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On a class of quasilinear problems with double-phase reaction and indefinite weight

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Abstract. We study a nonlinear eigenvalue problem which is perturbed by a term with double-phase growth. The main result establishes the existence of at least two nontrivial weak solutions in the case of high perturbations of the parameter. The proof combines variational tools with the Pucci-Serrin three critical points theorem.

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1. Introduction and the main result

In a remarkable paper, Szulkin and Willem [17] studied linear and nonlinear eigenvalue problems in the case of indefinite weights. In the quasilinear setting, they were concerned with the following boundary value problem

$$\begin{cases} -\Delta_p u = \lambda V(x)|u|^{p-2}u & \text{in } \Omega \\ u \in \mathcal{D}_0^{1,p}(\Omega) \setminus \{0\} & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded open set with smooth boundary and $1 < p < N$ is a real number.

Recall that Δ_p is the p-Laplace differential operator, that is,

$$\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u).$$

The main feature of the paper by Szulkin and Willem is that they assume that the weight function V may change sign in Ω. The main hypothesis on the indefinite potential V is the following:

(V) : $V \in L^1_{loc}(\Omega)$, $V^+ = V_1 + V_2 \neq 0$, $V_1 \in L^{N/p}(\Omega)$, and $\lim_{x\to y;\, x\in\Omega} |x-y|^p V_2(x) = 0$ for every $y \in \overline{\Omega}$.

As usual, in the above hypothesis we assume that $V = V^+ - V^-$, where $V^\pm(x) = \max\{\pm V(x), 0\}$.

Under hypothesis (V), Szulkin and Willem [17] proved the existence of a nondecreasing sequence of eigenvalues $\{\lambda_n\}_{n\geq 1}$ with associated eigenfunctions $\{e_n\}_{n\geq 1}$ such that

$$\lambda_n = \int_\Omega |\nabla e_n|^p dx \to +\infty \quad \text{as } n \to \infty.$$

Moreover, the first eigenfunction e_1 is nonnegative in Ω and e_1 is a solution of the minimization problem

$$\lambda_1 = \min\left\{\int_\Omega |\nabla u|^p dx;\ u \in \mathcal{D}_0^{1,p}(\Omega),\ \int_\Omega V(x)|u|^p dx = 1\right\}.$$

The main purpose of the present paper is to study problem (1) under the effect of a perturbation with unbalanced growth at zero and infinity. More precisely, we are concerned with the following quasilinear elliptic problem.

$$\begin{cases} -\Delta_p u = V(x)|u|^{p-2}u + \lambda f(u) & \text{in } \Omega \\ u \in \mathcal{D}_0^{1,p}(\Omega) \setminus \{0\} & \text{on } \partial\Omega, \end{cases} \tag{2}$$

We assume that λ is a real parameter and the reaction f satisfies the following conditions:

$(f1)$: f is continuous and there exists $\delta > 0$ such that f is nonnegative and nontrivial on $[0, \delta]$;

$(f2)$: there exist real numbers p_1 and p_2 such that $0 < p_1 < p-1 < p_2$ such that

$$\sup_{t\in\mathbb{R}} \frac{|f(t)|}{1+|t|^{p_1}} < \infty \quad \text{and} \quad \sup_{t\in\mathbb{R}} \frac{|f(t)|}{|t|^{p_2}} < \infty.$$

Roughly speaking, hypothesis $(f2)$ implies that f should have a $(p-1)$-superlinear growth near the origin and a $(p-1)$-sublinear growth at infinity. A function satisfying hypotheses $(f1)$ and $(f2)$ is

$$f(t) = \begin{cases} |t|^{p_1-1}t & \text{if } |t| \geq 1 \\ |t|^{p_2-1}t & \text{if } |t| < 1. \end{cases}$$

We are looking for nontrivial weak solutions of problem (2), that is, functions $u \in \mathcal{D}_0^{1,p}(\Omega) \setminus \{0\}$ such that for all $v \in \mathcal{D}_0^{1,p}(\Omega)$

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla v dx = \int_\Omega V(x)|u|^{p-2}uv dx + \lambda \int_\Omega f(u)v dx.$$

In such a case, we say that $\lambda \in \mathbb{R}$ is an eigenvalue of problem (2) and the corresponding $u \in \mathcal{D}_0^{1,p}(\Omega) \setminus \{0\}$ is an eigenfunction associated to this eigenvalue. These notions are in accordance with the related ones introduced by Fučik, Nečas, Souček and Souček [7, p. 117] in the context of nonlinear operators. Indeed, if we denote

$$S(u) := \frac{1}{p}\int_\Omega |\nabla u|^p dx - \frac{1}{p}\int_\Omega V(x)|u|^p dx \text{ and } T(u) := \int_\Omega F(u)dx,$$

where $F(t) = \int_0^t f(s)ds$, then λ is an eigenvalue for the pair (S, T) of nonlinear operators (in the sense introduced in [7]) if and only if there is a corresponding eigenfunction $u \in \mathcal{D}_0^{1,p}(\Omega) \setminus \{0\}$, which is a solution of problem (2) as described previously.

The main result of this paper establishes the existence of a continuous spectrum for the perturbed nonlinear eigenvalue problem (2). The semilinear version of our main result was established by Rădulescu [15]. We prove the following multiplicity property.

Theorem 1.1. *Assume that hypotheses (V), $(f1)$ and $(f2)$ are fulfilled. Let λ_1 be the principal eigenvalue of problem (1) and suppose that $\lambda_1 > 1$. Then there is a real number Λ such that problem (2) has at least two solutions for all $\lambda > \Lambda$.*

2. Proof of Theorem 1.1

The energy functional associated to problem (2) is $E : \mathcal{D}_0^{1,p}(\Omega) \to \mathbb{R}$ defined by
$$E(u) := A(u) + \lambda B(u),$$
where
$$A(u) := \frac{1}{p}\int_\Omega |\nabla u|^p dx - \frac{1}{p}\int_\Omega V(x)|u|^p dx \quad \text{for all } u \in \mathcal{D}_0^{1,p}(\Omega)$$
and
$$B(u) := -\int_\Omega F(u)dx \quad \text{for all } u \in \mathcal{D}_0^{1,p}(\Omega).$$

We first prove that E is well defined. Indeed, by $(f2)$, there is a positive constant C such that for all $t \in \mathbb{R}$
$$|F(t)| \le C \max\{|t| + |t|^{p_1+1}, |t|^{p_2+1}\}. \tag{3}$$
So, by the Sobolev embedding theorem (see Brezis [2, Corollary IX.14]), the functional $E : \mathcal{D}_0^{1,p}(\Omega) \to \mathbb{R}$ is well-defined.

Next, we prove that the nonlinear operator $\mathcal{D}_0^{1,p}(\Omega) \ni u \mapsto F(u)$ is not constant. Let $e_1 \ge 0$ be an eigenfunction associated to the principal eigenvalue λ_1 and let δ be the positive number introduced in hypothesis $(f1)$. Consider the open set
$$\Omega_\delta = [e_1 > \delta] := \{x \in \Omega;\ e_1(x) > \delta\}.$$
Replacing eventually e_1 by ρe_1 for $\rho > 0$ big enough, it follows that $\Omega_\delta \neq \emptyset$.

Define the function $g : \mathbb{R} \to \mathbb{R}$ by $g(t) = \min\{t, \delta\}$ and consider the function $w : \Omega \to \mathbb{R}$ defined by $w = g \circ e_1$. Thus, by [2, Proposition IX.5], we have $w \in \mathcal{D}_0^{1,p}(\Omega)$.

We also observe that
$$w(x) = \delta \quad \text{for all } x \in \Omega_\delta$$
and
$$0 \le w(x) \le \delta \quad \text{for all } x \in \Omega.$$

By hypothesis $(f1)$ we conclude that
$$F(w(x)) \ge 0 \quad \text{for all } x \in \Omega$$
and
$$F(w(x)) > 0 \quad \text{for all } x \in \Omega_\delta,$$
which shows that the operator $u \mapsto F(u)$ is not constant on $\mathcal{D}_0^{1,p}(\Omega)$.

We now prove that E is coercive. Recall that $E = A + \lambda B$. Since $\lambda_1 > 1$, there exists $\eta > 0$ such that for all $u \in \mathcal{D}_0^{1,p}(\Omega)$
$$A(u) \ge \eta \int_\Omega |\nabla u|^p dx.$$
By the Poincaré inequality, we can assume that $\mathcal{D}_0^{1,p}(\Omega)$ is endowed with the norm
$$\|u\| = \left(\int_\Omega |\nabla u|^p dx\right)^{1/p}.$$
It follows that for all $u \in \mathcal{D}_0^{1,p}(\Omega)$
$$A(u) \ge \eta \|u\|^p.$$

By (3) combined with hypothesis $(f2)$ we deduce that for all $u \in \mathcal{D}_0^{1,p}(\Omega)$ such that $\|u\|$ is large enough, we have

$$B(u) \geq -C\,(\|u\| + \|u\|^{p_1+1}).$$

So, for all $u \in \mathcal{D}_0^{1,p}(\Omega)$ such that $\|u\|$ is large enough,

$$E(u) \geq \eta\|u\|^p - C\,(\|u\| + \|u\|^{p_1+1}).$$

Thus, by $(f2)$, we deduce that

$$E(u) \to +\infty \quad \text{as } \|u\| \to \infty.$$

By Lemma V.4 of Brezis and Nirenberg [3], we obtain that E satisfies the Palais-Smale compactness condition. Thus, any sequence (u_n) in $\mathcal{D}_0^{1,p}(\Omega)$ such that $E(u_n)$ has a limit and $E'(u_n) \to 0$, contains a convergent subsequence.

To conclude the proof, we apply the three critical points theorem of Pucci and Serrin, see [14, Corollary 1]. This property says that if E is real-valued C^1-functional defined on a real Banach space having two local minima and satisfying the Palais-Smale condition, then E has at least three critical points.

In virtue of the Pucci-Serrin theorem, it remains to show that $E : \mathcal{D}_0^{1,p}(\Omega) \to \mathbb{R}$ has at least two local minima for big values of λ. For this purpose, we define the functionals $\mathcal{F}_1,\, \mathcal{F}_2 : \mathbb{R} \to \mathbb{R}$ by

$$\mathcal{F}_1(s) = \inf\left\{\frac{\inf_{B(v)=s} A(v) - A(u)}{B(u) - s};\ B(u) < s\right\}$$

and

$$\mathcal{F}_2(s) = \sup\left\{\frac{\inf_{B(v)=s} A(v) - A(u)}{B(u) - s};\ B(u) > s\right\}.$$

We observe that

$$\limsup_{s \nearrow 0} \mathcal{F}_1(s) \leq \mathcal{F}_1(0) \tag{4}$$

and

$$\liminf_{s \nearrow 0} \mathcal{F}_2(s) = +\infty. \tag{5}$$

Set $\Lambda := \mathcal{F}_1(0)$. Relations (4) and (5) combined with the Pucci-Serrin theorem imply that E has at least two distinct nontrivial critical points for all $\lambda > \Lambda$. The proof is now complete. □

Further comments. (i) Nonlinear problems with unbalanced growth have been studied studied starting with the pioneering papers by Marcellini [8, 9]. Important contributions are due to Mingione *et al.* [1, 5, 6] and Rădulescu *et al.* [4, 10, 16]. In all these papers, the double-phase problems are driven by *differential operators* with unbalanced growth, while in our case the setting is different and it corresponds to a *reaction* with unbalanced decay.

(ii) Analyzing the proof of Theorem 1.1 we observe that the same arguments can be used for all domains of $\mathbb{R}^N$ (even unbounded) such that the Poincaré inequality holds, for instance domains which are bounded in one direction (strips, domains with finite measure, etc.).

(iii) The three critical points theorem of Pucci and Serrin and the Palais-Smale compactness condition should be regarded in relationship with the mountain pass theorem.

We refer to [12, 13] for related properties involving the mountain pass theorem and applications to PDEs.
(iv) The main abstract tools used in this paper and various extensions can be found in the recent monograph [11].

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