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Commutative equality algebras and &-equality algebras

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Abstract. The notion of &-equality algebras is introduced, and related properties are investigated. Using &-equality algebras, (commutative ordered) semigroups are induced. Conditions for an equality algebra to be an &-equality algebra and commutative equality algebra are provided. The concept of terminal section of an element is introduced, and several properties are studied. Using the notion of terminal section, conditions for an equality algebra to be a commutative equality algebra are considered. Given a subset of an equality algebra, the $\circ$-set and the $*$-set are introduced, and then several related properties are displayed. Conditions for the $\circ$-set (resp., $*$-set) to be deductive systems are stated.

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1. Introduction

A new structure, called equality algebras, is introduced by Jenei in [4] and it continued in [1, 2, 3, 5, 7]. The study of equality algebras is motivated by EQ-algebras of Novák et al. [6]. The equality algebra has two connectives, a meet operation and an equivalence, and a constant. Novák et al. [6] introduced a closure operator in the class of equality algebras, and discussed relations between equality algebras and BCK-algebras. Zebardast et al. [8] shown that there are relations among equality algebras and some of other logical algebras such as residuated lattice, MTL-algebra, BL-algebra, MV-algebra, Hertz-algebra, Heyting-algebra, Boolean-algebra, EQ-algebra and hoop-algebra. They found that under which conditions, equality algebras are equivalent to these logical algebras. Zebardast et al. [8] also studied commutative equality algebras. They considered characterizations of commutative equality algebras.

In this paper, we introduce the notion of &-equality algebras, and investigate related properties. Using &-equality algebras, we induce (commutative ordered) semigroups. We provide conditions for an equality algebra to be an &-equality algebra. We also provide conditions for an equality algebra to be a commutative equality algebra. We introduce the notion of terminal section of an element in equality algebras, and investigate several properties. Using the notion of terminal section, we consider conditions for an equality algebra to be a commutative equality algebra. Given a subset of an equality algebra, we define an $\circ$-set and $*$-set, and then we investigate several related properties. We consider conditions for the $\circ$-set (resp., $*$-set) to be deductive systems.

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2. Preliminaries

Definition 2.1 ([4]). By an *equality algebra*, we mean an algebra $(E, \wedge, \sim, 1)$ satisfying the following conditions.

(E1) $(E, \wedge, 1)$ is a commutative idempotent integral monoid (i.e., meet semilattice with the top element 1),
(E2) The operation "$\sim$" is commutative,
(E3) $(\forall x \in E)(x \sim x = 1)$,
(E4) $(\forall x \in E)(x \sim 1 = x)$,
(E5) $(\forall x, y, z \in E)(x \leq y \leq z \;\Rightarrow\; x \sim z \leq y \sim z,\; x \sim z \leq x \sim y)$,
(E6) $(\forall x, y, z \in E)(x \sim y \leq (x \wedge z) \sim (y \wedge z))$,
(E7) $(\forall x, y, z \in E)(x \sim y \leq (x \sim z) \sim (y \sim z))$,

where $x \leq y$ if and only if $x \wedge y = x$.

In an equality algebra $(E, \wedge, \sim, 1)$, two operations "$\rightarrow$" and "$\leftrightarrow$" on $\mathcal{E}$ are defined as follows:

$$x \rightarrow y := x \sim (x \wedge y), \tag{2.1}$$
$$x \leftrightarrow y := (x \rightarrow y) \wedge (y \rightarrow x). \tag{2.2}$$

Proposition 2.1 ([4]). *Let $\mathcal{E} := (E, \wedge, \sim, 1)$ be an equality algebra. Then the following assertions are valid: for all $x, y, z \in E$,*

$$x \rightarrow y = 1 \;\Leftrightarrow\; x \leq y, \tag{2.3}$$
$$x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z), \tag{2.4}$$
$$1 \rightarrow x = x,\; x \rightarrow 1 = 1,\; x \rightarrow x = 1, \tag{2.5}$$
$$x \leq y \rightarrow z \;\Leftrightarrow\; y \leq x \rightarrow z, \tag{2.6}$$
$$x \leq y \rightarrow x, \tag{2.7}$$
$$x \leq (x \rightarrow y) \rightarrow y, \tag{2.8}$$
$$x \rightarrow y \leq (y \rightarrow z) \rightarrow (x \rightarrow z), \tag{2.9}$$
$$y \leq x \;\Rightarrow\; x \leftrightarrow y = x \rightarrow y = x \sim y, \tag{2.10}$$
$$x \sim y \leq x \leftrightarrow y \leq x \rightarrow y, \tag{2.11}$$
$$x \leq y \;\Rightarrow\; y \rightarrow z \leq x \rightarrow z,\; z \rightarrow x \leq z \rightarrow y. \tag{2.12}$$

A subset F of E is called a *deductive system* (or *filter*) of $\mathcal{E}$ (see [5]) if it satisfies:

$$1 \in F, \tag{2.13}$$
$$(\forall x, y \in E)(x \in F, x \leq y \;\Rightarrow\; y \in F), \tag{2.14}$$
$$(\forall x, y \in E)(x \in F, x \sim y \in F \;\Rightarrow\; y \in F). \tag{2.15}$$

Lemma 2.2 ([3]). *Let $\mathcal{E}$ be an equality algebra. A subset F of E is a deductive system of $\mathcal{E}$ if and only if it satisfies* (2.13) *and*

$$(\forall x, y \in E)(x \in F, x \rightarrow y \in F \;\Rightarrow\; y \in F). \tag{2.16}$$

3. &-equality algebra

Given an equality algebra $\mathcal{E} = (E, \wedge, \sim, 1)$ and $a, b \in E$, we define

$$E(a, b) := \{x \in E \mid a \leq b \to x\}. \tag{3.1}$$

It is clear that 1, a and b are contained in $E(a, b)$.

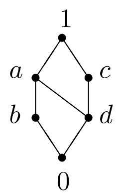
Definition 3.1. An equality algebra $\mathcal{E} = (E, \wedge, \sim, 1)$ is called an *&-equality algebra* if for all $a, b \in E$, the set $E(a, b)$ has the least element which is denoted by $a \odot b$.

It is easy to show that

$$(\forall a, b \in E)(a \odot b \leq a, a \odot b \leq b, a \odot 1 = 1 \odot a = a) \tag{3.2}$$

in the &-equality algebra $\mathcal{E}$.

Example 3.1. Let $E = \{0, a, b, c, d, 1\}$ be a set with the following Hasse diagram.



Then $(E, \wedge, 1)$ is a meet semilattice with top element 1. Define an operation $\sim$ on E by Table 1.

TABLE 1. Cayley table for the binary operation "$\sim$"

$\sim$	0	a	b	c	d	1
0	1	d	c	b	a	0
a	d	1	a	d	c	a
b	c	a	1	0	d	b
c	b	d	0	1	a	c
d	a	c	d	a	1	d
1	0	a	b	c	d	1

Then $\mathcal{E} = (E, \wedge, \sim, 1)$ is an equality algebra, and the implication "$\to$" is given by Table 2.

TABLE 2. Cayley table for the implication "$\to$"

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	a	c	c	1
b	c	1	1	c	c	1
c	b	a	b	1	a	1
d	a	1	a	1	1	1
1	0	a	b	c	d	1

It is routine to verify that $E(x,y)$ has the least element $x \odot y$ for all $x, y \in E$ and it is given by Table 3.

TABLE 3. Cayley table for the operation "$\odot$"

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	b	b	d	0	a
b	0	b	b	0	0	b
c	0	d	0	c	d	c
d	0	0	0	d	0	d
1	0	a	b	c	d	1

Proposition 3.1. *If $\mathcal{E} = (E, \wedge, \sim, 1)$ is an &-equality algebra, then*

$$(\forall a, b \in E)(a \odot b = b \odot a), \tag{3.3}$$

$$(\forall a, b, c \in E)((a \odot b) \odot c = a \odot (b \odot c)), \tag{3.4}$$

$$(\forall a, b, c \in E)(a \leq b \;\Rightarrow\; a \odot c \leq b \odot c). \tag{3.5}$$

Proof. Since $b \leq a \to x$ is equivalent to $a \leq b \to x$, we have $a \odot b = b \odot a$. For any $a, b, c \in E$, we have

$$a \leq b \to (a \odot b) \leq b \to (c \to ((a \odot b) \odot c)) = c \to (b \to ((a \odot b) \odot c)). \tag{3.6}$$

It follows that $a \odot c \leq b \to ((a \odot b) \odot c)$ and so that $(a \odot c) \odot b \leq (a \odot b) \odot c$. Similarly, $(a \odot b) \odot c \leq (a \odot c) \odot b$, and so (3.4) is valid. Let $a, b, c \in E$ be such that $a \leq b$. Then

$$c \leq b \to (c \odot b) \leq a \to (c \odot b),$$

and so $a \odot c = c \odot a \leq c \odot b = b \odot c$. □

By Proposition 3.1, we know that every &-equality algebra is a commutative ordered semigroup under the operation $\odot$.

Proposition 3.2. *In an &-equality algebra $\mathcal{E} = (E, \wedge, \sim, 1)$, we have the following assertions.*

(1) $(a \odot b) \to c = a \to (b \to c)$,
(2) $(a \to b) \odot (b \to c) \leq a \to c$,
(3) $a \odot b \leq a \to b \leq (c \odot a) \to (c \odot b)$.

Proof. Let $a, b, c \in E$. Using (2.4), (2.6), (2.8) and (2.9), we have

$$b \leq (b \to c) \to c \leq (a \to (b \to c)) \to (a \to c) = a \to ((a \to (b \to c)) \to c)$$

and so $a \odot b \leq (a \to (b \to c)) \to c$. It follows from (2.6) that $a \to (b \to c) \leq (a \odot b) \to c$. On the other hand, $a \leq b \to (a \odot b) \leq ((a \odot b) \to c) \to (b \to c)$, which implies from (2.6) that $(a \odot b) \to c \leq a \to (b \to c)$. Thus $(a \odot b) \to c = a \to (b \to c)$. Since $a \to b \leq (b \to c) \to (a \to c)$ by (2.9), we have $(a \to b) \odot (b \to c) \leq a \to c$. Since $a \leq (a \to b) \to b$ by (2.8), we get $a \odot (a \to b) \leq b$. It follows from (3.5) that

$$(c \odot a) \odot (a \to b) = c \odot (a \odot (a \to b)) \leq c \odot b.$$

Hence $a \to b \leq (c \odot a) \to ((c \odot a) \odot (a \to b)) \leq (c \odot a) \to (c \odot b)$. Since $a \leq 1$, it follows that $a \odot b \leq 1 \odot b = b \leq a \to b$. □

Theorem 3.3. *Let $\mathcal{E} = (E, \wedge, \sim, 1)$ be an equality algebra in which there exists a binary operation "&" such that*

$$(\forall a, b, c \in E)(a \to (b \to c) = (a\&b) \to c). \tag{3.7}$$

Then $\mathcal{E} = (E, \wedge, \sim, 1)$ is an &-equality algebra.

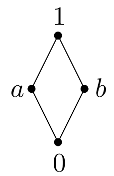
Proof. Let $a, b, c \in E$. The condition (3.7) implies that

$$b \to (c \to (b\&c)) = (b\&c) \to (b\&c) = 1$$

and so $b \leq c \to (b\&c)$, i.e., $b\&c \in E(b, c)$. Assume that $a \in E(b, c)$, that is, $b \leq c \to a$. Then $(b\&c) \to a = b \to (c \to a) = 1$, i.e., $b\&c \leq a$. Thus $b\&c$ is the least element of $E(b, c)$. Therefore $\mathcal{E} = (E, \wedge, \sim, 1)$ is an &-equality algebra. □

The following example illustrates Theorem 3.3.

Example 3.2. Let $E = \{0, a, b, 1\}$ be a set with the following Hasse diagram.



Then $(E, \wedge, 1)$ is a commutative idempotent integral monoid. We define a binary operation $\sim$ on E by Table 4. Then $\mathcal{E} = (E, \wedge, \sim, 1)$ is an equality algebra, and the

TABLE 4. Cayley table for the implication "$\sim$"

$\sim$	0	a	b	1
0	1	b	a	0
a	b	1	0	a
b	a	0	1	b
1	0	a	b	1

implication ($\to$) is given by Table 5. If we define a binary operation "&" on E by

TABLE 5. Cayley table for the implication "$\to$"

$\to$	0	a	b	1
0	1	1	1	1
a	b	1	b	1
b	a	a	1	1
1	0	a	b	1

Table 6, then $x \to (y \to z) = (x\&y) \to z$ for all $x, y, z \in E$. Therefore $\mathcal{E} = (E, \wedge, \sim, 1)$ is an &-equality algebra by Theorem 3.3.

Theorem 3.4. *If $\mathcal{E} = (E, \wedge, \sim, 1)$ is an equality algebra, then the operation "&" which satisfies the condition* (3.7) *is unique and $(E, \&)$ is a commutative semigroup.*

Table 6. Cayley table for the binary operation "&"

&	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

Proof. Let $\&_1$ and $\&_2$ be operations on E satisfying the condition (3.7). For any $x, y \in E$, we have

$$(x\&_1y) \to (x\&_2y) = x \to (y \to (x\&_2y)) = (x\&_2y) \to (x\&_2y) = 1, \tag{3.8}$$

and so $x\&_1y \leq x\&_2y$. Similarly, we get $x\&_2y \leq x\&_1y$. Therefore the operation "&" is unique. Let $x, y, z \in E$. Then

$$\begin{aligned}((x\&y)\&z) \to (x\&(y\&z)) &= (x\&y) \to (z \to (x\&(y\&z))) \\ &= x \to (y \to (z \to (x\&(y\&z)))) \\ &= x \to ((y\&z) \to (x\&(y\&z))) \\ &= (x\&(y\&z)) \to (x\&(y\&z)) = 1.\end{aligned}$$

Similay, we have $(x\&(y\&z)) \to ((x\&y)\&z) = 1$. Hence $((x\&y)\&z) = (x\&(y\&z))$. Also, we get

$$(x\&y) \to (y\&x) = x \to (y \to (y\&x)) = y \to (x \to (y\&x)) = (y\&x) \to (y\&x) = 1.$$

By the similar way, we get $(y\&x) \to (x\&y) = 1$, and so $x\&y = y\&x$. Therefore $(E, \&)$ is a commutative semigroup. □

Theorem 3.5. *For any elements a and b of an equality algebra $\mathcal{E} = (E, \wedge, \sim, 1)$, if $x \in E$ is the greatest element satisfying $x \odot a \leq b$, then $x = a \to b$.*

Proof. Since $a \to b \leq a \to b$, we get $(a \to b) \odot a \leq b$. Assume that $y \in E$ satisfies $y \odot a \leq b$. Then $y \leq a \to (y \odot a) \leq a \to b$. This completes the proof. □

4. Commutative equality algebras

Definition 4.1 ([8])**.** An equality algebra $\mathcal{E} = (E, \wedge, \sim, 1)$ is said to be *commutative* if it satisfies:

$$(\forall x, y \in E)((x \to y) \to y = (y \to x) \to x). \tag{4.1}$$

Lemma 4.1 ([8])**.** *For any equality algebra $\mathcal{E} = (E, \wedge, \sim, 1)$, the following are equivalent.*

(1) $\mathcal{E} = (E, \wedge, \sim, 1)$ *is commutative,*
(2) $(\forall a, b, c \in E)(c \leq a,\ a \to c \leq b \to c \ \Rightarrow\ b \leq a)$,
(3) $(\forall a, b \in E)(a \leq b \ \Rightarrow\ (b \to a) \to a = b)$,
(4) $(\forall a, b \in E)((a \to b) \to b \leq (b \to a) \to a)$.

We provide conditions for an equality algebra to be commutative.

Theorem 4.2. *If an equality algebra* $\mathcal{E} = (E, \wedge, \sim, 1)$ *satisfies:*

$$(\forall a, b \in E)((a \to b) \to b = (((a \to b) \to b) \to a) \to a), \tag{4.2}$$

then it is commutative.

Proof. Let $a, b \in E$ be such that $a \leq b$. Then

$$b = 1 \to b = (a \to b) \to b = (((a \to b) \to b) \to a) \to a)$$
$$= ((1 \to b) \to a) \to a = (b \to a) \to a$$

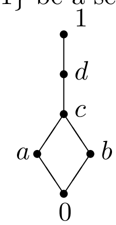
by (2.5) and (4.2). It follows from Lemma 4.1 that $\mathcal{E} = (E, \wedge, \sim, 1)$ is a commutative equality algebra. □

In an equality algebra $\mathcal{E} = (E, \wedge, \sim, 1)$, we consider the following equalities:

$$(x \to y) \to x = x, \tag{4.3}$$
$$x \to (x \to y) = x \to y, \tag{4.4}$$
$$x \to (y \to z) = (x \to y) \to (x \to z). \tag{4.5}$$

The following example shows that the above three equalities are not true in an equality algebra.

Example 4.1. Let $E = \{0, a, b, c, d, 1\}$ be a set with the following Hasse diagram.



Then $(E, \wedge, 1)$ is a commutative idempotent integral monoid. We define a binary operation $\sim$ on E by Table 7.

TABLE 7. Cayley table for the implication "$\sim$"

$\sim$	0	a	b	c	d	1
0	1	d	d	d	c	0
a	d	1	c	d	c	a
b	d	c	1	d	c	b
c	d	d	d	1	d	c
d	c	c	c	d	1	d
1	0	a	b	c	d	1

Then $\mathcal{E} = (E, \wedge, \sim, 1)$ is an equality algebra, and the implication $(\to)$ is given by Table 8.
Then $(a \to b) \to a = c \neq a$, $a \to (a \to b) = 1 \neq d = a \to b$ and

$$a \to (c \to b) = 1 \neq d = (a \to c) \to (a \to b).$$

Proposition 4.3. *Two equalities* (4.4) *and* (4.5) *are equivalent in an equality algebra.*

TABLE 8. Cayley table for the implication "$\to$"

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	d	1	1	1
b	d	d	1	1	1	1
c	d	d	d	1	1	1
d	c	c	c	d	1	1
1	0	a	b	c	d	1

Proof. Let $\mathcal{E} = (E, \wedge, \sim, 1)$ be an equality algebra, and assume that the equality (4.4) is valid. Using (2.4), (2.9) and (2.12), we have

$$u \to (y \to z) \leq u \to ((z \to x) \to (y \to x)) = (z \to x) \to (u \to (y \to x)) \tag{4.6}$$

for all $u, x, y, z \in E$. If we substitute $z \to x$ for x, $z \to y$ for y, $z \to (z \to x)$ for z and $z \to (y \to x)$ for u in (4.6), then

$$\begin{aligned}
&((z \to (z \to x)) \to (z \to x)) \to ((z \to (y \to x)) \to ((z \to y) \to (z \to x)))\\
&= (z \to (y \to x)) \to (((z \to (z \to x)) \to (z \to x)) \to ((z \to y) \to (z \to x)))\\
&\geq (z \to (y \to x)) \to ((z \to y) \to (z \to (z \to x)))\\
&\geq (z \to (y \to x)) \to (y \to (z \to x))\\
&= (z \to (y \to x)) \to (z \to (y \to x)) = 1.
\end{aligned}$$

It follows from (2.5) and (4.4) that $(z \to (y \to x)) \to ((z \to y) \to (z \to x)) = 1$, that is, $z \to (y \to x) \leq (z \to y) \to (z \to x)$. On the other hand,

$$\begin{aligned}
&((z \to y) \to (z \to x)) \to (z \to (y \to x)) = ((z \to y) \to (z \to x)) \to (y \to (z \to x))\\
&\geq y \to (z \to y) = z \to (y \to y) = 1,
\end{aligned}$$

and so $(z \to y) \to (z \to x) \leq z \to (y \to x)$. Therefore (4.5) is valid.

Conversely, if we put $y = x$ in (4.5) and use (2.5), then we have the equality (4.4). □

Proposition 4.4. *Three equalities* (4.3), (4.4) *and* (4.5) *are equivalent in a commutative equality algebra.*

Proof. Let $\mathcal{E} = (E, \wedge, \sim, 1)$ be a commutative equality algebra. Then two equalities (4.4) and (4.5) are equivalent by Proposition 4.3. Assume that (4.3) is valid. Using (4.1), (4.3) and (2.5), we have

$$(x \to (x \to y)) \to (x \to y) = ((x \to y) \to x) \to x = x \to x = 1,$$

and so $x \to (x \to y) \leq x \to y$ for all $x, y \in E$. Since $x \to y \leq x \to (x \to y)$, it follows that $x \to (x \to y) = x \to y$ for all $x, y \in E$. Now suppose (4.4) is valid and let $x, y \in E$. Then

$$((x \to y) \to x) \to x = (x \to (x \to y)) \to (x \to y) = (x \to y) \to (x \to y) = 1$$

by (4.1), (4.4) and (2.5). Hence $(x \to y) \to x \leq x$, and so $(x \to y) \to x = x$ by using (2.7). This completes proof. □

Definition 4.2. For an equality algebra $\mathcal{E} = (E, \wedge, \sim, 1)$, the *terminal section* of an element $a \in E$ is denoted by $\overrightarrow{a}$ and is defined by

$$\overrightarrow{a} = \{b \in E \mid a \leq b\}. \tag{4.7}$$

It is clear that $1, a \in \overrightarrow{a}$, $\overrightarrow{1} = \{1\}$ and $\overrightarrow{0} = E$.

Example 4.2. Let $E = \{0, a, b, c, d, 1\}$ be an equality algebra which is given in Example 3.1. Then $\overrightarrow{0} = E$, $\overrightarrow{a} = \{a, 1\}$, $\overrightarrow{b} = \{a, b, 1\}$, $\overrightarrow{c} = \{c, 1\}$, $\overrightarrow{d} = \{a, c, d, 1\}$ and $\overrightarrow{1} = \{1\}$.

Proposition 4.5. *In an equality algebra* $\mathcal{E} = (E, \wedge, \sim, 1)$*, we have*

$$(\forall a, b \in E)(\overrightarrow{(a \to b) \to b} \subseteq \overrightarrow{a} \cap \overrightarrow{b}). \tag{4.8}$$

Proof. Let $x \in \overrightarrow{(a \to b) \to b}$ for all $a, b \in E$. Then $(a \to b) \to b \leq x$, and so $a, b \leq x$, i.e., $x \in \overrightarrow{a} \cap \overrightarrow{b}$. □

The following example shows that the reverse inclusion in (4.8) is not true.

Example 4.3. Let $E = [0, 1]$ and define the operation $\sim$ on E as follows:

$$x \sim y = \begin{cases} 1 & \text{if } x = 1, \\ \max\{\frac{1}{2} - y, x\} & \text{if } x < y. \\ \max\{\frac{1}{2} - x, y\} & \text{if } x > y. \end{cases}$$

Then $(E, \wedge, \to, 1)$ is an equality algebra and the implication is given by

$$x \to y = \begin{cases} 1 & \text{if } x \leq y, \\ \max\{\frac{1}{2} - x, y\} & \text{if } x > y. \end{cases}$$

Let $a = \frac{1}{2}$ and $b = \frac{1}{3}$. It is clear that $\overrightarrow{a} = [\frac{1}{2}, 1]$ and $\overrightarrow{b} = [\frac{1}{3}, 1]$, and so $\overrightarrow{a} \cap \overrightarrow{b} = [\frac{1}{2}, 1]$. But we have

$$\overrightarrow{(a \to b) \to b} = \overrightarrow{(\tfrac{1}{2} \to \tfrac{1}{3}) \to \tfrac{1}{3}} = \overrightarrow{\tfrac{1}{3} \to \tfrac{1}{3}} = \overrightarrow{1} = \{1\}.$$

Proposition 4.6. *If* $\mathcal{E} = (E, \wedge, \sim, 1)$ *is a commutative equality algebra, then*

$$(\forall a, b \in E)(\overrightarrow{a} \cap \overrightarrow{b} \subseteq \overrightarrow{(a \to b) \to b}). \tag{4.9}$$

Proof. Let $x \in \overrightarrow{a} \cap \overrightarrow{b}$. Then $a \leq x$ and $b \leq x$, i.e., $a \to x = 1$ and $b \to x = 1$, which implies that

$$\begin{aligned}
((a \to b) \to b) \to x &= ((a \to b) \to b) \to (1 \to x) \\
&= ((a \to b) \to b) \to ((b \to x) \to x) \\
&= ((a \to b) \to b) \to ((x \to b) \to b) \\
&= (x \to b) \to (((a \to b) \to b) \to b) \\
&= (x \to b) \to (a \to b) \\
&= a \to ((x \to b) \to b) = a \to ((b \to x) \to x) \\
&= (b \to x) \to (a \to x) = 1 \to 1 = 1.
\end{aligned}$$

Hence $(a \to b) \to b \leq x$, i.e., $x \in \overrightarrow{(a \to b) \to b}$. Therefore (4.9) is valid. □

Theorem 4.7. *If an equality algebra* $\mathcal{E} = (E, \wedge, \sim, 1)$ *satisfies* (4.9)*, then* $\mathcal{E}$ *is commutative.*

Proof. Assume that $\mathcal{E} = (E, \wedge, \sim, 1)$ satisfies (4.9). Then $\overrightarrow{a} \cap \overrightarrow{b} = \overrightarrow{(a \to b) \to b}$ for all $a, b \in E$. Hence

$$\overrightarrow{(a \to b) \to b} = \overrightarrow{a} \cap \overrightarrow{b} = \overrightarrow{b} \cap \overrightarrow{a} = \overrightarrow{(b \to a) \to a},$$

and so $(a \to b) \to b \leq (b \to a) \to a$. It follows from Lemma 4.1 that $\mathcal{E} = (E, \wedge, \sim, 1)$ is a commutative equality algebra. □

Let $\mathcal{E} = (E, \wedge, \sim, 1)$ be an equality algebra. For any $a \in E$ and a subset F of E, we consider the set

$$\mathring{a} := \{x \in E \mid x \to a \leq a\},\ \breve{a} := \{x \in E \mid a \to x \leq x\} \tag{4.10}$$

and

$$F^{\circ} := \bigcap_{a \in F} \mathring{a},\ F^{*} := \bigcap_{a \in F} \breve{a}. \tag{4.11}$$

We say that the set F° (resp., F^{*}) is the $\circ$-*set* (resp., $*$-*set*) of F.

It is clear that

(1) $(\forall a \in E)(1 \in \mathring{a} \cap \breve{a})$,
(2) $\mathring{1} = E = \breve{1}$,
(3) if $\mathcal{E} = (E, \wedge, \sim, 1)$ is a commutative equality algebra, then $\mathring{a} = \breve{a}$ for all $a \in E$ and $F^{\circ} = F^{*}$.

Proposition 4.8. *Let G and F be subsets of an equality algebra $\mathcal{E} = (E, \wedge, \sim, 1)$. Then*

$$1 \in F \ \Rightarrow\ F \cap F^{\circ} = \{1\} = F \cap F^{*}, \tag{4.12}$$

$$(\forall x \in E)(x \in F^{\circ} \ \Leftrightarrow\ (\forall a \in F)(x \to a = a)), \tag{4.13}$$

$$(\forall x \in E)(x \in F^{*} \ \Leftrightarrow\ (\forall a \in F)(a \to x = x)), \tag{4.14}$$

$$G \subseteq F \ \Rightarrow\ F^{\circ} \subseteq G^{\circ},\ F^{*} \subseteq G^{*}. \tag{4.15}$$

Proof. It is obvious that $\{1\} \subseteq F \cap F^{\circ}$. Let $x \in F \cap F^{\circ}$. Then $x \in F$ and $x \in F^{\circ}$. Hence $x \in \mathring{x}$, i.e., $1 = x \to x \leq x$, and so $x = 1$. Therefore $F \cap F^{\circ} = \{1\}$. Similarly, we have $F \cap F^{*} = \{1\}$.

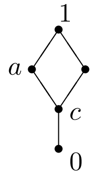
If $x \in F^{\circ}$, then $x \in \mathring{a}$, i.e., $x \to a \leq a$ for all $a \in F$. Since $a \leq x \to a$ by (2.7), it follows that $x \to a = a$ for all $a \in F$. Conversely, if $x \to a = a$ for all $a \in F$, then $x \in \mathring{a}$ for all $a \in F$. Hence $x \in F^{\circ}$. By the similarly way, we know that (4.14) is valid. It is clear that (4.15) is valid. □

Proposition 4.9. *If G and F are subsets of an equality algebra $\mathcal{E} = (E, \wedge, \sim, 1)$, then $(G \cup F)^{\circ} = G^{\circ} \cap F^{\circ}$ and $(G \cup F)^{*} = G^{*} \cap F^{*}$.*

Proof. Since G and F are subsets of $G \cup F$, we have $(G \cup F)^{\circ} \subset G^{\circ}, F^{\circ}$ and $(G \cup F)^{*} \subset G^{*}, F^{*}$ by (4.15). Hence $(G \cup F)^{\circ} \subseteq G^{\circ} \cap F^{\circ}$ and $(G \cup F)^{*} \subseteq G^{*} \cap F^{*}$. If $x \in G^{\circ} \cap F^{\circ}$, then $x \to a = a$ and $x \to b = b$ for all $a \in G$ and $b \in F$ by (4.13). It follows that $x \to c = c$ for all $c \in G \cup F$. Hence $x \in (G \cup F)^{\circ}$, and so $G^{\circ} \cap F^{\circ} \subseteq (G \cup F)^{\circ}$. Similarly, we have $G^{*} \cap F^{*} \subseteq (G \cup F)^{*}$. □

Given a subset F of an equality algebra $\mathcal{E} = (E, \wedge, \sim, 1)$, $F^{\circ\circ}$, $F^{\circ *}$, $F^{*\circ}$ and F^{**} are different in general as seen in the following example.

Example 4.4. Let $E = \{0, a, b, c, 1\}$ be a set with the following Hasse diagram.



Then $(E, \wedge, 1)$ is a commutative idempotent integral monoid. We define a binary operation $\sim$ on E by Table 9. Then $\mathcal{E} = (E, \wedge, \sim, 1)$ is an equality algebra which is

TABLE 9. Cayley table for the implication "$\sim$"

$\sim$	0	a	b	c	1
0	1	0	0	0	0
a	0	1	c	b	a
b	0	c	1	a	b
c	0	b	a	1	c
1	0	a	b	c	1

not commutative, and the implication $(\to)$ is given by Table 10. If we take $F = \{a\}$,

TABLE 10. Cayley table for the implication "$\to$"

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	b	b	1
b	0	a	1	a	1
c	0	1	1	1	1
1	0	a	b	c	1

then $F^\circ = \{b, 1\}$, $F^* = \{0, b, 1\}$, $F^{\circ *} = \{0, a, 1\}$, $F^{\circ\circ} = \{a, 1\}$, $F^{*\circ} = \{a, 1\}$, and $F^{**} = \{1\}$.

Proposition 4.10. *If F is subset of a commutative equality algebra $\mathcal{E} = (E, \wedge, \sim, 1)$, then $F \subseteq F^{\circ\circ} = F^{\circ *} = F^{*\circ} = F^{**}$.*

Proof. If $x \in F$, then $x \to a \leq a$ for all $a \in F^\circ$. Since $\mathcal{E}$ is commutative, it follows that $a \to x \leq x$ for all $a \in F^\circ$. This implies that $x \in F^{\circ *} = F^{\circ\circ} = F^{*\circ} = F^{**}$. □

In the following example, we know that there exists a subset F of a commutative equality algebra such that $F \neq F^{\circ *} = F^{*\circ} = F^{\circ\circ} = F^{**}$.

Example 4.5. Let $\mathcal{E} = (E, \wedge, \sim, 1)$ be the commutative equality algebra in Example 3.2. If we take $F = \{a\}$, then $F^\circ = \{b, 1\}$ and $F \neq \{a, 1\} = F^{\circ *} = F^{*\circ} = F^{\circ\circ} = F^{**}$.

Proposition 4.11. *Let $\mathcal{E} = (E, \wedge, \sim, 1)$ be a commutative equality algebra and F be a subset of E. If $F^\circ = \{1\}$, then $F^{\circ *} = F^{*\circ} = F^{\circ\circ} = F^{**} = E$.*

Proof. Straightforward. □

Theorem 4.12. *Let $\mathcal{E} = (E, \wedge, \sim, 1)$ be an equality algebra. If F is a deductive system of E, then so is F°.*

Proof. It is clear that $1 \in F^\circ$. Let $x, y \in E$ be such that $x \in F^\circ$ and $x \to y \in F^\circ$. Then $x \to a = a$ and $(x \to y) \to a = a$ for all $a \in F$. It follows from (2.4), (2.7) and (2.9) that

$$a \leq y \to a \leq (x \to y) \to (x \to a) = (x \to y) \to a = a.$$

Hence $y \to a = a$ and so $y \in \mathring{a}$ for all $a \in F$. Thus $y \in F^\circ$ and therefore F° is a deductive system of E. □

Question. Let $\mathcal{E} = (E, \wedge, \sim, 1)$ be an equality algebra. If F is a deductive system of E, then is F^* a deductive system of E?

The answer to the question above is negative as seen in the following example.

Example 4.6. Consider the equality algebra $\mathcal{E} = (E, \wedge, \sim, 1)$ in Example 4.4. Note that $F = \{a, b, c, 1\}$ is a deductive system of E. Then $\breve{a} = \{0, b, 1\}$, $\breve{b} = \{0, a, 1\}$, $\breve{c} = \{0, 1\}$ and $\breve{1} = E$. Hence $F^* = \breve{a} \cap \breve{b} \cap \breve{c} \cap \breve{1} = \{0, 1\}$ is not a deductive system of E since $1 \to a = 0 \in F^*$ and $1 \in F^*$, but $a \notin F^*$.

Theorem 4.13. *Let $\mathcal{E} = (E, \wedge, \sim, 1)$ be a commutative equality algebra. If F is a deductive system of E, then so is F^*.*

Proof. Straightforward. □

Proposition 4.14. *Let G and F be subsets of an equality algebra $\mathcal{E} = (E, \wedge, \sim, 1)$ containing the element 1. If $G \subseteq F^\circ$ (resp., $G \subseteq F^*$), then $G \cap F = \{1\}$.*

Proof. If $G \subseteq F^\circ$ (resp., $G \subseteq F^*$), then $G \cap F \subseteq F \cap F^\circ = \{1\}$ (resp., $G \cap F \subseteq F \cap F^* = \{1\}$) by (4.12). □

The following example shows that the converse of Proposition 4.14 is not true in general.

Example 4.7. Let $\mathcal{E} = (E, \wedge, \sim, 1)$ be the equality algebra in Example 3.2. If we take $G = \{0, 1\}$ and $F = \{a, 1\}$, then $G \cap F = \{1\}$ and $F^\circ = \{b, 1\} = F^*$, and so $G \nsubseteq F^\circ$ and $G \nsubseteq F^*$.

We provide conditions for the converse of Proposition 4.14 to be true.

Proposition 4.15. *Let G and F be deductive systems of an equality algebra $\mathcal{E} = (E, \wedge, \sim, 1)$. If $G \cap F = \{1\}$, then $G \subseteq F^\circ$ and $G \subseteq F^*$.*

Proof. Let $x \in G$ and assume that $x \notin F^\circ$. Then there exists $a \in F$ such that $x \notin \mathring{a}$. Hence $(x \to a) \to a \neq 1$, and so $(x \to a) \to a \notin G \cap F$. Since $x \leq (x \to a) \to a$ and G is a deductive system of E, we have $(x \to a) \to a \in G$. Also, since $a \leq (x \to a) \to a$ and F is a deductive system of E, we get $(x \to a) \to a \in F$. This is a contradiction, and therefore $x \in F^\circ$. Hence $G \subseteq F^\circ$. Similarly, we have $G \subseteq F^*$. □

Theorem 4.16. *If F is a subset of an equality algebra $\mathcal{E} = (E, \wedge, \sim, 1)$, then*

$$\{x \in E \mid \langle x \rangle \cap \langle F \rangle = \{1\}\} \subseteq F^\circ. \tag{4.16}$$

Proof. If $x \notin F^\circ$, then there exists $a \in F$ such that $x \to a \neq a$ and so $(x \to a) \to a \neq 1$. Since $a \leq (x \to a) \to a$ and $a \in F$, we have $(x \to a) \to a \in \langle F \rangle$. Also, since $x \leq (x \to a) \to a$, it follows that $(x \to a) \to a \in \langle x \rangle$. Hence $1 \neq (x \to a) \to a \in \langle x \rangle \cap \langle F \rangle$ which shows that (4.16) is valid. □

The following example shows that the reverse inclusion in (4.16) is not true in general.

Example 4.8. Let $E = \{0, a, b, c, 1\}$ be a set with $0 \leq a \leq b \leq c \leq 1$ and define an operation $\sim$ on E by Table 11. Then $\mathcal{E} = (E, \wedge, \sim, 1)$ is an equality algebra, and

TABLE 11. Cayley table for the binary operation "$\sim$"

$\sim$	0	a	b	c	1
0	1	0	0	0	0
a	0	1	b	b	a
b	0	b	1	c	b
c	0	b	c	1	c
1	0	a	b	c	1

the implication "$\to$" is given by Table 12. If we take $F = \{0, 1\}$, then F is not a

TABLE 12. Cayley table for the implication "$\to$"

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	1	1	1
b	0	b	1	1	1
c	0	b	c	1	1
1	0	a	b	c	1

deductive system of E and

$$F^{\circ} = \{a, b, c, 1\} \nsubseteq \{1\} = \{x \in E \mid \langle x \rangle \cap \langle F \rangle = \{1\}\}.$$

Theorem 4.17. *Let $\mathcal{E} = (E, \wedge, \sim, 1)$ be a commutative equality algebra. For any subset F of E, if F° is a deductive system of E, then the reverse inclusion in* (4.16) *is valid.*

Proof. Let $x \in F^{\circ}$. Then $\langle x \rangle \subseteq F^{\circ}$ since F° is a deductive system of E. It is sufficient to show that $F^{\circ} \cap \langle F \rangle = \{1\}$. If $y \in F^{\circ} \cap \langle F \rangle$, then $y \in F^{\circ}$ and $y \in \langle F \rangle$. Hence

$$(\forall a \in F)(y \in \mathring{a}, \text{ that is, } y \to a = a) \tag{4.17}$$

and

$$(\exists a_1, a_2, \cdots, a_n \in F)(a_1 \to (a_2 \to \cdots (a_n \to y) \cdots) = 1). \tag{4.18}$$

Since $\mathcal{E}$ is commutative, (4.17) implies that $a \to y = y$ for all $a \in F$. It follows from (4.18) that $y = 1$. Hence $F^{\circ} \cap \langle F \rangle = \{1\}$, and so $\langle x \rangle \cap \langle F \rangle = \{1\}$. This completes the proof. □

Corollary 4.18. *Let $\mathcal{E} = (E, \wedge, \sim, 1)$ be a commutative equality algebra. For any subset F of E, if F is a deductive system of E, then the reverse inclusion in* (4.16) *is valid.*

Theorem 4.19. *If F is a subset of an equality algebra $\mathcal{E} = (E, \wedge, \sim, 1)$, then*

$$\{x \in E \mid \langle x\rangle \cap \langle F\rangle = \{1\}\} \subseteq F^*. \tag{4.19}$$

Proof. If $x \notin F^*$, then $(a \to x) \to x \neq 1$ for some $a \in F$. Since $x \leq (a \to x) \to x$, it follows that $(a \to x) \to x \in \langle x\rangle$. Also, since $a \leq (a \to x) \to x$ and $a \in F$, we have $(a \to x) \to x \in \langle F\rangle$. Thus $1 \neq (a \to x) \to x \in \langle x\rangle \cap \langle F\rangle$ which shows that (4.19) is valid. □

The following example shows that the reverse inclusion in (4.19) is not true in general.

Example 4.9. Consider the equality algebra $\mathcal{E} = (E, \wedge, \sim, 1)$ in Example 4.8. If we take $F = \{a, 1\}$ which is not a deductive system of E, then

$$F^* = \{0, 1\} \nsubseteq \{1\} = \{x \in E \mid \langle x\rangle \cap \langle F\rangle = \{1\}\}.$$

Theorem 4.20. *Let $\mathcal{E} = (E, \wedge, \sim, 1)$ be a commutative equality algebra. For any subset F of E, if F^* is a deductive system of E, then the reverse inclusion in* (4.19) *is valid.*

Proof. If $y \in F^*$, then $\langle y\rangle \subseteq F^*$ since F^* is a deductive system. Let $z \in F^* \cap \langle F\rangle$. Then $a \to z = z$ for all $a \in F$ by (4.14), and $a_1 \to (a_2 \to \cdots (a_n \to z) \cdots) = 1$ for some $a_1, a_2, \cdots, a_n \in F$. It follows that $z = 1$. Hence $\langle y\rangle \cap \langle F\rangle \subseteq F^* \cap \langle F\rangle = \{1\}$, and so $y \in \{x \in E \mid \langle x\rangle \cap \langle F\rangle = \{1\}\}$. Therefore $F^* \subseteq \{x \in E \mid \langle x\rangle \cap \langle F\rangle = \{1\}\}$. □

Corollary 4.21. *Let $\mathcal{E} = (E, \wedge, \sim, 1)$ be a commutative equality algebra. For any subset F of E, if F is a deductive system of E, then the reverse inclusion in* (4.19) *is valid.*

The following example illustrates Theorems 4.16, 4.17, 4.19 and 4.20.

Example 4.10. (i) Let $\mathcal{E} = (E, \wedge, \sim, 1)$ be the equality algebra which is given in Example 4.4. Suppose $F = \{a, b\}$. It is clear that $F^\circ = \{1\}$. Since $\langle 0\rangle = E, \langle a\rangle = \{a, 1\}, \langle b\rangle = \{1, b\}, \langle c\rangle = \{a, b, c, 1\}$ and $\langle 1\rangle = \{1\}$, we can see that $\{1\} = \{x \in E \mid \langle x\rangle \cap \langle F\rangle = \{1\}\} \subseteq F^\circ = \{1\}$. So, this is the example of Theorem 4.16.

(ii) Let $\mathcal{E}$ be the commutative equality algebra as in Example 3.2. Suppose $F = \{a\}$. Then it is clear that $F^\circ = \{b, 1\}$. Since $\langle 0\rangle = E, \langle a\rangle = \{a, 1\}, \langle b\rangle = \{1, b\}$ and $\langle 1\rangle = \{1\}$, we have $\langle F\rangle = \{1, a\}$, we can see that $\{1, b\} = F^\circ \subseteq \{x \in E \mid \langle x\rangle \cap \langle F\rangle = \{1\}\} = \{1, b\}$. So, this is the example of Theorem 4.17. Since in any commutative equality algebra, $F^\circ = F^*$, so this example is true for Theorem 4.20.

(iii) Let $\mathcal{E}$ be the equality algebra as in Example 4.4. Suppose $F = \{b, c\}$. It is clear that $F^* = \{0, 1\}$ and $\langle F\rangle = \{c, a, b, 1\}$. By (i), we can see that $\{1\} = \{x \in E \mid \langle x\rangle \cap \langle F\rangle = \{1\}\} \subseteq F^* = \{0, 1\}$. So, this is the example of Theorem 4.19.

5. Conclusion

The notion of &-equality algebras is introduced, and by using &-equality algebras, (commutative ordered) semigroups are induced. Also, the concept of terminal section of an element is introduced, and several properties are studied. Using the notion of terminal section, conditions for an equality algebra to be a commutative equality algebra are considered. Given a subset of an equality algebra, the $\circ$-set and the $*$-set are introduced, and then several related properties are displayed. Conditions for the $\circ$-set (resp., $*$-set) to be deductive systems are stated. In future work, we will introduce positive implicative equality algebras and investigate that under which condition an &-equality algebras will be an EQ-algebras.

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