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Nonholonomic frames for Finsler space with infinite series of (α, β) metric

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Abstract. The purpose of present paper to determine the Finsler spaces due to deformation of special Finsler (α, β)-metric. Consequently, we obtained the non-holonomic frame with the help of Riemannian metric $\alpha^2 = a_{ij}(x)y^i y^j$, one form metric $\beta = b_i(x)y^i$ and infinite series of (α, β) metric such as the forms

I. $(\frac{\beta^2}{\beta-\alpha})\alpha = \frac{\alpha\beta^2}{\beta-\alpha}$ i.e. product of the infinite series of (α, β)-metric and Riemannian metric

II. $(\frac{\beta^2}{\beta-\alpha})\beta = \frac{\beta^3}{\beta-\alpha}$ i.e. product of the infinite series of (α, β)-metric and 1-form metric.

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1. Introduction

In 1982 Holland [6] studied about the nonholonomic frame on space time which by considering a charged particle moving in an external electromagnetic field. Again in 1987 Ingarden [7] was the first person who point out that the Lorentz force law can be written in above case as geodesic equation on a Finsler space which was known as Randers space. Further in 1993 and 1995 Beil [2, 3] in his two consecutive papers have studied a gauge transformation by considering as a nonholonomic frame on the tangent bundle of a four dimensional base manifold. In the continuation of the above work so many result are obtained in the nonholonomic frames by various authors [4, 11, 12, 13].

In 2004 Lee and Park [8] introduced a r-th series (α, β)-metric

$$L(\alpha, \beta) = \beta \sum_{k=0}^{r} (\frac{\alpha}{\beta})^k, \tag{1}$$

where they assume $\alpha < \beta$. If $r = 1$ then $L = \alpha + \beta$ is a Randers metric. If $r = 2$ then $L = \alpha + \beta + \frac{\alpha^2}{\beta}$ is a combination of Randers metric and Kropina metric. If $r = \infty$ then above metric is expressed as

$$L(\alpha, \beta) = \frac{\beta^2}{\beta - \alpha} \tag{2}$$

and the metric (2) named as infinite series (α, β)-metric. This metric was very remarkable because it represent the difference between the Randers and Matsumoto metric.

In the present paper we have used the common Finsler idea to study the existence of a nonholonomic frame on the vertical sub bundle VTM of the tangent bundle of a base manifold M. In this case we have considered the fundamental tensor field might be the deformation of two different special Finsler spaces from the combination of (α,β)- metrics which form the special case of generlaized Lagrange space. In first case we consider a nonholonomic frame for a Finsler space with (α,β)- metrics such as first product of the infinite series of (α,β)-metric and Riemannian metric, and in second case it is the product of the infinite series of (α,β)-metric and 1-form metric. Both the above deformations are the generalized case of infinite series metric by using Riemannian and one-form metric respectively. Further we obtain a corresponding frame for each of these two Finsler deformations.

2. Preliminaries

Finsler spaces F^n equipped with $(\alpha,\ \beta)$-metrics are defined as [9].

Definition 2.1. A Finsler space $F^n = \{M, F(x,y)\}$ is called with (α,β)-metric if there exists a 2-homogeneous function L of two variables such that the Finsler metric $F: TM \to R$ is given by

$$F^2(x,y) = L\{\alpha(x,y), \beta(x,y)\} \tag{3}$$

where $\alpha^2(x,y) = a_{ij}(x)y^iy^j, \alpha$ is a Riemannian metric on the manifold M, and $\beta(x,y) = b_i(x)y^i$ is a 1-form on M.

In 1993 Beil suggested a more general case by considering, $a_{ij}(x)$ as the components of a Riemannian metric on the base manifold M, $a\,(x,y) > 0$ and $b\,(x,y) \geq 0$ two functions on TM, and $B\,(x,y) = B_i\,(x,y)\,(dx^i)$ a vertical 1-form on TM. Then

$$g_{ij}(x,y) = a(x,y)a_{ij}(x) + b(x,y)B_i(x,y)B_j(x,y)$$

Nowadays the above generalized Lagrange metric is known as the Beil metric. The metric tensor g_{ij} is also known as a Beil deformation of the Riemannian metric a_{ij}. It has been studied and applied in General Relativity for $a(x,y) = \exp\{2\sigma(x,y)\}$ and b = 0. The case $a(x,y) = 1$ with various choices of b and B_i was introduced and studied by Beil for constructing a new unified field theory [3]. Further in 2002 Bucataru [5] considered the class of Lagrange spaces with $(\alpha,\ \beta)$-metric and obtained some new and interesting results.

A unified formalism which uses a nonholonomic frame on space time, a sort of plastic deformation, arising from consideration of a charged particle moving in an external electromagnetic field in the background space time viewed as a strained mechanism studied by Holland. If we do not ask for the function L to be homogeneous of order two with respect to the $(\alpha,\ \beta)$ variables, then we have a Lagrange space with $(\alpha,\ \beta)$-metric. Next we defined some different Finsler space with (α,β)-metrics.

Further consider $g_{ij} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}$ the fundamental tensor of the Randers space (M,F). Taking into account the homogeneity of a and F we have the following formulae:

$$p^i = \frac{1}{a}y^i = a^{ij}\frac{\partial \alpha}{\partial y^j}; \quad p_i = a_{ij}p^j = \frac{\partial \alpha}{\partial y^i};$$

$$l^i = \frac{1}{L}y^i = g^{ij}\frac{\partial l}{\partial y^i}; l_i = g_{ij}l^j = \frac{\partial L}{\partial y^i} = P_i + b_i \tag{4}$$

$$l^i = \frac{1}{L}p^i; l^i l_i = p^i p_i = 1; l^i p_i = \frac{\alpha}{L}; p^i l_i = \frac{L}{\alpha};$$

$$b_i P^i = \frac{\beta}{\alpha}; b_i l^i = \frac{\beta}{L}$$

with respect to these notations, the metric tensors a_{ij} and g_{ij} are related by

$$g_{ij}(x,y) = \frac{L}{\alpha}a_{ij} + b_i P_j + P_i b_j - \frac{\beta}{\alpha}p_i p_j = \frac{L}{\alpha}(a_{ij} - p_i p_j) + l_i l_j \tag{5}$$

Theorem 2.1. [4] *For a Finsler space (M,F) consider the metric with the entries:*

$$Y^i_j = \sqrt{\frac{\alpha}{L}}(\delta^i_j - l^i l_j + \sqrt{\frac{\alpha}{L}}p^i p_j) \tag{6}$$

defined on TM. Then $Y_j = Y^i_j(\frac{\partial}{\partial y^i}), \quad j \in 1, 2, 3, \ldots, n$ *is a non holonomic frame.*

Theorem 2.2. [10] *With respect to frame the holonomic components of the Finsler metric tensor* $a_{\alpha\beta}$ *is the Randers metric* g_{ij}*, i.e,*

$$g_{ij} = Y^\alpha_i Y^\beta_j a_{\alpha\beta}. \tag{7}$$

Throughout this section we shall rise and lower indices only with the Riemannian metric $a_{ij}(x)$ that is $y_i = a_{ij}y^j, \beta^i = a^{ij}b_j$, and so on. For a Finsler space with (α,β)–metric $F^2(x,y) = L\{\alpha(x,y), \beta(x,y)\}$ we have the Finsler invariants.

$$\rho = \frac{1}{2\alpha}\frac{\partial L}{\partial \alpha}; \rho_0 = \frac{1}{2}\frac{\partial^2 L}{\partial \beta^2}; \rho_{-1} = \frac{1}{2\alpha}\frac{\partial^2 L}{\partial \alpha \partial \beta}; \rho_{-2} = \frac{1}{2\alpha^2}(\frac{\partial^2 L}{\partial \alpha^2} - \frac{1}{\alpha}\frac{\partial L}{\partial \alpha}) \tag{8}$$

where subscripts 1, 0, -1, -2 gives us the degree of homogeneity of these invariants. For a Finsler space with (α, β)-metric we have,

$$\rho_{-1}\beta + \rho_{-2}\alpha^2 = 0 \tag{9}$$

with respect to the notations we have that the metric tensor g_{ij} of a Finsler space with (α,β)–metric is given by

$$g_{ij}(x,y) = \rho a_{ij}(x) + \rho_0 b_i(x) + \rho_{-1}\{b_i(x)y_j + b_j(x)y_i\} + \rho_{-2}y_i y_j \tag{10}$$

From (10) we can see that $g_{ij}$is the result of two Finsler deformations:

$$I. \qquad a_{ij} \to h_{ij} = \rho a_{ij} + \frac{1}{\rho_{-2}}(\rho_{-1}b_i + \rho_{-2}y_i)(\rho_{-1}b_j + \rho_{-2}y_j)$$

$$II. \quad h_{ij} \to g_{ij} = h_{ij} + \frac{1}{\rho_{-2}}(\rho_0\rho_{-1} - \rho^2_{-1})b_i b_j \tag{11}$$

The nonholonomic Finsler frame that corresponding to the I^{st} deformation (11) is according to the theorem (7.9.1) in [4], given by,

$$X^i_j = \sqrt{\rho}\delta^i_j - \frac{1}{\beta^2}\{\sqrt{\rho} + \sqrt{\rho + \frac{\beta^2}{\rho_{-2}}}\}(\rho_{-1}b^i + \rho_{-2}y^i)(\rho_{-1}b_j + \rho_{-2}y_j) \tag{12}$$

where $B^2 = a_{ij}(\rho_{-1}b^i + \rho_{-2}y^i)(\rho_{-1}b_j + \rho_{-2}y_j) = \rho^2_{-1}b^2 + \beta\rho_{-1}\rho_{-2}$.

This metric tensor a_{ij} and h_{ij} are related by,

$$h_{ij} = X^k_i X^l_j a_{kl} \tag{13}$$

Again the frame that corresponds to the II_{nd} deformation (10) given by,

$$Y^i_j = \delta^i_j - \frac{1}{C^2}\{1 \pm \sqrt{1 + \left(\frac{\rho_{-2}C^2}{\rho_0\rho_{-2} - \rho^2_{-1}}\right)}\}b^i b_j \tag{14}$$

where $C^2 = h_{ij}b^i b^j = \rho b^2 + \frac{1}{\rho_{-2}}(\rho_{-1}b^2 + \rho_{-2}\beta)^2$.

The metric tensor h_{ij} and g_{ij} are related by the formula;

$$g_{mn} = Y^i_m Y^j_n h_{ij} \tag{15}$$

Theorem 2.3. [10] *Let $F^2(x, y) = L\{\alpha(x, y), \beta(x, y)\}$ be the metric function of a Finsler space with (α,β) metric for which the condition (9) is true. Then*

$$V^i_j = X^i_k Y^k_j$$

is a nonholonomic Finsler frame with X^i_k and Y^k_j are given by (12) and (14) respectively.

3. Nonholonomic frames for Finsler space with infinite series of (α, β)-metric

In this section we consider two cases of nonholonomic Finlser frames with special (α, β)-metrics, such a I^{st} Finsler frame product of the infinite series of (α, β)-metric and Riemannian metric and II^{nd} Finsler frame product of the infinite series of (α, β)-metric and 1-form metric.

3.1. Nonholonomic frame for $L = (\frac{\beta^2}{\beta-\alpha})\alpha = \frac{\alpha\beta^2}{\beta-\alpha}$. In the first case, for a Finsler space with the fundamental function $L = (\frac{\beta^2}{\beta-\alpha})\alpha = \frac{\alpha\beta^2}{\beta-\alpha}$ the Finsler invariants (8) are given by

$$\rho = \frac{\beta^3}{2\alpha(\beta-\alpha)^2}, \qquad \rho_0 = \frac{\alpha^3}{(\beta-\alpha)^3},$$
$$\rho_{-1} = \frac{\beta^3 - 3\alpha\beta^2}{2\alpha(\beta-\alpha)^3}, \quad \rho_{-2} = \frac{\beta^3(3\alpha-\beta)}{2\alpha^3(\beta-\alpha)^3},$$
$$B^2 = \frac{\beta^4(\beta-3\alpha)^2(\alpha^2b^2-\beta^2)}{4\alpha^4(\beta-\alpha)^6} \tag{16}$$

Using (16) in (12) we have,

$$X^i_j = \sqrt{\frac{\beta^3}{2\alpha(\beta-\alpha)^2}}\delta^i_j - \frac{\beta^2(\beta-3\alpha)^2}{4\alpha^2(\beta-\alpha)^6}[\sqrt{\frac{\beta^3}{2\alpha(\beta-\alpha)^2}} + \sqrt{\frac{\beta^3}{2\alpha(\beta-\alpha)^2} + \frac{2\alpha^3(\beta-\alpha)^3}{\beta(3\alpha-\beta)}}](b^i - \frac{\beta}{\alpha^2}y^i)(b_j - \frac{\beta}{\alpha^2}y_j) \tag{17}$$

Again using (16) in (14) we have,

$$Y^i_j = \delta^i_j - \frac{1}{C^2}\{1 \pm \sqrt{1 + \frac{2(\beta-\alpha)^3C^2}{2\alpha^3 - 3\alpha^2\beta + \alpha\beta^2}}\}b^i b_j \tag{18}$$

where $C^2 = \frac{\beta^3}{2\alpha(\beta-\alpha)^2}b^2 - \frac{\beta(\beta-3\alpha)}{2\alpha^3(\beta-\alpha)^3}(\alpha^2b^2 - \beta^2)^2$.

Theorem 3.1. *Let $L = (\frac{\beta^2}{\beta-\alpha})\alpha = \frac{\alpha\beta^2}{\beta-\alpha}$ be the metric function of a Finsler space with (α, β) metric for which the condition (9) is true. Then*

$$V_j^i = X_k^i Y_j^k$$

is nonholonomic Finsler Frame with X_k^i and Y_j^k are given by (17) and (18) respectively.

3.2. Nonholonomic frame for $L = (\frac{\beta^2}{\beta-\alpha})\beta = \frac{\beta^3}{\beta-\alpha}$. In the second case, for a Finsler space with the fundamental function $L = (\frac{\beta^2}{\beta-\alpha})\beta = \frac{\beta^3}{\beta-\alpha}$ the Finsler invariants (8) are given by

$$\rho = \frac{\beta^3}{2\alpha(\beta-\alpha)^2}, \quad \rho_0 = \frac{2\beta^3 - 6\alpha\beta^2 + 6\alpha^2\beta}{(\beta-\alpha)^3},$$
$$\rho_{-1} = \frac{\beta^2(\beta-3\alpha)}{2\alpha(\beta-\alpha)^3}, \quad \rho_{-2} = \frac{\beta^3(3\alpha-\beta)}{2\alpha^3(\beta-\alpha)^3},$$
$$B^2 = \frac{\beta^4(\beta-3\alpha)^2(\alpha^2 b^2 - \beta^2)}{4\alpha^4(\beta-\alpha)^6} \tag{19}$$

Using (19) in (12) we have,

$$X_j^i = \sqrt{\frac{\beta^3}{2\alpha(\beta-\alpha)^2}}\delta_j^i - \frac{\beta^2(\beta-3\alpha)^2}{4\alpha^2(\beta-\alpha)^6}[\sqrt{\frac{\beta^3}{2\alpha(\beta-\alpha)^2}} + \sqrt{\frac{\beta^3}{2\alpha(\beta-\alpha)^2} + \frac{2\alpha^3(\beta-\alpha)^3}{\beta(3\alpha-\beta)}}](b^i - \frac{\beta}{\alpha^2}y^i)(b_j - \frac{\beta}{\alpha^2}y_j) \tag{20}$$

Again using (19) in (14) we have,

$$Y_j^i = \delta_j^i - \frac{1}{C^2}\{1 \pm \sqrt{1 + \frac{2(\beta-\alpha)^3 C^2}{4\beta^3 + 9\alpha^2\beta - \alpha\beta^2}}\}b^i b_j \tag{21}$$

where $C^2 = \frac{\beta^3}{2\alpha(\beta-\alpha)^2}b^2 - \frac{\beta(\beta-3\alpha)}{2\alpha^3(\beta-\alpha)^3}(\alpha^2 b^2 - \beta^2)^2$.

Theorem 3.2. *Let $L = (\frac{\beta^2}{\beta-\alpha})\beta = \frac{\beta^3}{\beta-\alpha}$ be the metric function of a Finsler space with (α,β) metric for which the condition (9) is true. Then*

$$V_j^i = X_k^i Y_j^k$$

is nonholonomic Finsler Frame with X_k^i and Y_j^k are given by (20) and (21) respectively.

4. Conclusions

Non-holonomic frame relates a semi-Riemannian metric with an induced Finsler metric. Antonelli and Bucataru had been determined such a non-holonomic frame for two important classes of Finsler spaces that are dual in the sense of Randers and Kropina spaces. In the present paper, we have considered two different special Finsler spaces from the combination of (α, β)- metrics which form the special case of generlaized Lagrange space. In first case we consider a nonholonomic frame for a Finsler space with (α, β)- metrics such as first product of the infinite series of (α, β)-metric and Riemannian metric, and in second case it is the product of the infinite series of (α, β)-metric and 1-form metric. Both the above deformations are the generalized case of infinite series metric by using Riemannian and one-form metric respectively. Further we have obtained the nonholonomic Finsler frames in these two cases of the Finsler spaces with deformed infinite series metrics. But, in Finsler geometry, there are many (α, β) - metrics, in future work we can determine the frames for them also.

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