} u^2 + \int_{\Omega} {u_{0,n}}^2}{\min(\frac{1}{2}, \mu(C_1))} \leq C'_u$$
$$\int_{Q_T} v^2 + |\nabla v|^2 \leq \frac{M_g + \int_{Q_T} v^2 + \int_{\Omega} {v_{0,n}}^2}{\min(\frac{1}{2}, d_v)} \leq C'_v\ . \tag{16}$$

Setting $C_2 = \max(C'_u, C'_v)$, we conclude that

$$||(u,v)||_{L^2(0,T;(H^1)')^2} \leq C_2. \tag{17}$$

Next we estimate the $\partial_t u$ and $\partial_t v$ in $L^2(0,T;(H^1)')$. We know that

$$\partial_t u = \operatorname{div}\left(D\left(J_\rho\left(\nabla w_{1_\sigma}\right)\right)\nabla u\right) + f_n(t,x,w_1,w_2)$$
$$\partial_t v = d_v\ \Delta v + g_n(t,x,w_1,w_2). \tag{18}$$

It follows that

$$||\partial_t u||_{L^2(0,T;(H^1)')} \leq C\ ||\nabla u||_{L^2(Q_T)} + M_f\ T$$
$$||\partial_t v||_{L^2(0,T;(H^1)')} \leq d_v||\nabla v||_{L^2(Q_T)} + M_g\ T\,. \tag{19}$$

Thereafter

$$||\partial_t u||_{L^2(0,T;(H^1)')} \leq C\ C_1 + M_f\ T$$
$$||\partial_t v||_{L^2(0,T;(H^1)')} \leq d_v C_1 + M_g\ T\,. \tag{20}$$

Eventually,

$$||(\partial_t u, \partial_t v)||_{L^2(0,T;(H^1)')^2} \leq \max(C\ C_1 + M_f\ T, d_v\ C_1 + M_g\ T) := C_3. \tag{21}$$

Now we are in position to apply Schauder fixed point in the functional space:

$$\mathcal{W}_0(0,T) = \{u, v \in L^2(0,T;H^1) \cap L^\infty(0,T;L^2)\ /\ ||(u,v)||_{L^\infty(0,T;L^2} \leq C_1$$
$$||(u,v)||_{L^2(0,T;H^1(\Omega))^2} \leq C_2 \quad ||(\partial_t u, \partial_t v)||_{L^2(0,T;H^1(\Omega)')^2} \leq C_3$$
$$u(\cdot,0) = u_{0,n} \quad v(\cdot,0) = v_{0,n}\}. \tag{22}$$

We can easily verify that $\mathcal{W}_0(0,T)$ is a nonempty closed convex in $\mathcal{W}(0,T)$. In order to use Schauder's theorem we will show that the application

$$F : w \in \mathcal{W}_0(0,T) \to F(w) = (u,v) \in \mathcal{W}_0(0,T)$$

is weakly continuous.

Let us consider a sequence $w_k \in \mathcal{W}_0(0,T)$ such that w_k converges weakly in $\mathcal{W}_0(0,T)$ toward w, and let $F(w_k) = (u_k, v_k)$. Thus,

$$\partial_t u_k = \operatorname{div}\left(D\left(J_\rho\left(\nabla w_{1_{k\,\sigma}}\right)\right)\nabla u_k\right) + f_n(t,x,w_{1_k},w_{2_k})$$
$$\partial_t v_k = d_v\ \Delta v_k + g_n(t,x,w_{1_k},w_{2_k}). \tag{23}$$

Based on the previous estimations, (u_k, v_k) is bounded in $\left(L^2(0,T;H^1(\Omega))\right)^2$ and $(\partial_t u_k, \partial_t v_k)$ is bounded in $\left(L^2(0,T;(H^1(\Omega))')\right)^2$ then by Aubin-Simon compactness [8] (u_k, v_k) is relatively compact on $\left(L^2(Q_T)\right)^2$; which means we can extract a subsequence denoted $w_k = (u_k, v_k)$ such that

- $u_k \rightharpoonup u$ in $L^2(0,T;H^1(\Omega))$,
- $v_k \rightharpoonup v$ in $L^2(0,T;H^1(\Omega))$,
- $f(t,x,w_k) \longrightarrow f(t,x,w)$ in $L^2(Q_T)$,
- $g(t,x,w_k) \longrightarrow g(t,x,w)$ in $L^2(Q_T)$,
- $u_k \longrightarrow u$ in $L^2(0,T;L^2(\Omega))$ and a.e in Q_T,
- $v_k \longrightarrow v$ in $L^2(0,T;L^2(\Omega))$ and a.e in Q_T,
- $\nabla u_k \rightharpoonup \nabla u$ in $L^2(0,T;L^2(\Omega))$,
- $\nabla v_k \rightharpoonup \nabla v$ in $L^2(0,T;L^2(\Omega))$,
- $w_k \longrightarrow w$ in $L^2(0,T;\mathcal{H})$ and a.e in Q_T,
- $J_\rho\left(\nabla w_{1_{k\,\sigma}}\right) \to J_\rho\left(\nabla w_{1\sigma}\right)$ in $L^2(Q_T)$,
- $\partial_t u_k \rightharpoonup \partial_t u$ in $L^2(0,T;H^1(\Omega)')$,
- $\partial_t v_k \rightharpoonup \partial_t v$ in $L^2(0,T;H^1(\Omega)')$.

Using these convergence, we can pass to the limit in (23) and show that the limit u and v are solutions of the following problem (18). We conclude that $F(w) = (u,v)$ therefore F is weakly continuous which proves the desired results. □

3. Existence result for integrable nonlinearities

In this section, we focus on proving the existence of solution to the original problem (1). To achieve our goal we will need the following properties of the approximate problem (7).

Lemma 3.1. [15] *Let (u_n, v_n) be the solution of the approximate problem (7). Then,*
(1) *There exists a constant M depending only on $\int_\Omega u_0, \int_\Omega v_0, L_1, T$ and $|\Omega|$ such that*

$$\int_{Q_T} (u_n + v_n) \leq M \qquad \forall t \in [0, T] \tag{24}$$

(2) *There exists $C_2 > 0$ such that*

$$\int_{Q_T} |f_n| + |g_n| \leq C_2 \tag{25}$$

Proof. (1) The triangular structure of problem (1) implies that

$$(u_n + v_n)_t - \operatorname{div}\left(D\left(J_\rho\left(\nabla u_{n\sigma}\right)\right)\nabla u_n\right) - d_v\ \Delta v_n \leq L_1(u_n + v_n + 1), \tag{26}$$

integrating over Q_t, $0 < t \leq T$ leads to

$$\int_\Omega (u_n + v_n)(t) \leq \int_\Omega (u_0 + v_0) + L_1 \int_{Q_t} (u_n + v_n + 1), \tag{27}$$

using a standard Gronwall argument we get

$$\int_{Q_T} (u_n + v_n)(t) \leq \left[\int_\Omega (u_0 + v_0) + L_1 |Q_T|\right] \exp\left(L_1\ T\right), \tag{28}$$

and therefore the desired result is proven.

(2) For v_n solution of

$$\partial_t v_n - d_v\ \Delta v_n = g_n \leq L_2(1 + u_n + v_n), \tag{29}$$

we can write

$$\partial_t v_n - d\Delta v_n + L_2(1 + u_n + v_n) - g_n = L_2(1 + u_n + v_n), \tag{30}$$

which implies

$$\int_{Q_T} \partial_t v_n + \int_{Q_T} (L_2(1 + u_n + v_n) - g_n) \leq \int_{Q_T} L_2(1 + u_n + v_n), \tag{31}$$

then

$$\int_\Omega v_n(T) - \int_\Omega v_n(0) + \int_{Q_T} (L_2(1 + u_n + v_n) - g_n) \leq \int_{Q_T} L_2(1 + u_n + v_n), \tag{32}$$

we know that $\int_{Q_T} L_2(1 + u_n + v_n)$ is bounded, which follows that

$$\| L_2(1 + u_n + v_n) - g_n \|_{L^1(Q_T)} \leq C, \tag{33}$$

therefore

$$\| g_n \|_{L^1(Q_T)} \leq C_g. \tag{34}$$

Since $L_1(1 + u_n + v_n) - f_n - g_n \geq 0$, we obtain the same for $f_n + g_n$, hence

$$\| f_n \|_{L^1(Q_T)} \leq C_f. \tag{35}$$

□

Now we deduce the following compactness result.

Theorem 3.2. *The sequence (u_n, v_n) given by the solution of the approximate problem (7) is relatively compact in $L^1(Q_T)$.*

Proof. We will employ a duality argument in order to prove that the sequence u_n is bounded in $L^1\left(0,T;W^{1,1}(\Omega)\right)$ and $\partial_t u_n$ is bounded in $L^1\left(0,T;(W^{1,1}(\Omega))'+L^1(\Omega)\right)$. Let $h\in\mathcal{C}_0^\infty(Q_T)$ and $\phi\in L^2(0,T;W_0^{1,2}(\Omega))\cap W^{2,1}(Q_T)$ a solution of the dual problem:

$$-\partial_t\phi-\operatorname{div}\left(D\left(J_\rho\left(\nabla u_{n\sigma}\right)\right)\nabla\phi\right)=\operatorname{div}(h)$$
$$\phi(T)=0. \tag{36}$$

It is well known according to [14] that for all $\bar{s},\bar{q}>1$ satisfying $\frac{2}{\bar{s}}+\frac{N}{\bar{q}}<1$:

$$||\phi||_{L^\infty(Q_T)}\le C||h||_{L^{\bar{s}}(0,T;L^{\bar{q}}(\Omega))}. \tag{37}$$

C is independent of n since the term $D\left(J_\rho\left(\nabla u_{n\sigma}\right)\right)$ is uniformly bounded in L^∞ (thanks to the convolution with G_σ). The previous estimate implies that

$$\begin{aligned}|\int_{Q_T}u_n\operatorname{div}(h)|&\le C(||u_{0,n}||_{L^1(\Omega)},||f_n||_{L^1(\Omega)})\ ||\phi||_{L^\infty(Q_T)}\\&\le C(||u_{0,n}||_{L^1(\Omega)},||f_n||_{L^1(\Omega)})\ ||h||_{L^{\bar{s}}(0,T;L^{\bar{q}}(\Omega))},\end{aligned}$$

and by integration by parts and the definition of the dual norm we get :

$$||\nabla u_n||_{L^s(0,T;L^q(\Omega))}\le C(||u_{0,n}||_{L^1(\Omega)},||f_n||_{L^1(\Omega)}), \tag{38}$$

such that $\frac{2}{s}+\frac{N}{q}>N+1$.

Since $||u_{0,n}||_{L^1(\Omega)}\le M_0$ and f_n is uniformly bounded in $L^1(Q_T)$ from lemma 3.1, we finally reach the conclusion that u_n is bounded in $L^1\left(0,T;W^{1,1}(\Omega)\right)$:

$$||u_n||_{L^1(0,T;W^{1,1}(\Omega))}\le C(M_0,C_2). \tag{39}$$

It is also clear that $D\left(J_\rho\left(\nabla u_{n\sigma}\right)\right)\nabla u_n$ is bounded in $L^1(Q_T)$ coupled with the fact that $D\left(J_\rho\left(\nabla u_{n\sigma}\right)\right)$ is uniformly bounded in $L^\infty(Q_T)$ leads to the estimate:

$$||\partial_t u_n||_{L^1(0,T;(W^{1,1}(\Omega))'+L^1(\Omega))}\le C.$$

We deduce the desired result from Aubin-Simon compactness theorem which concludes the proof. □

Using the compactness result in Theorem (3.2) and the well known compactness of the standard heat equation operator, we can extract a sub-sequence also denoted (u_n,v_n) such that

- $u_n\rightharpoonup u$ in $L^1(0,T;W^{1,1}(\Omega))$,
- $v_n\rightharpoonup v$ in $L^1(0,T;W^{1,1}(\Omega))$,
- $f(t,x,u_n,v_n)\longrightarrow f(t,x,u,v)$ a.e. in Q_T,
- $g(t,x,u_n,v_n)\longrightarrow g(t,x,u,v)$ a.e. in Q_T,
- $u_n\longrightarrow u$ in $L^1(Q_T)$ and a.e in Q_T,
- $v_n\longrightarrow v$ in $L^1(Q_T)$ and a.e in Q_T,
- $\nabla u_n\rightharpoonup\nabla u$ in $L^1(0,T;L^1(\Omega))$,
- $\nabla v_n\rightharpoonup\nabla v$ in $L^1(0,T;L^1(\Omega))$,
- $D(J_\rho\left(\nabla u_{1_n\sigma}\right))\nabla u_n\to D(J_\rho\left(\nabla u_{1\sigma}\right))\nabla u$ in $L^1(Q_T)$,
- $\partial_t u_n\rightharpoonup\partial_t u$ in $L^1(0,T;W^{1,1}(\Omega)')$,
- $\partial_t v_n\rightharpoonup\partial_t v$ in $L^2(0,T;W^{1,1}(\Omega)')$.

The above convergence are enough to pass to the limit in the left hand side of the equation (7). The problem lies in the right hand side. For this almost everywhere convergence is not sufficient. We actually need to prove that $f_n(t,x,u_n,v_n)$ converges strongly toward $f(t,x,u,v)$ in $L^1(Q_T)$ and this convergence is given by the following Lemma as established in [15].

Lemma 3.3. *Let x_n be a sequence in $L^1(Q_T)$. Then the following statements are equivalent:*

(1) *x_n is equi-integrable in $L^1(Q_T)$*

(2) *There exists $J:(0,\infty)\longrightarrow(0,\infty)$ such that $J(0^+)=0$ and*

$$\begin{aligned}&(a)\quad J \text{ is convex, } J' \text{ is concave, } J'\geq 0\\ &(b)\lim_{r\to+\infty}\frac{J(r)}{r}=+\infty\\ &(c)\quad \sup_n \textstyle\int_{Q_T} J(|x_n|)<\infty\end{aligned}$$

We choose a convex function as in Lemma (3.3) such that

$$\sup_n \int_{Q_T} J\left(L_1(1+u_n+v_n)\right)<\infty, \qquad \sup_n \int_{Q_T} J\left(L_1(1+u_{0,n}+v_{0,n})\right)<\infty, \quad (40)$$

and we set

$$j(r)=\int_0^r \min\Big(J'(s),\left(J^*\right)^{-1}(s)\Big)\,ds,$$

where J^* is the conjugate of J, which satisfies:

$$\forall r\geq 0,\quad j(r)\leq J(r),\quad J^*\left(j'(r)\right)\leq r.$$

Let $R_n=L_1(1+u_n+v_n)-f_n-g_n\geq 0$ and $S_n=L_2(u_n+v_n+1)-g_n\geq 0$.
We have

$$\partial_t(u_n+v_n)-B_n+R_n+S_n=L_1(u_n+v_n+1)+L_2(u_n+v_n+1), \qquad (41)$$

where $B_n=d\Delta v_n+\operatorname{div}(D((J_\rho(\nabla(u_n)_\sigma))\,\nabla u_n)$.

Let $M_n=u_n+v_n$ multiplying (41) to $j'(M_n)$ and integrating over Q_T, we obtain

$$\int_\Omega j(M_n)+\int_{Q_T} j'(M_n)\left(R_n+S_n\right)=\int_\Omega j(M_{0,n})+\int_{Q_T} j'(M_n)B_n+$$
$$\int_{Q_T} j'(M_n)\left((L_1+L_2)(M_n+1)\right), \qquad (42)$$

using the Fenchel inequality $j'(r)\cdot s\leq J(s)+J^*(j'(r))\leq J(s)+r$ and the properties (40), we can prove that

$$\int_{Q_T} j'(M_n)\left((L_1+L_2)(M_n+1)\right)\leq\int_{Q_T} J((L_1+L_2)(M_n+1))+M_n\leq C$$
$$\int_\Omega j(M_{0,n})\leq\int_{Q_T} J(M_{0,n})\leq C,$$

also from (39) we can deduce that

$$\int_{Q_T} j'(M_n)B_n\leq C.$$

This enables us to obtain the

$$\int_{Q_T} j'(M_n)\left(R_n+S_n\right)\leq C. \qquad (43)$$

Theorem 3.4. *The sequences f_n and g_n are equi-integrable in $L^1(Q_T)$.*

Proof. We know that f_n, g_n converge almost everywhere toward f, g. We will show that f_n and g_n are equi-integrable in $L^1(Q_T)$. The proof will be given for f_n, however the same result holds for g_n. For this, we let $\varepsilon > 0$ and prove that there exists $\delta > 0$ such that $|E| < \delta$ implies that $\int_E f_n < \varepsilon$. We have,

$$\int_E |f_n| = \int_{E\cap[M_n\leq k]} |f_n| + \int_{E\cap[M_n>k]} |f_n|$$
$$\leq \frac{1}{j'(k)} \int_E j'(M_n)|f_n| + |E| \sup_{|u_n|+|v_n|\leq k} |f_n(t,x,u_n,v_n)|.$$

Since equation (43) insures that $\int_E j'(M_n)|f_n|$ is bounded. We can choose δ small enough and a larger k such that $\int_E |f_n| \leq \varepsilon$. The same thing holds for g_n as well. □

References

[1] P. Perona, T. Shiota, J. Malik, Anisotropic diffusion, *Geometry-driven diffusion in computer vision*, Springer (1994), 73–92.

[2] N. Alaa, M. Aitoussous, W. Bouarifi, D. Bensikaddour, Image restoration using a reaction-diffusion process *Electronic Journal of Differential Equations* **2014** (2014), no. 197, 1–12.

[3] M. Pierre, Global existence in reaction-diffusion systems with control of mass: a survey, *Milan Journal of Mathematics* **78** (2010), no. 2, 417–455.

[4] D. Schmitt, Existence globale ou explosion pour les systemes de réaction-diffusion avec contrôle de masse, Ph.D. thesis, Nancy 1, (1995).

[5] J.L. Lions, *Equations differentielles operationnelles: et problémes aux limites*, Springer-Verlag, 111, (2013).

[6] H. Amann, Dynamic theory of quasilinear parabolic equations. II. Reaction-diffusion systems, *Differential Integral Equations* **3** (1990), no. 1, 13–75.

[7] P. Benilan, H. Brezis, Solutions faibles d'équations d'évolution dans les espaces de Hilbert, *Ann. Inst. Fourier (Grenoble)* **22** (1972), no. 2, 311–329.

[8] J. Simon, Compact sets in the space $L^p(O,T;B)$, *Annali di Matematica pura ed applicata* **146** (1986), no. 1, 65–96.

[9] N.E. Alaa, M. Zirhem, Bio-Inspired Reaction Diffusion System Applied to Image Restoration, *International Journal of Bio-Inspired Computation (IJBIC)* **12** (2018), no. 2.

[10] M. Ebihara, H. Mahara, T. Sakurai, A. Nomura, H. Miike, Image processing by a discrete reaction-diffusion system, *Proc. Visualization, Imaging, Image Process.* **396** (2003), 145–150.

[11] X. Li, T. Chen, Nonlinear diffusion with multiple edginess thresholds, *Pattern Recognition* **27** (1994), no. 8, 1029–1037.

[12] J. Joachim, B. Benhamouda, A semidiscrete nonlinear scale-space theory and its relation to the Perona–Malik paradox, *Advances in computer vision*, Springer (1997), 1–10.

[13] S. Morfu, Traitement d'images inspiré des processus de réaction-diffusion, *XXIIe colloque GRETSI (traitement du signal et des images)*, Dijon (FRA), 8-11 septembre 2009, GRETSI, Groupe d'Etudes du Traitement du Signal et des Images, (2009).

[14] O.A. Ladyzhenskaya, V.A. Solonnikov, N.N. Ural'tseva, Linear and Quasilinear Equations of Parabolic Type, Moscow, *Google Scholar* (1967).

[15] C.H. Le, Etude de la classe des opérateurs m-accrétifs dans $L^1(\Omega)$ et accretifs dans $L^\infty(\Omega)$, *These de 3^o cycle*, Paris VI, (1977).

[16] J. Weickert Anisotropic diffusion in image processing, *Teubner Stuttgart* **1** (1998).

(H. Fahim, H. Khalfi, N. Alaa) Faculty of Science and Technology, Laboratory LAMAI, Marrakesh, Morocco

E-mail address: hamza.khalfi@edu.uca.ma, houda.fahim@edu.uca.ma, n.alaa@uca.ac.ma