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Invariant Submanifolds of (ϵ)-Sasakian Manifolds

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Abstract. In this paper, we consider invariant submanifolds of an (ϵ)-Sasakian manifolds. We show that if the second fundamental form of an invariant submanifold of a (ϵ)-Sasakian manifold is recurrent then the submanifold is totally geodesic. We also prove that, invariant submanifolds of an Einstein (ϵ)-Sasakian manifolds satisfying the conditions $\widetilde{\mathcal{C}}(X,Y)\cdot\sigma = 0$ and $\widetilde{\mathcal{C}}(X,Y)\cdot\widetilde{\nabla}\sigma = 0$ with $\epsilon r \neq n(n-1)$ are also totally geodesic.

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1. Introduction

It is well known that the properties of a manifold depend on the nature of metric defined on it. In Riemannian geometry, we study manifolds with metric which is positive definite. Since manifolds with indefinite metric have significant use in Physics, and it is interesting to study such manifolds equipped with different structures. In 1969, Takahashi [22] introduced almost contact manifolds equipped with associated pseudo-Riemannian metric. These indefinite almost contact metric manifolds and indefinite Sasakian manifolds are also known as (ϵ)-almost contact metric manifolds and (ϵ)-Sasakian manifols, respectively. The concept of (ϵ)-Sasakian manifolds was further enriched by Bejancu and Duggal [3], Xufend and Xiaoli [24], Rakesh Kumar et al [16] and many others.

In modern analysis, the geometry of submanifolds have become a subject of growing interest for its significant application in applied mathematics and theoretical physics. For instance, the notion of invariant submanifold is used to discuss properties of non-linear autonomous system [12]. The study of geometry of invariant submanifolds was initiated by Bejancu and Papaghuic [4]. In general, the geometry of an invariant submanifold inherits almost all properties of the ambient manifold. On the other hand, a number of works on the geometry of submanifolds of various kinds of almost contact metric manifolds have been carried out in the papers [1, 7, 13, 23] and the references therein. Based on this background, in this paper we consider to study an invariant submanifolds of a class of indefinite almost contact manifold, in particular, (ϵ)-Sasakian manifold.

The present paper is organized as follows: in section 2, we give necessary information about submanifolds and conformal curvature tensor. In section 3, some

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definitions and notions about (ϵ)-Sasakian manifold and their invariant submanifolds are given. In section 4, we consider invariant submanifolds of an (ϵ)-Sasakian manifolds whose second fundamental form is recurrent and show that these type of submanifolds are totally geodesic. We also prove that invariant submanifolds of an (ϵ)-Sasakian manifold with parallel third fundamental form is again a totally geodesic. In the last section, we prove that for an n-dimensional invariant submanifold M of an Einstein (ϵ)-Sasakian manifold $\widetilde{M}$ such that $\epsilon r \neq n(n-1)$ satisfying the conditions $\widetilde{\mathcal{C}}(X,Y)\cdot\sigma = 0$ and $\widetilde{\mathcal{C}}(X,Y)\cdot\widetilde{\nabla}\sigma = 0$ imply that M is totally geodesic, where $\mathcal{C}$ is a conformal curvature tensor.

2. Preliminaries

Let $\widetilde{M}$ be an n-dimensional differentiable manifold endowed with an almost contact structure (ϕ,ξ,η), where ϕ is a $(1,1)$-tensor field, ξ is a vector field and η is a 1-form on $\widetilde{M}$ satisfying

$$\eta(\xi) = 1, \quad \phi^2 X = -X + \eta(X)\xi. \tag{1}$$

It follows that

$$\eta\cdot\phi = 0, \quad \phi(\xi) = 0, \quad rank\ \ \phi = 2n; \tag{2}$$

then $\widetilde{M}$ is called an almost contact manifold. If there exists a psedo-Riemannian metric g satisfying

$$g(\phi X,\phi Y) = g(X,Y) - \epsilon\eta(X)\eta(Y), \quad \forall X,Y \in \chi(\widetilde{M}), \tag{3}$$

where $\epsilon = \pm 1$, then the structure (ϕ,ξ,η,g) is called an (ϵ)-almost contact metric structure and $\widetilde{M}$ is called an (ϵ)-almost contact metric manifold. For an (ϵ)-almost contact metric manifold $\widetilde{M}$, we have

$$\eta(X) = \epsilon g(X,\xi)\ \ and\ \ \epsilon = g(\xi,\xi)\ \ \forall X \in \chi(\widetilde{M}), \tag{4}$$

hence, ξ is never a light-like vector field on $\widetilde{M}$ and according to the casual character of ξ we have two classes of (ϵ)-almost contact metric manifolds. When $\epsilon = -1$ and index of g is odd, then $\widetilde{M}$ is a time-like almost contact metric manifold and when $\epsilon = -1$ and index of g is even, then $\widetilde{M}$ is a space-like almost contact metric manifold. Further, $\widetilde{M}$ is usual almost contact metric manifold for $\epsilon = 1$ and index of g is 0 and $\widetilde{M}$ is a Lorentz-almost contact metric manifold for $\epsilon = -1$ and index of g is 1.

If $d\eta(X,Y) = g(\phi X,Y)$, then $\widetilde{M}$ is said to have (ϵ)-contact metric structure (ϕ,ξ,η,g). If moreover, this structure is normal, that is, if

$$[\phi X,\phi Y] + \phi^2[X,Y] - \phi[X,\phi Y] - \phi[\phi X,Y] = -2d\eta(X,Y)\xi, \tag{5}$$

then the (ϵ)-contact metric structure is called an (ϵ)-Sasakian structure and the manifold endowed with this structure is called an (ϵ)-Sasakian manifold. Generally, an (ϵ)-almost contact metric structure (ϕ,ξ,η,g) is said to be an (ϵ)-Sasakian manifold $\widetilde{M}$ if and only if

$$(\widetilde{\nabla}_X\phi)Y = g(X,Y)\xi - \epsilon\eta(Y)X, \quad \forall X,Y \in \chi(\widetilde{M}), \tag{6}$$

where $\widetilde{\nabla}$ is the Levi-Civita connection with respect to g. Also, from (6) we have

$$\widetilde{\nabla}_X \xi = -\epsilon\phi X \quad \forall X \in \chi(\widetilde{M}). \tag{7}$$

In an (ϵ)-Sasakian manifold $\widetilde{M}$, the following relations hold: (see [24])

$$\begin{aligned} \widetilde{R}(\xi, X)Y &= \epsilon g(X,Y)\xi - \eta(Y)X, \\ \widetilde{S}(X,\xi) &= \epsilon(n-1)\eta(X), \\ \widetilde{Q}\xi &= (n-1)\xi \end{aligned} \tag{8}$$

for all vector fields X, Y on $\widetilde{M}$.
An (ϵ)-Sasakian manifold $\widetilde{M}$ is called Einstein if we have

$$\widetilde{S}(X,Y) = \frac{\widetilde{r}}{n} g(X,Y) \tag{9}$$

for all X, Y tangent to $\widetilde{M}$. This gives

$$\widetilde{S}(X,\xi) = \epsilon\frac{\widetilde{r}}{n}\eta(X), \;\; \widetilde{S}(\xi,\xi) = \epsilon\frac{\widetilde{r}}{n}, \;\; \widetilde{Q}X = \frac{\widetilde{r}}{n}X, \;\; \widetilde{Q}\xi = \frac{\widetilde{r}}{n}\xi \tag{10}$$

for each X tangent to $\widetilde{M}$.

For an n-dimensional ($n \geq 3$) Riemannian manifold (M, g), the conformal curvature tensor C of M is defined by [5]

$$\begin{aligned} \mathcal{C}(X,Y)Z =& \widetilde{R}(X,Y)Z - \frac{1}{n-2}\{g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X \\ &- S(X,Z)Y\} + \frac{r}{(n-1)(n-2)}\{g(Y,Z)X - g(X,Z)Y\}, \end{aligned} \tag{11}$$

where Q is the Ricci operator defined by $g(QX,Y) = S(X,Y)$ and r is the scalar curvature. Moreover, in an Einstein (ϵ)-Sasakian manifold from (11), (8) and (10) we also have

$$\begin{aligned} \mathcal{C}(\xi, X)Y &= \left(\epsilon - \frac{r}{n(n-1)}\right)\{g(X,Y)\xi - \epsilon\eta(Y)X\} \\ &= \left(1 - \frac{\epsilon r}{n(n-1)}\right)\{\epsilon g(X,Y)\xi - \eta(Y)X\} \end{aligned} \tag{12}$$

and

$$\begin{aligned} \mathcal{C}(\xi, X)\xi &= \epsilon\left(\epsilon - \frac{r}{n(n-1)}\right)\{-X + \eta(X)\xi\} \\ &= \left(1 - \frac{\epsilon r}{n(n-1)}\right)\{-X + \eta(X)\xi\}. \end{aligned} \tag{13}$$

3. Invariant submanifolds of (ϵ)-Sasakian manifolds

Let M be a submanifold of an (ϵ)-Sasakian manifold $\widetilde{M}$ with induced metric g. Also let ∇ and $\nabla^{\perp}$ be the induced connection on the tangent bundle TM and the normal bundle $T^{\perp}M$ of M respectively. Then the formulas of Gauss and Weingarten are given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + \sigma(X,Y), \tag{14}$$

$$\widetilde{\nabla}_X N = -A_N X + \nabla^{\perp}_X N \tag{15}$$

for tangent vectors X, Y and $N \in \chi(T^{\perp}M)$, where σ and A_N are second fundamental form and the shape operator respectively for the immersion of M into $\widetilde{M}$. If the second fundamental form σ is identically zero then the submanifold is said to be *totally geodesic*. The second fundamental form σ and A_N are related by

$$\widetilde{g}(\sigma(X,Y),N) = g(A_N X, Y)$$

for tangent vector fields X, Y. We note that $\sigma(X,Y)$ is bilinear and since $\nabla_{fX}Y = f\nabla_X Y$ for any smooth function f on a manifold, we have $\sigma(fX,Y) = f\sigma(X,Y)$.

Definition 3.1. [4] A submanifold M of an (ϵ)-Sasakian manifold $\widetilde{M}$ is called an *invariant submanifold* of $\widetilde{M}$ if for each $x \in M$, $\phi(T_xM) \subset T_xM$. As a consequence, ξ becomes tangent to M. The submanifold M of the (ϵ)-Sasakian manifold $\widetilde{M}$ is called totally geodesic if $\sigma(X,Y) = 0$ for any tangent vectors X, Y.

The first and second covariant derivatives of the second fundamental form σ are given by

$$(\widetilde{\nabla}_X\sigma)(Y,Z) = \nabla^{\perp}_X(\sigma(Y,Z)) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z) \tag{16}$$

and

$$\begin{aligned}(\widetilde{\nabla}^2\sigma)(Z,W,X,Y) =&(\widetilde{\nabla}_X\widetilde{\nabla}_Y\sigma)(Z,W),\\ =&\nabla^{\perp}_X((\widetilde{\nabla}_Y\sigma)(Z,W)) - (\widetilde{\nabla}_Y\sigma)(\nabla_X Z, W)\\ &- (\widetilde{\nabla}_X\sigma)(Z,\nabla_Y W) - (\widetilde{\nabla}_{\nabla_X Y}\sigma)(Z,W),\end{aligned} \tag{17}$$

respectively, where $\widetilde{\nabla}$ is called the *van der Waerden-Bortolotti connection* of $\widetilde{M}$ [6]. If $\widetilde{\nabla}\sigma = 0$, then M is said to have *parallel second fundamental form* [6].

An immersion is said to be *semiparallel* if

$$\widetilde{R}(X,Y)\cdot\sigma = (\widetilde{\nabla}_X\widetilde{\nabla}_Y - \widetilde{\nabla}_Y\widetilde{\nabla}_X - \widetilde{\nabla}_{[X,Y]})\sigma = 0, \tag{18}$$

holds for all vector fields X, Y tangent to $\widetilde{M}$ [8], where $\widetilde{R}$ denotes the curvature tensor of the connection $\widetilde{\nabla}$. Semiparallel immersions have been studied by various authors, see for example ([9], [10], [11], [15], [17]). An immersion is said to be 2-*semiparallel* if $\widetilde{R}(X,Y)\cdot\widetilde{\nabla}\sigma = 0$ holds for all vector fields X, Y tangent to M. 2-semiparallel immersions have been studied in ([2], [18]). From the Gauss and Weingarten formulas we obtain

$$\begin{aligned}(\widetilde{R}(X,Y)Z)^T =&R(X,Y)Z - A_{\sigma(Y,Z)}X + A_{\sigma(X,Z)}Y\\ &+ (\nabla_X\sigma)(Y,Z) - (\nabla_Y\sigma)(X,Z).\end{aligned} \tag{19}$$

By (18), we have

$$(\widetilde{R}(X,Y)\cdot\sigma)(U,V) = R^{\perp}(X,Y)\sigma(U,V) - \sigma(R(X,Y)U,V) - \sigma(U,R(X,Y)V) \tag{20}$$

for all vector fields X, Y, U and V tangent to M, where

$$R^{\perp}(X,Y) = [\nabla^{\perp}_X, \nabla^{\perp}_Y] - \nabla^{\perp}_{[X,Y]}. \tag{21}$$

Similarly, we have

$$\begin{aligned}(\widetilde{R}(X,Y)\cdot\widetilde{\nabla}\sigma)(U,V,W) =& R^{\perp}(X,Y)(\widetilde{\nabla}\sigma)(U,V,W) - (\widetilde{\nabla}\sigma)(R(X,Y)U,V,W)\\ &- (\widetilde{\nabla}\sigma)(U,R(X,Y)V,W) - (\widetilde{\nabla}\sigma)(U,V,R(X,Y)W)\end{aligned} \tag{22}$$

for all vector fields X, Y, U, V, W tangent to M, where $(\widetilde{\nabla}\sigma)(U,V,W) = (\widetilde{\nabla}_U\sigma)(V,W)$ [2]. Again for the conformal curvature tensor $\mathcal{C}$ we have

$$(\widetilde{\mathcal{C}}(X,Y)\cdot\sigma)(U,V) = R^{\perp}(X,Y)\sigma(U,V) - \sigma(\mathcal{C}(X,Y)U,V) - \sigma(U,\mathcal{C}(X,Y)V) \tag{23}$$

and

$$\begin{aligned}(\widetilde{\mathcal{C}}(X,Y)\cdot\widetilde{\nabla}\sigma)(U,V,W) =& R^{\perp}(X,Y)(\widetilde{\nabla}\sigma)(U,V,W) - (\widetilde{\nabla}\sigma)(\mathcal{C}(X,Y)U,V,W)\\ &- (\widetilde{\nabla}\sigma)(U,\mathcal{C}(X,Y)V,W) - (\widetilde{\nabla}\sigma)(U,V,\mathcal{C}(X,Y)W),\end{aligned} \tag{24}$$

respectively.

Now M is an invariant submanifold of an (ϵ)-Sasakian manifold $\widetilde{M}$. By Gauss formula, we have

$$-\epsilon\phi X = \widetilde{\nabla}_X\xi = \nabla_X\xi + \sigma(X,\xi), \tag{25}$$

this implies that

$$(a)\ \nabla_X\xi = -\epsilon\phi X \quad and \quad (b)\ \sigma(X,\xi) = 0 \tag{26}$$

for each vector field X tangent to M. Next,

$$\begin{aligned}\widetilde{\nabla}_X\phi Y &= \nabla_X\phi Y + \sigma(X,\phi Y)\\ &= (\nabla_X\phi)Y + \phi\nabla_X Y + \sigma(X,\phi Y)\end{aligned} \tag{27}$$

and

$$\begin{aligned}\widetilde{\nabla}_X\phi Y &= (\widetilde{\nabla}_X\phi)Y + \phi\widetilde{\nabla}_X Y\\ &= g(X,Y)\xi - \epsilon\eta(Y)X + \phi\nabla_X Y + \phi\sigma(X,Y)\end{aligned} \tag{28}$$

for each vector field X and Y tangent to M. On solving (27) and (28), we get

$$(\nabla_X\phi)Y = g(X,Y)\xi - \epsilon\eta(Y)X, \tag{29}$$

$$\sigma(X,\phi Y) = \phi\sigma(X,Y). \tag{30}$$

From the Gauss formula (14), we have

$$\widetilde{R}(X,Y)\xi = R(X,Y)\xi + A_{\sigma(X,\xi)}Y - A_{\sigma(Y,\xi)}X. \tag{31}$$

Then using second equation of (26), we find

$$\widetilde{R}(X,Y)\xi = R(X,Y)\xi. \tag{32}$$

Contracting (32), we obtain

$$\widetilde{S}(X,\xi) = S(X,\xi). \tag{33}$$

Hence, we state the following theorem:

Theorem 3.1. *An invariant submanifold M of an (ϵ)-Sasakian manifold $\widetilde{M}$ is an (ϵ)-Sasakian manifold.*

4. Recurrent invariant submanifolds of (ϵ)-Sasakian manifolds

In [14], Kon showed that a submanifold M of a Kenmotsu manifold $\widetilde{M}$ has parallel second fundamental form if and only if M is totally geodesic. In [21], Sular and Ozgur showed that in a submanifold M of a Kenmotsu manifold $\widetilde{M}$ tangent to ξ, the second fundamental form is recurrent if and only if M is totally geodesic. Also, in [20], Sarkar and Sen studied the geometry of submanifolds of trans-Sasakian manifolds and they proved that an invariant submanifold of a trans-Sasakian manifold is totally geodesic if and only if its second fundamental form is recurrent.
The covariant differential of the p^{th} order, $p \geq 1$ of a $(0,k)$-tensor field T, $k \geq 1$ denoted by $\nabla^p T$, defined on a Riemannian manifold (M,g) with the Levi-Civita connection ∇. The tensor T is said to be *recurrent* [19], if the following condition holds on M:

$$(\nabla T)(X_1, ..., X_k; X)T(Y_1, ..., Y_k) = (\nabla T)(Y_1, ..., Y_k; X)T(X_1, ..., X_k), \tag{34}$$

where $X, X_1, Y_1, ..., X_k, Y_k \in TM$. From (34) it follows that at a point $x \in M$, if the tensor T is non-zero then there exists a unique 1-form ω or a $(0,2)$-tensor ψ, defined on a neighborhood U of x, such that [20]

$$\nabla T = T \otimes \omega, \quad \omega = d(log\|T\|) \tag{35}$$

or

$$\nabla^2 T = T \otimes \psi, \tag{36}$$

respectively holds on U, where $\|T\|$ denotes the norm of T and $\|T\|^2 = g(T,T)$.

In this section, we begin with the following:

Theorem 4.1. *Let M be an invariant submanifold of an (ϵ)-Sasakian manifold $\widetilde{M}$. Then the second fundamental form σ is recurrent if and only if M is totally geodesic.*

Proof. Let us suppose that σ is recurrent, from (35) we get

$$(\widetilde{\nabla}_X \sigma)(Y,Z) = A(X)\sigma(Y,Z), \tag{37}$$

where A is a 1-form on M. Then in view of (16), the above equation can be written as

$$\nabla^{\perp}_X \sigma(Y,Z) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z) = A(X)\sigma(Y,Z). \tag{38}$$

Setting $Z = \xi$ in (38) we have

$$\nabla^{\perp}_X \sigma(Y,\xi) - \sigma(\nabla_X Y, \xi) - \sigma(Y, \nabla_X \xi) = A(X)\sigma(Y,\xi). \tag{39}$$

Making use of relation (26(b)) in above equation, we obtain

$$\sigma(Y, \nabla_X \xi) = 0,$$

from which it follows that

$$\sigma(Y, -\epsilon\phi X) = -\epsilon\sigma(Y, \phi X) = 0. \tag{40}$$

Then by virtue of (30), we have from (40) that

$$\sigma(Y, X) = 0.$$

Therefore, it shows that M is totally geodesic. The converse statement is trivial. This completes the proof of the theorem. □

Corollary 4.2. *Let M be an invariant submanifold of an (ϵ)-Sasakian manifold. Then M has parallel second fundamental form if and only if M is totally geodesic.*

Proof. If the 1-form A vanishes, then the relation (37) reduces to

$$(\widetilde{\nabla}_X \sigma)(Y, Z) = 0, \tag{41}$$

that is, M has parallel second fundamental form. So the similar calculation as we did for the Theorem(4.1) shows that M is totally geodesic. The converse statement is trivial. This completes the proof of the corollary. □

Theorem 4.3. *Let M be an invariant submanifold of an (ϵ)-Sasakian manifold $\widetilde{M}$. Then M has parallel third fundamental form if and only if M is totally geodesic.*

Proof. Suppose that M has parallel third fundamental form. Then we can write

$$(\widetilde{\nabla}_X \widetilde{\nabla}_Y \sigma)(Z, W) = 0. \tag{42}$$

Setting $W = \xi$ in (42) and using (17) we obtain

$$\nabla^{\perp}_X((\widetilde{\nabla}_Y \sigma)(Z, \xi)) - (\widetilde{\nabla}_Y \sigma)(\nabla_X Z, \xi) - (\widetilde{\nabla}_X \sigma)(Z, \nabla_Y \xi) - (\widetilde{\nabla}_{\nabla_X Y} \sigma)(Z, \xi) = 0. \tag{43}$$

By (16) and (26(b)) we have the following equalities

$$\nabla^{\perp}_X((\widetilde{\nabla}_Y \sigma)(Z, \xi)) = -\epsilon \nabla^{\perp}_X \sigma(Z, \phi Y), \tag{44}$$

$$(\widetilde{\nabla}_Y \sigma)(\nabla_X Z, \xi) = \epsilon \sigma(\nabla_X Z, \phi Y), \tag{45}$$

$$(\widetilde{\nabla}_X \sigma)(Z, \nabla_Y \xi) = -\epsilon(\widetilde{\nabla}_X \sigma)(Z, \phi Y), \tag{46}$$

$$(\widetilde{\nabla}_{\nabla_X Y} \sigma)(Z, \xi) = \epsilon \sigma(Z, \phi \nabla_X Y). \tag{47}$$

Then substituting (44) - (47) into (43) we have

$$\epsilon \sigma(Z, (\nabla_X \phi) Y) = 0. \tag{48}$$

Setting Y by ξ in (48) and using (26(b)) and (29), we get

$$\sigma(Z, X) = 0.$$

It shows that M is totally geodesic. The converse statement is trivial. Hence, the proof of the theorem is completed. □

5. Invariant submanifolds of Einstein (ϵ)-Sasakian manifolds satisfying $\widetilde{\mathcal{C}}(X,Y) \cdot \sigma = 0$ and $\widetilde{\mathcal{C}}(X,Y) \cdot \widetilde{\nabla}\sigma = 0$

Recently, the authors Ozgur and Murathan [18] considered invariant submanifolds of Lorentzian para-Sasakian manifolds satisfying the conditions $\mathcal{Z}(X,Y) \cdot \sigma = 0$ and $\mathcal{Z}(X,Y) \cdot \widetilde{\nabla}\sigma = 0$, where $\mathcal{Z}$ is the concircular curvature tensor. As a continuation of this study, in this section we consider invariant submanifold of Einstein (ϵ)-Sasakian manifolds satisfying the conditions $\widetilde{\mathcal{C}}(X,Y) \cdot \sigma = 0$ and $\widetilde{\mathcal{C}}(X,Y) \cdot \widetilde{\nabla}\sigma = 0$, where $\mathcal{C}$ is the conformal curvature tensor.

Theorem 5.1. *Let M be an invariant submanifold of an Einstein (ϵ)-Sasakian manifold $\widetilde{M}$. Then the condition $\widetilde{\mathcal{C}}(X,Y) \cdot \sigma = 0$ holds on M if and only if M is totally geodesic provided $\epsilon r \neq n(n-1)$.*

Proof. Suppose that M satisfies the condition $\widetilde{\mathcal{C}}(X,Y)\cdot\sigma=0$. Then, from (23), it follows that

$$R^{\perp}(X,Y)\sigma(U,V)-\sigma(\mathcal{C}(X,Y)U,V)-\sigma(U,\mathcal{C}(X,Y)V)=0. \tag{49}$$

By plugging $X=V=\xi$ in (49) we get

$$R^{\perp}(\xi,Y)\sigma(U,\xi)-\sigma(\mathcal{C}(\xi,Y)U,\xi)-\sigma(U,\mathcal{C}(\xi,Y)\xi)=0.$$

By virtue of (26(b)) the above equation is reduces to

$$\sigma(U,\mathcal{C}(\xi,Y)\xi)=0.$$

Using (13), one can get

$$(1-\frac{\epsilon r}{n(n-1)})\sigma(U,-Y+\eta(Y)\xi)=0.$$

From the assumption, since $\epsilon r\neq n(n-1)$, by virtue of (26), we obtain $\sigma(Y,U)=0$, which gives us that M is totally geodesic. The converse statement is trivial. This completes the proof of the theorem. □

Theorem 5.2. *Let M be an invariant submanifold of an Einstein (ϵ)-Sasakian manifold $\widetilde{M}$. Then the condition $\widetilde{\mathcal{C}}(X,Y)\cdot\widetilde{\nabla}\sigma=0$ holds on M if and only if M is totally geodesic provided $\epsilon r\neq n(n-1)$.*

Proof. Suppose that M satisfies the condition $\widetilde{\mathcal{C}}(X,Y)\cdot\widetilde{\nabla}\sigma=0$. Then, from (24), we obtain

$$\begin{aligned}R^{\perp}(X,Y)(\widetilde{\nabla}\sigma)(U,V,W)&-(\widetilde{\nabla}\sigma)(\mathcal{C}(X,Y)U,V,W)\\&-(\widetilde{\nabla}\sigma)(U,\mathcal{C}(X,Y)V,W)-(\widetilde{\nabla}\sigma)(U,V,\mathcal{C}(X,Y)W)=0.\end{aligned} \tag{50}$$

By putting $X=V=\xi$ in (50) we have

$$\begin{aligned}R^{\perp}(\xi,Y)(\widetilde{\nabla}\sigma)(U,\xi,W)&-(\widetilde{\nabla}\sigma)(\mathcal{C}(\xi,Y)U,\xi,W)\\&-(\widetilde{\nabla}\sigma)(U,\mathcal{C}(\xi,Y)\xi,W)-(\widetilde{\nabla}\sigma)(U,\xi,\mathcal{C}(\xi,Y)W)=0.\end{aligned} \tag{51}$$

Then, in view of (16), (7) and (26(b)) we have

$$(\widetilde{\nabla}\sigma)(U,\xi,W)=\epsilon\sigma(\phi U,W). \tag{52}$$

Also, from (16), (12), (13) and (26), we get the following equalities:

$$\begin{aligned}(\widetilde{\nabla}\sigma)(\mathcal{C}(\xi,Y)U,\xi,W)=&(\widetilde{\nabla}_{\mathcal{C}(\xi,Y)U}\sigma)(\xi,W)=\nabla^{\perp}_{\mathcal{C}(\xi,Y)U}(\sigma(\xi,W))\\&-\sigma(\nabla_{\mathcal{C}(\xi,Y)U}\xi,W)-\sigma(\xi,\nabla_{\mathcal{C}(\xi,Y)U}W)\\=&-\epsilon\left(1-\frac{\epsilon r}{n(n-1)}\right)\eta(U)\sigma(\phi Y,W),\end{aligned} \tag{53}$$

$$\begin{aligned}(\widetilde{\nabla}\sigma)(U,\mathcal{C}(\xi,Y)\xi,W) =&(\widetilde{\nabla}_U\sigma)(\mathcal{C}(\xi,Y)\xi,W) = \nabla^{\perp}_U(\sigma(\mathcal{C}(\xi,Y)\xi,W))\\ &- \sigma(\nabla_U\mathcal{C}(\xi,Y)\xi,W) - \sigma(\mathcal{C}(\xi,Y)\xi,\nabla_U W)\\ =& -\nabla^{\perp}_U\left(\left(1-\frac{\epsilon r}{n(n-1)}\right)\sigma(Y,W)\right)\\ &-\sigma\left(\nabla_U\left[\left(1-\frac{\epsilon r}{n(n-1)}\right)(-Y+\eta(Y)\xi)\right],W\right)\\ &+\left(1-\frac{\epsilon r}{n(n-1)}\right)\sigma(Y,\nabla_U W) \end{aligned}\tag{54}$$

and

$$\begin{aligned}(\widetilde{\nabla}\sigma)(U,\xi,\mathcal{C}(\xi,Y)W) &= (\widetilde{\nabla}_U\sigma)(\xi,\mathcal{C}(\xi,Y)W)\\ &= \nabla^{\perp}_U\sigma(\xi,\mathcal{C}(\xi,Y)W) - \sigma(\nabla_U\xi,\mathcal{C}(\xi,Y)W) - \sigma(\xi,\nabla_U\mathcal{C}(\xi,Y)W)\\ &= -\epsilon\left(1-\frac{\epsilon r}{n(n-1)}\right)\sigma(\phi U,Y)\eta(W). \end{aligned}\tag{55}$$

From (50), it follows by virtue of equalities (52)-(55) that

$$\begin{aligned}&\epsilon R^{\perp}(\xi,Y)\sigma(\phi U,W) + \epsilon\left(1-\frac{\epsilon r}{n(n-1)}\right)\eta(U)\sigma(\phi Y,W)\\ &+\nabla^{\perp}_U\left(\left(1-\frac{\epsilon r}{n(n-1)}\right)\sigma(Y,W)\right) + \sigma\left(\nabla_U\left(1-\frac{\epsilon r}{n(n-1)}\right)(-Y+\eta(Y)\xi),W\right)\\ &-\left(1-\frac{\epsilon r}{n(n-1)}\right)\sigma(Y,\nabla_U W) + \epsilon\left(1-\frac{\epsilon r}{n(n-1)}\right)\eta(W)\sigma(\phi U,Y) = 0. \end{aligned}\tag{56}$$

Substituting W by ξ in (56) and using (26(b)), we obtain

$$\left(1-\frac{\epsilon r}{n(n-1)}\right)\{\epsilon\sigma(\phi U,Y) - \sigma(Y,\nabla_U\xi)\} = 0,\tag{57}$$

In view of (7), (57) yields

$$\sigma(\phi U,Y) = 0 \quad with \quad \epsilon r \neq n(n-1).\tag{58}$$

Replacing U by ϕU and using (1) and (26(b)), we get from (58)

$$\sigma(U,Y) = 0.$$

This shows that, M is totally geodesic. The converse statement is trivial. Hence our theorem is proved. □

In view of Theorems (4.1), (4.3), (5.1) and (5.2) we concluded

Corollary 5.3. *For an invariant submanifold M of an (ϵ)-Sasakian manifold $\widetilde{M}$ the following conditions are equivalent:*

(1) *the second fundamental form of M is parallel;*
(2) *the second fundamental form of M is recurrent;*
(3) *the second fundamental form of M is parallel;*
(4) *the condition $\widetilde{\mathcal{C}}(X,Y)\cdot\sigma = 0$ with $\epsilon r \neq n(n-1)$ holds on M and M is Einstein;*
(5) *the condition $\widetilde{\mathcal{C}}(X,Y)\cdot\widetilde{\nabla}\sigma = 0$ with $\epsilon r \neq n(n-1)$ holds on M and M is Einstein;*
(6) *M is totally geodesic.*

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